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Varieties of algebras as a lattice with an additional operation

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1. Introduction

Let $\mathfrak{k}$ be a non-trivial associative and commutative ring with 1. In the present paper we are concerned with varieties (equational classes) of $\mathfrak{k}$-algebras that are not necessarily associative and not necessarily with 1. These are classes of algebras closed under the formation of subalgebras, homomorphic images and Cartesian products; equivalently, classes of all algebras satisfying given sets of polynomial identities. Two basic properties of free groups enhanced the theory of group varieties: a subgroup of a free group is free, and a fully invariant subgroup of a fully invariant subgroup of a free group is fully invariant. Given two group varieties $\mathscr{U}$, $\mathscr{V}$, $\mathscr{U}\cdot\mathscr{V}$ is the class of all groups that are Schreier-extensions of a group in $\mathscr{U}$ by a group in $\mathscr{V}$. It turns out that $\mathscr{U}\cdot\mathscr{V}$ is a variety. Under this multiplication, the groupoid of group varieties is a free monoid with zero. This was shown independently by B. H. NEUMANN, HANNA NEUMANN and P. M. NEUMANN [15] and by A. L. ŠMELKIN [21]. A similar result holds for the groupoid of Lie algebra varieties over a field of characteristic 0; this is due to V. A. PARFENOV [18]. A subalgebra of a free associative algebra need not be free, P. M. COHN [4]. A T-ideal of a T-ideal of a free associative algebra may not be a T-ideal, A. I. MAL'CEV [13], A. A. ISKANDER [11]. It turns out that the groupoid of ring varieties is not associative and certainly not relatively free. It is not even power associative. The groupoid of varieties of $\mathfrak{k}$-algebras contains infinite submonoids. This groupoid has some sort of decomposition. The minimal varieties are determined. If $\mathfrak{k}$ has exactly 2 idempotent ideals, then a family of identities that is attainable on all power associative algebras is equivalent to $x=x$ or $x=y$.

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The word "algebra" will mean "$\mathfrak{k}$-algebra". The word "variety" will mean "variety of $\mathfrak{k}$-algebras". An algebra is called power-associative if every subalgebra generated by one element is associative. By a theorem of A. A. ALBERT [1], [2], if $\mathfrak{k}$ is a field of characteristic not 2, 3 or 5, then an algebra is power associative if it satisfies $(xx)x=x(xx)$ and $((xx)x)x=(xx)(xx)$. Let $\mathscr{A}i\mathfrak{k}$, $i=0,1,2,3$, denote, respectively, the varieties of all algebras, all power-associative algebras, all associative algebras and all associative and commutative algebras. If $\mathscr{V}$ is a variety, we denote by $L\mathscr{V}$ the set of all subvarieties of $\mathscr{V}$. Under class inclusion $L\mathscr{V}$ is a complete modular lattice. Under an additional operation $L\mathscr{V}$ is a partially ordered groupoid with zero ($\mathscr{V}$) and 1 ($\mathscr{E}$); where $\mathscr{E}$ is the trivial variety of one-element algebras.

For an account of the variety theory, the reader may consult [3], [5], [10], [14], [16], [17].

Definition 1. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C} \in \ 0\mathfrak{k}$. Then $\mathfrak{C}$ is an extension of $\mathfrak{A}$ by $\mathfrak{B}$ if $\mathfrak{C}$ possesses an ideal isomorphic to $\mathfrak{A}$ whose factor is isomorphic to $\mathfrak{B}$. If $\mathscr{U}$, $\mathscr{V}$, $\mathscr{K}$ are classes of algebras, then $\mathscr{U} \cdot_{\mathscr{K}} \mathscr{V}$ is the class of all algebras of $\mathscr{K}$ that are extensions of an algebra of $\mathscr{U}$ by an algebra of $\mathscr{V}$.

We will write $\mathscr{U} \cdot_i \mathscr{V}$ for $\mathscr{U} \cdot_{\mathscr{A}i\mathfrak{k}} \mathscr{V}$, $i=0,1,2,3$.

Ring extensions were introduced by C. J. EVERETT [8]. It is the analogue of O. SCHREIER's group extensions [20]. The concept of class multiplication for groups may be found in HANNA NEUMANN [16]. A. I. MAL'CEV [13] generalized class multiplication and proved the following theorem for algebraic systems.

Theorem 1. *If $\mathscr{U}$, $\mathscr{V}$, $\mathscr{K}$ are varieties, then $\mathscr{U} \cdot_{\mathscr{K}} \mathscr{V}$ is a subvariety of $\mathscr{K}$. $\langle L\mathscr{K}; \cdot_{\mathscr{K}}\rangle$ is a partially ordered groupoid with zero and 1; $\mathscr{K}$ is the zero-element and the trivial variety $\mathscr{E}$ is 1. If $\mathscr{A}, \mathscr{B}, \mathscr{C} \in L\mathscr{K}$, then $\mathscr{A} \cdot_{\mathscr{K}} (\mathscr{B} \cdot_{\mathscr{K}} \mathscr{C}) \subseteqq (\mathscr{A} \cdot_{\mathscr{K}} \mathscr{B}) \cdot_{\mathscr{K}} \mathscr{C}$. If $\mathscr{A} \subseteqq \mathscr{B}$, then $\mathscr{C} \cdot_{\mathscr{K}} \mathscr{A} \subseteqq \mathscr{C} \cdot_{\mathscr{K}} \mathscr{B}$ and $\mathscr{A} \cdot_{\mathscr{K}} \mathscr{C} \subseteqq \mathscr{B} \cdot_{\mathscr{K}} \mathscr{C}$.*

Although the lattice of group varieties has a complicated structure, the groupoid of group varieties has a very simple structure: a free monoid with zero. However, from A. A. ISKANDER [11], $\langle L\mathscr{A}2\mathbf{Z}; \cdot\rangle$ ($\mathbf{Z}$ is the ring of integers) contains a denumerable set of idempotents. Thus, it is far from being free. We will show that $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$ are not power-associative and, under some restrictions on $\mathfrak{k}$, a decomposition is valid in $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$.

Theorem 2. *Let $i=0,1,2$. Then $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$ is not power-associative; in fact, $(\mathscr{C} \cdot_i \mathscr{C}) \cdot_i \mathscr{C} \neq \mathscr{C} \cdot_i (\mathscr{C} \cdot_i \mathscr{C})$, where $\mathscr{C}=\mathscr{A}3\mathfrak{k}$. If $\mathfrak{k}$ is a field of characteristic 0, then $\langle L\mathscr{A}3\mathfrak{k}; \cdot_i\rangle$ is isomorphic to the multiplicative monoid of natural numbers.*

However, we will show that $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$ contains infinite associative subgroupoids.

Definition 2. Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{E}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. $\mathscr{V}$ is i-indecomposable if $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U}$, $\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$ implies $\mathscr{U}=\mathscr{E}$ or $\mathscr{V}=\mathscr{E}$. $\mathscr{V}$ is i-pseudo-indecomposable if $\mathscr{V}$ contains a non-trivial algebra satisfying $xy=0$ and $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U}$, $\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$ implies $\mathscr{U}$ or $\mathscr{W}$ does not contain any non-trivial algebras satisfying $xy=0$, $i=0, 1, 2, 3$.

Theorem 3. *Let $\mathfrak{k}$ be a Dedekind domain, $i=0, 1, 2, 3$. If $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, then either $\mathscr{V}$ does not contain any non-trivial algebras satisfying $xy=0$ or $\mathscr{V}$ is a product of a finite number of i-pseudo-indecomposable varieties; if $\mathfrak{k}$ is a field of characteristic* 0, *then $\mathscr{V}=\mathscr{E}$ or $\mathscr{V}$ is a product of a finite number of i-indecomposable varieties.*

An equationally complete variety is a variety whose lattice of subvarieties contains exactly 2 elements; i.e., it is a minimal non-trivial variety. A. TARSKI [24] determined the equationally complete associative ring varieties: they are those determined by $px=0$, $xy=0$ for some prime p or by $px=0$, $x-x^p=0$ for some prime p. The following theorem determines the minimal varieties in $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$:

Theorem 4. *The equationally complete varieties of $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$, are exactly the equationally complete varieties of $L\mathscr{A}3\mathfrak{k}$. They are the varieties determined by one of the following sets of identities:*

(1) *for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, $ax=0$ for all $a\in\mathfrak{m}$, $xy=0$;*

(2) *for some maximal ideal $\mathfrak{m}$ of finite index in $\mathfrak{k}$, $ax=0$ for all $a\in\mathfrak{m}$ and $x-x^n=0$ where $n=|\mathfrak{k}/\mathfrak{m}|$.*

Thus the minimal varieties of $L\mathscr{A}i\mathfrak{k}$, $i=1, 2, 3$, are those generated by $\mathfrak{k}/\mathfrak{m}$ for some maximal ideal of $\mathfrak{k}$ of finite index in $\mathfrak{k}$ or by the zero algebra on $\mathfrak{k}/\mathfrak{m}$ where $\mathfrak{m}$ is a maximal ideal of $\mathfrak{k}$.

Definition 3. Let I be a set of polynomial identities. An algebra $\mathfrak{R}$ is I-indecomposable if A is an ideal of $\mathfrak{R}$ such that $\mathfrak{R}/A$ satisfies I implies $A=\mathfrak{R}$. I is attainable on $\mathscr{K}\subseteq\mathscr{A}0\mathfrak{k}$ if for every $\mathfrak{R}\in\mathscr{K}$, the least ideal of $\mathfrak{R}$ whose factor satisfies I is I-indecomposable.

This concept is due to T. TAMURA [22] where he determined the sets of identities attainable on the class of all semigroups. As shown by T. TAMURA and F. M. YAQUB [23], the sets $\{xy-yx\}$, $\{px=0, x=x^p\}$, p is prime, are not attainable on the class of all associative rings. It was shown by A. A. ISKANDER [11] that a family of identities that is attainable on the variety of all associative rings or on the variety of all commutative and associative rings is equivalent to $x=x$ or $x=y$. In general

Theorem 5. *Let $\mathfrak{k}$ contain no idempotent ideals other than $\mathfrak{o}$, $\mathfrak{k}$, and suppose $i=1, 2, 3$. If I is a set of polynomial identities that is attainable on $\mathscr{A}i\mathfrak{k}$, then I is equivalent on $\mathscr{A}i\mathfrak{k}$ to $x=x$ or $x=y$.*

M. V. Volkov [25] introduced and successfully used the concept of "S-joined varieties", where S is a submonoid of the multiplicative monoid of $\mathfrak{k}$ containing no zero-divisors, to gain information about the lattice of subvarieties of a variety $\mathscr{V}$ by studying the corresponding lattice of varieties of $\mathfrak{k}'$-algebras, where $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}$ relative to S. In the present paper, we study a slightly more general case and show that the S-joined subvarieties of a variety $\mathscr{V}$ form a subgroupoid of $\langle L\mathscr{V}; \cdot_{\mathscr{V}}\rangle$.

2. Relatively free algebras and T-ideals

Before we prove Theorems 2, 3, 4 and 5, we will need some preliminaries and prove some other results.

For every cardinal number $n>0$, $X(n)$ is a set of cardinality n and $F(n, \mathscr{V})$ is the free algebra of $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ whose free generating set is $X(n)$. Let $X=\{x_0, x_1, \dots\}$ be a denumerable set. $F\mathscr{V}$ will denote the free algebra of $\mathscr{V}$ whose free generating set is X; $Fi=F\mathscr{A}i\mathfrak{k}$, $i=0, 1, 2, 3$. Let $G0, G1, G2$ and $G3$ be, respectively, the free groupoid, the free power-associative groupoid, the free semigroup and the free commutative semigroup whose set of free generators is X. The following lemma is in the literature:

Lemma 6. *The $\mathfrak{k}$-module structure of Fi is the free unital $\mathfrak{k}$-module with basis Gi. The multiplication in Fi is defined by $(au)(bv)=(ab)(uv)$, $a(bv)=(ab)v$ and distributivity, where $a, b\in\mathfrak{k}$, $u, v\in Gi$, $i=0, 2, 3$.*

For $i=0$, cf. J. M. Osborn [17], p. 167. For $i=2$, cf. P. M. Cohn [6], p. 30 and [7], p. 63. $i=3$ is similar.

If $f\neq 0, f\in F0$, $d(f)$ denotes the degree of f, i.e., the highest among the lengths of elements of $G0$ with non-zero coefficients in f. $o(f)$ denotes the order of f, i.e., the least among the lengths of elements of $G0$ with non-zero coefficients in f. $f(x_1, \dots, x_n)$ will mean that the elements of X occurring at least once in f are among $x_1, \dots, x_n$. f is called homogeneous of degree r in x_i if every element of $G0$ with non-zero coefficient in f has exactly r entries of x_i; f is called homogeneous if it is homogeneous in every $x_i\in X$. f is called multilinear if f is homogeneous of degree at most 1 in every x_i. Every variety of algebras is determined by a set of identities. If $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, then the set of all $f\in F0$, such that $f=0$ is an identity in $\mathscr{V}$, is a T-ideal of $F0$; that is an ideal of $F0$ closed under all endomorphisms of $F0$, cf. [6], [17]. In fact, if $\mathscr{W}\in L\mathscr{V}$, then the identities $f=0$ of $\mathscr{W}$ relative to $\mathscr{V}$ form a T-ideal of $F\mathscr{V}$. The correspondence between $L\mathscr{V}$ and the T-ideals of $F\mathscr{V}$ is an antiisomorphism of the lattice $\langle L\mathscr{V}; \wedge, \vee\rangle$ onto the lattice of T-ideals of $F\mathscr{V}$. Script capital letters will denote classes or varieties of algebras; the corresponding Latin capitals will denote the T-ideals of $F0$ determined by them. Algebras will be denoted by German capitals and ideals of $\mathfrak{k}$ will be denoted by lower case German letters. Homomorphisms will be denoted by lower case Greek letters and will be applied to the right.

If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $A\subseteqq F0$, $B\subseteqq\mathfrak{k}$, then $B\mathfrak{R}$ is the set of all finite sums of elements of $\mathfrak{R}$ of the type bx, $b\in B$, $x\in\mathfrak{R}$ and $A(\mathfrak{R})$ is the set of all elements of $\mathfrak{R}$ that are equal to $f(r_1, \ldots, r_n)$ where $r_1, \ldots, r_n\in\mathfrak{R}$ and $f\in A$.

Lemma 7. *If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $\mathfrak{a}$ is an ideal of $\mathfrak{k}$, V is a T-ideal of $F0$, then $\mathfrak{a}\mathfrak{R}$ is an ideal of $\mathfrak{R}$ and $V(\mathfrak{R})$ is a T-ideal of $\mathfrak{R}$. $V(\mathfrak{R})$ is the least ideal of $\mathfrak{R}$ whose factor belongs to $\mathscr{V}$. $F\mathscr{V}\cong F0/V$.*

Cf. [5], [10], [14], [17].

The following lemma is a special case of a result of A. I. Mal'cev [13]:

Lemma 8. *Let $\mathscr{K}, \mathscr{U}, \mathscr{V}, \mathscr{W}\in L\mathscr{A}0\mathfrak{k}$, $\mathscr{W}\subseteqq\mathscr{K}$. Then $(\mathscr{U}\cap\mathscr{W})\cdot_{\mathscr{W}}(\mathscr{V}\cap\mathscr{W})= =(\mathscr{U}\cdot_{\mathscr{K}}\mathscr{V})\cap\mathscr{W}$. Furthermore $\mathfrak{R}\in\mathscr{U}\cdot_{\mathscr{K}}\mathscr{V}$ iff $\mathfrak{R}\in\mathscr{K}$ and $V(\mathfrak{R})\in\mathscr{U}$.*

Lemma 9. *If A is a basis of identities for $\mathscr{U}\in L\mathscr{A}0\mathfrak{k}$, $A\subseteqq F0$ and $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, then $A(V)$ is a basis of identities for $\mathscr{U}\cdot_0\mathscr{V}$. The T-ideal of $F0$ determined by $\mathscr{U}\cdot_0\mathscr{V}$ is the ideal of $F0$ generated by $U(V)$.*

The T-ideal of $F0$ determined by $\mathscr{U}\cdot_0\mathscr{V}$ will be denoted by $U\circ V$.

Proof. By Lemma 8, $\mathfrak{R}\in\mathscr{U}\cdot_0\mathscr{V}$ iff $V(R)\in\mathscr{U}$, i.e., iff $A(V(\mathfrak{R}))=0$. Thus $A(V)$ is a basis of identities for $\mathscr{U}\cdot_0\mathscr{V}$. Hence $U(V)$ is a basis for $\mathscr{U}\cdot_0\mathscr{V}$. However, the ideal of $F0$ generated by $U(V)$ is a T-ideal of $F0$ since it is the set of all finite sums of $w, fw, wg, (fw)g, f(wg), (f(gw))h, \ldots$ where $w\in U(V), f, g, h, \ldots\in F0$.

Proposition 10. *If $\mathscr{U}, \mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ are defined by multilinear identities, then $\mathscr{U}\cdot_0\mathscr{V}$ is definable by multilinear identities. If $\mathscr{U}$ is defined by a finite set of multilinear identities, $\mathscr{V}$ is finitely based, $\mathscr{U}, \mathscr{V}\in L\mathscr{A}2\mathfrak{k}$, then $\mathscr{U}\cdot_2\mathscr{V}$ is finitely based.*

Proof. Suppose $0\neq g\in F0$ and the number of elements of X occurring in g is r. Let $\bar{g}=g(x_1, \ldots, x_r)$. Let $D(g)$ be the set of all elements of $F0$ obtained from $\bar{g}$ by a finite number of applications of the following: If $h(x_1, \ldots, x_m)\in D(g)$ and every x_i, $1\leqq i\leqq m$, occurs in h, then $x_{m+1}h, hx_{m+1}\in D(g)$. Suppose $h_1, \ldots, h_r$ are non-zero elements of $F0$. $g*(h_1, \ldots, h_r)$ is the set of all $h(x_1, \ldots, x_n)=g(h_1', \ldots, h_r')$ such that h_i' is obtained from an element of $D(h_i)$ by renaming the elements of X so that h_i' and h_j' have no elements of X in common if $i\neq j$, and n is the number of elements of X occurring in $h_1', \ldots, h_r'$. If $A\subseteqq U$ is a basis for $\mathscr{U}$, $B\subseteqq V$ is a basis for $\mathscr{V}$ and every element in A is multilinear, then $D=\bigcup\{g*(h_1, \ldots, h_r)\colon g\in A, h_1, \ldots, h_r\in B\}$ is a basis for $\mathscr{U}\cdot_0\mathscr{V}$. This is true since every element of $\mathscr{V}$ is a finite sum of $h(f_1, \ldots, f_t)$, $h\in D(g)$, $g\in B$, $f_1, \ldots, f_t\in F0$. If g is multilinear, then $g(v_1, \ldots, v_r)$, where $v_1, \ldots, v_r\in V$, is a finite sum of elements of the form

$g(h_1(f_1, \ldots, f_l), \ldots, h_r(f_s, \ldots, f_t))$, where $h_1, \ldots, h_r \in \bigcup \{D(g)\colon g \in B\}$, $f_1, \ldots, f_t \in F0$, i.e., every element in $A(V)$ is a finite sum of $h(f_1, \ldots, f_t)$, where $h \in D, f_1, \ldots, f_t \in F0$. Since $A(V)$ is a basis for $\mathcal{U} \cdot_0 \mathcal{V}$ (by Lemma 9), $D \subseteq A(V)$, D is a basis for $\mathcal{U} \cdot_0 \mathcal{V}$. If every element in B is multilinear, then $D(g)$ contains only multilinear elements for every $g \in B$ and D is composed of multilinear elements. If $\mathcal{U}, \mathcal{V} \in L\mathcal{A}2\mathfrak{k}$, then in $F2$, $D'(g) = \{\bar{g}, x_{r+1}\bar{g}, \bar{g}x_{r+1}, x_{r+1}\bar{g}x_{r+2}\}$. Thus D' can be chosen finite in case A is finite and B is finite. Thus $\mathcal{U} \cdot_2 \mathcal{V}$ is finitely based; $D' = \bigcup \{g *'(h_1, \ldots, h_r)\colon h_1, \ldots, h_r \in B, g \in A\}$, where $*'$ is similar to $*$, using $D'(h_i)$ instead of $D(h_i)$.

For example, $\mathcal{C} \cdot_2 \mathcal{C}$ has the following basis:

$$[[x_1, x_2], [x_3, x_4]],$$

$$[[x_1, x_2], [x_3, x_4]x_5], \ [[x_1, x_2], x_5[x_3, x_4]], \ [[x_1, x_2], x_5([x_3, x_4]x_6)],$$

$$[[x_1, x_2]x_3, [x_4, x_5]x_6], \ [[x_1, x_2]x_3, x_6[x_4, x_5]], \ [[x_1, x_2]x_3, x_6([x_4, x_5]x_7)],$$

$$[x_3[x_1, x_2], x_6[x_4, x_5]], \ [x_3[x_1, x_2], x_6([x_4, x_5]x_7)], \ [x_3([x_1, x_2]x_4), x_5([x_6, x_7]x_8)],$$

$$x_1(x_2x_3) - (x_1x_2)x_3,$$

where $\mathcal{C} = \mathcal{A}3\mathfrak{k}$, $[x_1, x_2] = x_1x_2 - x_2x_1$.

3. Proof of Theorem 2, first part

By Lemma 8, $\langle L\mathcal{A}2\mathfrak{k}; \cdot_2 \rangle$ is a homomorphic image of $\langle L\mathcal{A}i\mathfrak{k}; \cdot_i \rangle$, $i = 0, 1$. Thus, it is sufficient to show that $(\mathcal{C} \cdot_2 \mathcal{C}) \cdot_2 \mathcal{C} \neq \mathcal{C} \cdot_2 (\mathcal{C} \cdot_2 \mathcal{C})$. This will be done by showing that $((C \circ C) \circ C)(F) \neq (C \circ (C \circ C))(F)$, where F is the free associative algebra on 2 generators a, b. We will show that

$$p = [a[[a, b], [a, b]a], [[a, b], [a, b]a]] \in (C \circ (C \circ C))(F)$$

but $p \notin ((C \circ C) \circ C)(F)$. Let T be the free semigroup on $\{a, b\}$. By Lemma 6, the $\mathfrak{k}$-module structure of F is a free unital module over $\mathfrak{k}$ with basis T. Thus, every element of F is a unique $\mathfrak{k}$-linear combination of elements of T. Let $N = ((C \circ C) \circ C)(F)$, $L = C(F) = [F, F]$ and $M = C(L)$. L is the ideal of F generated by $[f, g] = fg - gf$, $f, g \in F$. M is the ideal of L generated by all $[u, v]$, $u, v \in L$. By Lemma 9, N is the ideal of F generated by $(C \circ C)(C(F))$; i.e., N is the ideal of F generated by $C(C(C(F)))$. Thus N is the ideal of F generated by $[M, M]$, i.e., N is the ideal of F generated by $[u, v]$, $u, v \in M$. Let $c = ab - ba = [a, b]$. Elements of L are $\mathfrak{k}$-linear combinations of $[u, v]$, $s[u, v]$, $[u, v]t$, $s[u, v]t$; $s, t, u, v \in T$. By induction on the length of uv, $[u, v]$ is a $\mathfrak{k}$-linear combination of c, sc, ct, sct; $s, t \in T$; i.e., every element in L is a $\mathfrak{k}$-linear combination of sct, where $s, t \in T \cup \{1\}$, $1c = c = c1$. Elements of M are $\mathfrak{k}$-linear combinations of $w[sct, ucv]z$, where $s, t, u, v \in T \cup \{1\}$, $w, z \in L \cup \{1\}$.

The elements of M of least degree are of degree 5 and they are $\mathfrak{k}$-linear combinations of

(i) $[c, ac], \quad [c, ca], \quad [c, bc], \quad [c, cb].$

Elements of M of order 6 and degree 6 are $\mathfrak{k}$-linear combinations of

(ii)
$$\begin{array}{llll} [c, a^2c], & [c, aca], & [c, ca^2], & [ac, ca], \\ [c, b^2c], & [c, bcb], & [c, cb^2], & [bc, cb], \\ [c, abc], & [c, bac], & [c, acb], & [c, bca], \\ [c, cab], & [c, cba], & [ac, bc], & [ac, cb], \\ [ca, bc], & [ca, cb]. & & \end{array}$$

The ideal N of F generated by $[M, M]$ is generated by all $[u, v]$, $u, v \in M$. The elements of least degree in N are of degree 10. The elements of N of degree 10 are $\mathfrak{k}$-linear combinations of

(iii)
$$\begin{array}{lll} [[c, ac], [c, ca]], & [[c, ac], [c, cb]], & [[c, ac], [c, bc]], \\ [[c, ca], [c, bc]], & [[c, ca], [c, cb]], & [[c, bc], [c, cb]]. \end{array}$$

The elements of N of order 11 and degree 11 are $\mathfrak{k}$-linear combinations of ad, da, bd, db and $[g, h]$ where d belongs to the set (iii), i.e., d is of degree 10, g belongs to the set (i), i.e., g is of degree 5, and h belongs to the set (ii), i.e., h is of degree 6.

$F(2, \mathscr{C} \cdot_2 (\mathscr{C} \cdot_2 \mathscr{C})) \cong F/K$, where $K = (C \circ (C \circ C))(F)$. Thus K is the ideal of F generated by $C((C \circ C)(F))$. That is K is the ideal of F generated by $[(C \circ C)(F), (C \circ C)(F)] = [\overline{M}, \overline{M}]$, where $\overline{M}$ is the ideal of F generated by M. Now $a[c, ca] \in \overline{M}$, $[c, ca] \in M \subseteq \overline{M}$. Hence $p = [a[c, ca], [c, ca]] \in K$. We will be through if we show that $p \notin N$. Since p is homogeneous of degree 7 in a and 4 in b, and by Lemma 6, F is a free $\mathfrak{k}$-module whose basis is T, $p \in N$ iff p is a $\mathfrak{k}$-linear combination of

$$\begin{array}{ll} u1 = [[c, ac], [c, a^2c]], & u2 = [[c, ac], [c, ca^2]], \\ u3 = [[c, ac], [c, aca]], & u4 = [[c, ac], [ac, ca]], \\ u5 = [[c, ca], [c, a^2c]], & u6 = [[c, ca], [c, ca^2]], \\ u7 = [[c, ca], [c, aca]], & u8 = [[c, ca], [ac, ca]], \\ u9 = a[[c, ac], [c, ca]], & u10 = [[c, ac], [c, ca]]a. \end{array}$$

The homogeneous elements of F of degree 7 in a and 4 in b with 0 form a free $\mathfrak{k}$-submodule P of rank $\binom{11}{4} = 330$. The basis of P is the set of all words of T of length 11 in which exactly 7 entries are a. Let R be the submodule of P spanned by $\{ui\colon 1 \leqq i \leqq 10\}$, and let S be the submodule of P spanned by $R \cup \{p\}$. $p \in N$ iff $p \in R$, i.e., iff $S = R$. Let B be a subset of the basis of P, then if $p \in N$, the images of

R and S under the module homomorphism $\sum\{a_i s_i : s_i \in \text{basis of } P\} \to \sum\{a_i s_i : s_i \in B\}$ coincide. Table I below gives the coefficients of $p, u1, \ldots, u10$ as $\mathfrak{k}$-linear combinations of elements of T in

$$x1 = ba^2ba^3ba^2b, \qquad x2 = ba^4ba^2bab,$$
$$x3 = ba^3baba^3b, \qquad x4 = baba^3ba^3b,$$
$$x5 = baba^5bab, \qquad x6 = a^2b^2a^4bab,$$
$$x7 = ababa^3ba^2b, \qquad x8 = a^3b^2aba^3b,$$
$$x9 = ba^2ba^3b^2a^2, \qquad x10 = a^2bababa^3b.$$

Table I

	$x1$	$x2$	$x3$	$x4$	$x5$	$x6$	$x7$	$x8$	$x9$	$x10$
p	-2	0	-1	-1	0	0	1	0	-2	2
$u1$	1	-1	0	0	0	-1	-2	1	0	-2
$u2$	2	1	0	1	1	0	-2	0	0	1
$u3$	-3	0	-2	0	0	0	4	0	0	0
$u4$	0	1	0	0	0	-1	-2	0	-1	2
$u5$	-2	0	0	1	-1	0	2	-1	0	1
$u6$	-1	0	0	-2	0	0	1	0	-1	0
$u7$	3	0	2	1	0	0	-3	0	1	-1
$u8$	0	0	0	-1	0	0	1	0	2	-1
$u9$	0	0	0	0	0	-1	-2	1	0	0
$u10$	0	0	0	0	0	0	0	0	2	0

Table II

	$x1$	$x2$	$x3$	$x4$	$x5$	$x6$	$x7$	$x8$	$x9$	$x10$
$-u6$	1	0	0	2	0	0	-1	0	1	0
$u4$	0	1	0	0	0	-1	-2	0	-1	2
$-u3+3u6$	0	0	2	-6	0	0	-1	0	-3	0
$-u8$	0	0	0	1	0	0	-1	0	-2	1
$u2-u4+2u6-3u8$	0	0	0	0	1	1	-1	0	-7	2
$-u9$	0	0	0	0	0	1	2	-1	0	0
$u1+u3+u4+u6+u7-$ $-u8-2u9$	0	0	0	0	0	0	1	-1	-3	0
$-2u1-u3-2u4-2u6-$ $-u7+3u8+4u9$	0	0	0	0	0	0	0	2	9	-2
$u10$	0	0	0	0	0	0	0	0	2	0
$4p+u3-2u6+3u7+$ $+3u8-u10$	0	0	0	0	0	0	0	0	1	2
$8p+2u3-4u6+6u7+$ $+6u8-3u10$	0	0	0	0	0	0	0	0	0	4
$u2-u3-u4+u5-u7+$ $+u8+u9-u10$	0	0	0	0	0	0	0	0	0	0

The image of R into the submodule $\mathfrak{k}x3$ is $(2\mathfrak{k})x3$; the image of S is $\mathfrak{k}x3$. If 2 is not invertible in $\mathfrak{k}$, then $2\mathfrak{k}\neq\mathfrak{k}$ and $R\neq S$. If 2 is invertible in $\mathfrak{k}$, from Table II we get bases for the images of R and S in the free $\mathfrak{k}$-module $\sum\{\mathfrak{k}xi\colon 1\leqq i\leqq 10\}$.

The image of S into $\sum\{\mathfrak{k}xi\colon 1\leqq i\leqq 10\}$ is the whole submodule, i.e., it is a free $\mathfrak{k}$-module of rank 10. The image of R is a submodule generated by 9 elements. If $R=S$, we get a free $\mathfrak{k}$-module of two distinct ranks: 9 and 10. This is impossible since $\mathfrak{k}$ is a nontrivial commutative and associative ring with 1 and by reduction to $\mathfrak{k}/\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, we get a vector space with two dimensions: 9 and 10, cf. P. M. Cohn [6], p. 6. This concludes the proof of the first part of Theorem 2.

4. Multinilpotent varieties

In this section we prove the second part of Theorem 2 and some results of interest in their own right.

Let $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. If $\mathscr{V}\supseteqq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})=\infty$, otherwise $d(i,\mathscr{V})$ is the least degree of elements of $V(Fi)$, $\mathfrak{c}(i,n,\mathscr{V})$ is the ideal of $\mathfrak{k}$ generated by the coefficients of elements of $V(Fi)$ of degree n, $\mathfrak{c}(i,\mathscr{V})=\mathfrak{c}(i,d(i,\mathscr{V}),\mathscr{V})$, $i=0,1,2,3$.

Since V contains with every element of $F0$ all its linearizations, i.e.,

$$f(x_1,\ldots,x_j+x_{n+1},\ldots,x_n)-f(x_1,\ldots,x_j,\ldots,x_n)-f(x_1,\ldots,x_{n+1},\ldots,x_n),$$

$1\leqq j\leqq n$, cf. J. Goldman and S. Kass [9] and J. M. Osborn [17], $d(i,\mathscr{V})$ is achieved by multilinear identities.

Lemma 11. *If* $\mathscr{V},\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$, *then* $d(i,\mathscr{V}\cdot_i\mathscr{W})=d(i,\mathscr{V})\,d(i,\mathscr{W})$. *Thus, if* $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, $\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, *then* $\mathscr{V}\cdot_i\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, *i.e.,* $\langle L\mathscr{A}i\mathfrak{k};\cdot_i\rangle$ *has no zero-divisors. Furthermore,* $\mathfrak{c}(i,n,\mathscr{V})=\mathfrak{o}$ *iff* $d(i,\mathscr{V})>n$. *If* $\mathscr{V}\subseteqq\mathscr{W}$, *then* $d(i,\mathscr{V})\leqq d(i,\mathscr{W})$ *and* $\mathfrak{c}(i,n,\mathscr{V})\supseteqq\mathfrak{c}(i,n,\mathscr{W})$, $n\geqq 1$, $i=0,1,2,3$.

Proof. By Lemma 9, $(V\circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$, in the sense of the proof of Lemma 9. Thus the elements of least degree in $(V\circ W)(Fi)$ belong to $V(W(Fi))$. Let $f\in V(W(Fi))$. Then $f=g(w_1,\ldots,w_n)$, where $g(x_1,\ldots,x_n)\in V$, $w_1,\ldots,w_n\in W$. $o(f)\geqq o(g)\min\{o(w_1),\ldots,o(w_n)\}\geqq d(i,\mathscr{V})\cdot d(i,\mathscr{W})$. If g is multilinear of degree $d(i,\mathscr{V})$, each of $w_1,\ldots,w_n$ are multilinear of degree $d(i,\mathscr{W})$, $w_1,\ldots,w_n$ involves exactly $nd(i,\mathscr{W})$ elements of X, then $n=d(i,\mathscr{V})$, f is multilinear and $o(f)=d(f)=d(i,\mathscr{V})\,d(i,\mathscr{W})$. If $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, $\mathscr{W}\neq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})$, $d(i,\mathscr{W})<\infty$, and $d(i,\mathscr{V}\cdot_i\mathscr{W})=d(i,\mathscr{V})\,d(i,\mathscr{W})<\infty$. $\mathscr{V}\subseteqq\mathscr{W}$ iff $V\supseteqq W$, from which the rest of the lemma follows.

Definition 4. A variety $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ is i-multinilpotent if $V(Fi)=\sum\{\mathfrak{a}_n Fi^n\colon n\geqq 1\}$ where $\mathfrak{a}_1,\mathfrak{a}_2,\ldots$ are ideals of $\mathfrak{k}$ and Fi^n is the set of all finite sums of all possible products of n elements of Fi, $i=0,1,2,3$.

It is clear that $Fi^{n+1} \subseteq Fi^n$, $n \geqq 1$. Thus we can assume $\mathfrak{a}_1, \mathfrak{a}_2, \dots$ an ascending chain of ideals of $\mathfrak{k}$.

Lemma 12. *Let Mi be the set of all i-multinilpotent varieties. Then Mi is a complete sublattice of $\langle L\mathscr{A}i\mathfrak{k}; \wedge, \vee\rangle$, $i=0, 1, 2, 3$.*

Proof. Let $\mathscr{V}a$, $a \in I$, be i-multinilpotent varieties. Then there are ascending chains of ideals of $\mathfrak{k}$: $(\mathfrak{a}a_n)$, $n \geqq 1$, $a \in I$, such that $Va(Fi) = \sum \{\mathfrak{a}a_n Fi^n \colon n \geqq 1\}$, $a \in I$.

$$\left(\sum \{Va \colon a \in I\}\right)(Fi) = \sum \{Va(Fi) \colon a \in I\} =$$

$$= \sum \left\{\sum \{\mathfrak{a}a_n Fi^n \colon n \geqq 1\} \colon a \in I\right\} = \sum \left\{\sum \{\{\mathfrak{a}a_n \colon a \in I\} Fi^n \colon n \geqq 1\right\}.$$

Thus, the intersection of any family of i-multinilpotent varieties is i-multinilpotent.

$$\left(\bigcap \{Va \colon a \in I\}\right)(Fi) = \bigcap \{Va(Fi) \colon a \in I\} =$$

$$= \bigcap \left\{\sum \{\mathfrak{a}a_n Fi^n \colon n \geqq 1\} \colon a \in I\right\} \supseteq \sum \left\{\bigcap \{\mathfrak{a}a_n \colon a \in I\} Fi^n \colon n \geqq 1\right\}.$$

If $f \in Va(Fi)$ for all $a \in I$, $f = f_1 + \dots + f_r$ where each f_j is of order and of degree n_j, $n_1 < n_2 < \dots < n_r$, then $f_j \in Fi^{n_j}$ and $f_j \in \mathfrak{a}a_{n_j} Fi^{n_j}$ for all $a \in I$, $1 \leqq j \leqq r$. Hence $f_j \in \bigcap \bigcap \{\mathfrak{a}a_{n_j} \colon a \in I\} Fi^{n_j}$, $1 \leqq j \leqq r$, i.e., $f \in \sum \left\{\bigcap \{\mathfrak{a}a_n \colon a \in I\} Fi^n \colon n \geqq 1\right\}$. Thus, the join of any family of i-multinilpotent varieties is i-multinilpotent.

Lemma 13. *Let $\mathscr{V}$ be i-multinilpotent, $\mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$. Then $(V \circ W)(Fi) = V(W(Fi))$, $i=2, 3$.*

Proof. Since $(V \circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$ (from Lemma 9), we need to show that $V(W(Fi))$ is an ideal of Fi. $W(Fi)$ is an ideal of Fi. Hence $W(Fi)^n$ is an ideal of Fi and consequently $\mathfrak{a}_n W(Fi)^n$ is an ideal of Fi, where $\mathfrak{a}_n$ is an ideal of $\mathfrak{k}$, $n \geqq 1$. If $V(Fi) = \sum \{\mathfrak{a}_n Fi^n \colon n \geqq 1\}$, then $V(W(Fi)) = \sum \{\mathfrak{a}_n W(Fi)^n \colon n \geqq 1\}$ is an ideal of Fi.

Corollary 14. *If $\mathscr{U}, \mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}$ is i-multinilpotent, then $(\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W} = \mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$, $i=2, 3$.*

Proof. $(U \circ (V \circ W))(Fi)$ is the ideal of Fi generated by $U((V \circ W)(Fi))$ (from Lemma 9). From Lemma 13, $(V \circ W)(Fi) = V(W(Fi))$. Thus $U((V \circ W)(Fi)) = U(V(W(Fi)))$. $((U \circ V) \circ W)(Fi)$ is the ideal of Fi generated by $(U \circ V)(W(Fi))$. This is also the ideal of Fi generated by $U(V(W(Fi)))$. Hence $((U \circ V) \circ W)(Fi) = (U \circ (V \circ W))(Fi)$.

Since $\mathscr{A}i\mathfrak{k}$ and $\mathscr{E}$ are i-multinilpotent, Mi generates a submonoid with zero of $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i\rangle$, $i=2, 3$.

By Lemma 12, if $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, the join of all i-multinilpotent varieties contained in $\mathscr{V}$ is i-multinilpotent. We will denote the largest i-multinilpotent variety contained in $\mathscr{V}$ by $\mathscr{V}'$.

Lemma 15. *Suppose* $\mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $V(Fi)=\sum\{\mathfrak{a}_n Fi^n\colon n\geqq 1\}$, $W(Fi)=$ $=\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}$, $(\mathfrak{a}_n)$, $(\mathfrak{b}_n)$ *are ascending chains of ideals of* $\mathfrak{k}$. *If* $\mathscr{U}=(\mathscr{V}\cdot_i\mathscr{W})$, *then* $U(Fi)=\sum\{\mathfrak{c}_n Fi^n\colon n\geqq 1\}$, *where* $\mathfrak{c}_n=\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}$, $i=0, 1, 2, 3$.

Proof. $(V\circ W)(Fi)$ is the ideal of Fi generated by $V(W(Fi))$.

$$V(W(Fi)) = V\left(\sum\{\mathfrak{a}_n W(Fi)^n\colon n\geqq 1\}\right) =$$

$$= \mathfrak{a}_1\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}+\mathfrak{a}_2\sum\{\mathfrak{b}_n Fi^n\colon n\geqq 1\}^2+\dots$$

$$= \sum\left\{\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}Fi^{t1}\dots Fi^{tr}\colon t1+\dots+tr=n\}\colon n\geqq 1\right\}\subseteqq\sum\{\mathfrak{c}_n Fi^n\colon n\geqq 1\}.$$

If $\sum\{\mathfrak{d}_n Fi^n\colon n\geqq 1\}\supseteqq(V\circ W)(Fi)$, then $\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}\subseteqq\mathfrak{d}_n$, $n\geqq 1$.

Proposition 16. *Let* $i=2, 3$. *Then* Mi *is a submonoid with zero of* $\langle L\mathscr{A}i\mathfrak{k};\ \cdot\rangle$, *and* $\langle Mi;\ \wedge, \vee, \cdot\rangle$ *is isomorphic to the partially ordered monoid of ascending chains of ideals of* $\mathfrak{k}$, *where* $(\mathfrak{a}_n)\leqq(\mathfrak{b}_n)$ *iff* $\mathfrak{a}_n\supseteqq\mathfrak{b}_n$ *for all* $n\geqq 1$ *and* $(\mathfrak{a}_n)\cdot(\mathfrak{b}_n)=(\mathfrak{c}_n)$, *where* $\mathfrak{c}_n=\sum\{\mathfrak{a}_r\mathfrak{b}_{t1}\dots\mathfrak{b}_{tr}\colon t1+\dots+tr=n\}$.

Proof. For $i=2, 3$, Fi is associative. Thus $Fi^m Fi^n=Fi^{m+n}$. From the proof of Lemma 15, the product of i-multinilpotent varieties is i-multinilpotent. Proposition 16 then follows from Lemmas 12 and 15.

Proof of Theorem 2, second part. If $\mathfrak{k}$ is a field of characteristic 0, every identity is equivalent to multilinear identities, cf. J. M. Osborn [17], p. 181. Hence, in $\mathscr{A}3\mathfrak{k}$ every variety is 3-multinilpotent. In fact every variety in $\mathscr{A}3\mathfrak{k}$ is either $\mathscr{A}3\mathfrak{k}$ or defined by $x_1\dots x_n=0$ for some $n\geqq 1$. By Proposition 16, $\langle L\mathscr{A}3\mathfrak{k};\ \cdot\rangle$ is isomorphic to the monoid of ascending chains of ideals of $\mathfrak{k}$; this is isomorphic to the multiplicative monoid of natural numbers.

5. Subgroupoids of varieties and minimal varieties

Lemma 17. *If* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'\neq\mathscr{E}$, *then* $V\subseteqq\mathfrak{m}F0+F0^2$ *for some maximal ideal* $\mathfrak{m}$ *of* $\mathfrak{k}$, $i=0, 1, 2, 3$.

Proof. Since $V(Fi)\subseteqq V'(Fi)=\mathfrak{a}_1 Fi+\mathfrak{a}_2 Fi^2+\dots$ and $\mathscr{V}'\neq\mathscr{E}$, then $\mathfrak{a}_1\neq\mathfrak{k}$. Thus $V(Fi)\subseteqq\mathfrak{a}_1 Fi+Fi^2\subseteqq\mathfrak{m}Fi+Fi^2$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$ containing $\mathfrak{a}_1$. Since $F0/\mathfrak{m}F0+F0^2\cong Fi/\mathfrak{m}Fi+Fi^2$, $V\subseteqq\mathfrak{m}F0+F0^2$.

Definition 5. A set P of non-trivial algebras is verbally closed if for every $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathfrak{R}\in P$, $V(\mathfrak{R})\in P$ or $\mathfrak{R}/V(\mathfrak{R})\in P$. $N(i, P)$ is the set of all subvarieties of $\mathscr{A}i\mathfrak{k}$ containing no members of P.

Any family of algebras with precisely 2 T-ideals (i.e., T-simple) is verbally closed. Any family of simple algebras is verbally closed.

Lemma 18. *Let* $M \subseteq L\mathscr{A}i\mathfrak{k}$. *Then* M *is a subgroupoid of* $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ *and a lattice ideal of* $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$ *iff* $M = N(i, P)$ *for some verbally closed set of non-trivial algebras* P, $i = 0, 1, 2, 3$.

Proof. Let P be verbally closed, $\mathscr{V}, \mathscr{W} \in N(i, P)$, $\mathscr{U} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{U} \subseteq \mathscr{V}$. Then $\mathscr{U} \in N(i, P)$. Since $\mathscr{V} \vee \mathscr{W} \subseteq \mathscr{V} \cdot_i \mathscr{W}$, $N(i, P)$ is a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$ if $\mathscr{V} \cdot_i \mathscr{W} \in N(i, P)$. $\mathfrak{R} \in \mathscr{V} \cdot_i \mathscr{W}$ iff $W(\mathfrak{R}) \in \mathscr{V}$, $\mathfrak{R}/W(\mathfrak{R}) \in \mathscr{W}$ and $\mathfrak{R} \in \mathscr{A}i\mathfrak{k}$. Thus $\mathscr{V} \cdot_i \mathscr{W}$ does not contain any member $\mathfrak{R}$ of P, otherwise $W(\mathfrak{R}) \in P$ or $\mathfrak{R}/W(\mathfrak{R}) \in P$ contradicting $\mathscr{V} \in N(i, P)$, $\mathfrak{W} \in N(i, P)$. Conversely, let M be a subgroupoid of $\langle L\mathscr{A}i\mathfrak{k};\ \cdot_i\rangle$ and a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$. Let K be the set of all non-trivial algebras obtained from $\{F\mathscr{V}\colon \mathscr{V} \in L\mathscr{A}i\mathfrak{k}\}$ by a finite number of applications of: If $\mathfrak{R} \in K$, $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, $V(\mathfrak{R}) \neq 0$, then $V(\mathfrak{R}) \in K$ and if $\mathfrak{R} \neq V(\mathfrak{R})$, $\mathfrak{R}/V(\mathfrak{R}) \in K$. Let P be the set of all algebras $\mathfrak{R}$ of K such that $\operatorname{var}\mathfrak{R}$, i.e., the variety generated by $\mathfrak{R}$, does not belong to M. We claim that $M = N(i, P)$. Let $\mathscr{V} \in M$. If $\mathfrak{R} \in \mathscr{V}$, then $\operatorname{var}\mathfrak{R} \subseteq \mathscr{V}$. Hence, $\operatorname{var}\mathfrak{R} \in M$ as M is a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k};\ \wedge, \vee\rangle$. Thus, $\mathfrak{R} \notin P$, i.e., $M \subseteq N(i, P)$. Let $\mathscr{V} \in N(i, P)$. Then $F\mathscr{V} \notin P$. Since $\mathscr{V} = \operatorname{var} F\mathscr{V}$, $\mathscr{V} \in M$. It remains to check that P is verbally closed. Let $\mathfrak{R} \in \mathscr{A}i\mathfrak{k}$, $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$. If neither $V(\mathfrak{R})$ nor $\mathfrak{R}/V(\mathfrak{R})$ belongs to P, then $\operatorname{var} V(\mathfrak{R})$, $\operatorname{var}\mathfrak{R}/V(\mathfrak{R}) \in M$. But

$$\mathfrak{R} \in \operatorname{var} V(\mathfrak{R}) \cdot_i \operatorname{var}\mathfrak{R}/V(\mathfrak{R}).$$

Hence $\operatorname{var}\mathfrak{R} \subseteq \operatorname{var} V(\mathfrak{R}) \cdot_i \operatorname{var}\mathfrak{R}/V(\mathfrak{R})$. Since M is a subgroupoid and a lattice ideal, $\operatorname{var}\mathfrak{R} \in M$, i.e., $\mathfrak{R} \notin P$.

Lemma 19. *The following conditions on a variety* $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, $i = 0, 1, 2, 3$, *are equivalent:*

(1) $x_1 + f(x_1) \in V$ *for some* $f \in F_0^2$.

(2) $\mathscr{V}' = \mathscr{E}$, *i.e.*, $\mathscr{V}$ *does not contain any nontrivial i-multinilpotent varieties.*

(3) $\mathscr{V} \in N(i, \{O(\mathfrak{m})\colon \mathfrak{m}$ *is a maximal ideal of* $\mathfrak{k}\})$, *where* $O(\mathfrak{m})$ *is the algebra with zero multiplication on* $\mathfrak{k}/\mathfrak{m}$ *as a* $\mathfrak{k}$-*module.*

Proof. Let $x_1 + f(x_1) \in V$, $f \in F0^2$. If $\mathscr{V}' \neq \mathscr{E}$, then $V \subseteq \mathfrak{m}F0 + F0^2$ (by Lemma 17), for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Thus $x_1 + f(x_1) \in \mathfrak{m}F0 + F0^2$, i.e., $x_1 \in \mathfrak{m}F0$ — a contradiction. If $\mathscr{V}' = \mathscr{E}$, then $O(\mathfrak{m}) \notin \mathscr{V}$. In fact, if $\mathscr{W} = \operatorname{var} O(\mathfrak{m}) =$ = the variety generated by $O(\mathfrak{m})$, then $W = \mathfrak{m}F0 + F0^2$, $\mathscr{W} \subseteq \mathscr{V}$. Hence $V \subseteq \mathfrak{m}F0 +$ $+ F0^2$ and $V' \subseteq \mathfrak{m}F0 + F0^2$, i.e., $\mathscr{V}' \neq \mathscr{E}$. Finally, if $O(\mathfrak{m}) \notin \mathscr{V}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$, then $V + F0^2 = F0$. Otherwise, $V + F0^2$ is i-multinilpotent, $V + F0^2 \neq$ $\neq F0$. Hence $V \subseteq V + F0^2 \subseteq \mathfrak{m}F0 + F0^2$ for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Thus $F0/\mathfrak{m}F0 + F0^2 \in \mathscr{V}$. This implies $O(\mathfrak{m}) \in \mathscr{V}$ since the subalgebra of $F0/\mathfrak{m}F0 + F0^2$ generated by $x_1 + \mathfrak{m}F0 + F0^2$ is isomorphic to $O(\mathfrak{m})$. Now $x_1 \in F0 = V + F0^2$. Hence, there are $v \in V$, $f \in F0^2$ such that $x_1 = v - f$. By substituting 0 for all $x_i \neq x_1$, we can assume $f = f(x_1)$. Thus $x_1 + f(x_1) \in V$, $f \in F0^2$.

Corollary 20. *The set of varieties $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}' = \mathscr{E}$ is a subgroupoid of $\langle L\mathscr{A}i\mathfrak{k}; \cdot_i \rangle$ and a lattice ideal of $\langle L\mathscr{A}i\mathfrak{k}; \wedge, \vee \rangle$, $i = 0, 1, 2, 3$.*

This follows from Lemmas 18 and 19.

Corollary 21. *Let $\mathscr{U}, \mathscr{V} \in L\mathscr{A}i\mathfrak{k}$. Then $(\mathscr{U} \cdot_i \mathscr{V})' = \mathscr{E}$ iff $\mathscr{U}' = \mathscr{V}' = \mathscr{E}$, $i = 0, 1, 2, 3$.*

This follows from Corollary 20 and Lemma 19 since $\mathscr{U}' \vee V' \subseteqq (\mathscr{U} \vee \mathscr{V})' \subseteqq \subseteqq (\mathscr{U} \cdot_i \mathscr{V})'$.

Let G be a commutative non-trivial ring with 1 and let α be a homomorphism of $\mathfrak{k}$ into G preserving 1. Then G has a natural $\mathfrak{k}$-algebra structure: $ag = (a\alpha)g$, $a \in \mathfrak{k}$, $g \in G$. This $\mathfrak{k}$-algebra structure on G will be denoted by $\mathfrak{G}\alpha$.

Lemma 22. *Let G, H be commutative non-trivial rings with 1 and α, β homomorphisms of $\mathfrak{k}$ into G, H, respectively, preserving 1. Then $\mathfrak{G}\alpha$ is isomorphic to a subalgebra of $\mathfrak{H}\beta$ iff there is an injective ring homomorphism γ of G into H such that $\alpha\gamma = \beta$, γ preserves 1.*

Proof. If γ is an injective ring homomorphism of G into H and $\alpha\gamma = \beta$, then γ is an injective homomorphism of $\mathfrak{k}$-algebras. Conversely, if there is an injective homomorphism γ of $\mathfrak{G}\alpha$ into $\mathfrak{H}\beta$ and γ preserves 1, then γ is a ring homomorphism and $a\alpha\gamma = (a\alpha)\gamma = ((a1)\alpha)\gamma = (a\alpha \cdot 1\alpha)\gamma = a((1\alpha)\gamma) = a(1\beta) = a\beta$ for every $a \in \mathfrak{k}$.

Lemma 23. *Let $\mathscr{V} \in L\mathscr{A}2\mathfrak{k}$, $\mathscr{V}' = \mathscr{E}$, $\mathscr{V} \neq \mathscr{E}$. Then $\mathscr{V}$ satisfies $x - x^m = 0$ for some $m > 1$. There are a finite number of non-isomorphic finite fields G_j, $1 \leqq j \leqq n$, and sets Ij of homomorphisms of $\mathfrak{k}$ into G_j preserving 1, $1 \leqq j \leqq n$, such that $\mathscr{V}$ is generated by $\{\mathfrak{G}j\alpha\colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$.*

Proof. Let $\mathfrak{R} = F(1, \mathscr{V})$. $\mathfrak{R}$ is associative and commutative. Since $\mathscr{V}' = \mathscr{E}$, by Lemma 19, $\mathscr{V}$ satisfies $x + f(x) = 0$ where $f(x)$ is of order $\geqq 2$. Thus $\mathscr{V}$ satisfies $x = x^2h(x)$ where $h(x) \in \mathfrak{k}[x]$, the ring of polynomials in x over $\mathfrak{k}$. Thus $\mathfrak{R}$ is a commutative and associative von Neumann regular ring. Hence $\mathfrak{R}$ is a ring subdirect product of fields. If G is one of these fields, there is a ring homomorphism γ of R onto G. G inherits a $\mathfrak{k}$-algebra structure: $ag = (ag_1)\gamma$, $a \in \mathfrak{k}$, $g_1 \in \mathfrak{R}$, $g_1\gamma = g \in G$. There is a homomorphism α of $\mathfrak{k}$ into G preserving 1: $a\alpha = ae$ where e is the identity of G. $\mathfrak{G}\alpha$ is a homomorphic image of $\mathfrak{R}$ as $\mathfrak{k}$-algebras. Thus $\mathfrak{G}\alpha$ satisfies $x = x^2h(x)$. Thus G is finite since all its elements are roots of $x^2h(x) - x = 0$, $|G| \leqq \text{degree}\, x^2h(x) = \text{degree} f$. There is only a finite number of non-isomorphic fields satisfying $x = x^2h(x)$. Let $G1, \ldots, Gn$ be all the finite fields such that if G is a field and G is a homomorphic image of $\mathfrak{R}$, then G is an isomorphic copy of $G1, \ldots,$ or Gn and $Gi \not\cong Gj$ if $i \neq j$, $1 \leqq i, j \leqq n$, and let Ij be the set of all homomorphisms α of $\mathfrak{k}$ into

Gj such that $\mathfrak{G}j\alpha$ is a homomorphic image of $\mathfrak{R}$, $1 \leqq j \leqq n$. Then there is $m>1$ such that $x=x^m$ in $G1 \times \ldots \times Gn$. Thus $\mathscr{V}$ satisfies $x-x^m=0$. Thus, by Jacobson's Theorem $\mathscr{V}$ is commutative and FV is a ring subdirect sum of fields satisfying $x-x^m=0$, i.e., finite fields. If H is one of these fields, then as above, there is a homomorphism α of $\mathfrak{k}$ into H such that $\mathfrak{H}\alpha$ is a homomorphic image of $F\mathscr{V}$. Thus $\mathfrak{H}\alpha \in \mathscr{V}$. But $\mathfrak{H}\alpha$ is generated by one element. Hence, $\mathfrak{H}\alpha$ is a homomorphic image of $F(1, \mathscr{V})=\mathfrak{R}$. Thus $H \cong Gj$ for some $1 \leqq j \leqq n$, and $\mathfrak{H}\alpha \cong \mathfrak{G}j\beta$ for some $\beta \in Ij$. Thus $\mathscr{V}$ is generated by $\{\mathfrak{G}j\alpha\colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$. That $\mathfrak{H}\alpha \cong \mathfrak{G}j\beta$ follows from Lemma 22.

It may be noted that although the non-isomorphic fields in Lemma 23 are finitely many, the non-isomorphic algebras $\mathfrak{G}j\alpha$, $\alpha \in Ij$, $1 \leqq j \leqq n$, can be infinitely many. For instance, if $\mathfrak{k}$ is an infinite Boolean ring, $\mathscr{V}$ is the variety of associative $\mathfrak{k}$-algebras satisfying $x+x^2=0$, then $F(1, \mathscr{V})$ is ring isomorphic to an infinite subdirect power of $\mathbf{Z}_2$, the prime field of 2 elements; $F(1, \mathscr{V}) \cong \mathfrak{k}$. However, $\mathfrak{k}$ is a subdirect product of $\{\mathfrak{k}/\mathfrak{m}\colon \mathfrak{m}$ is a maximal ideal of $\mathfrak{k}\}$. $\mathfrak{k}/\mathfrak{m}$ is ring isomorphic to $\mathbf{Z}_2$, but $\mathfrak{k}/\mathfrak{m} \cong \mathfrak{k}/\mathfrak{m}'$ as $\mathfrak{k}$-algebras iff $\mathfrak{m}=\mathfrak{m}'$.

Corollary 24. *Let* $\mathscr{V} \in L\mathscr{A}1\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$, $\mathscr{V} \neq \mathscr{E}$. *Then* $\mathscr{V}$ *satisfies* $x-x^m=0$ *for some* $m>1$. *There are a finite number of non-isomorphic finite fields* $G1, \ldots, Gn$ *and sets* Ij *of homomorphisms of* $\mathfrak{k}$ *into* Gj *preserving* 1, $1 \leqq j \leqq n$, *such that* $F(1, \mathscr{V})$ *is a subdirect product of* $\{\mathfrak{G}j\alpha\colon \alpha \in Ij,\ 1 \leqq j \leqq n\}$.

This follows from Lemma 23 since $\operatorname{var} F(1, \mathscr{V}) \in L\mathscr{A}2\mathfrak{k}$ and $\operatorname{var} F(1, \mathscr{V})' \subseteqq \subseteqq \mathscr{V}'=\mathscr{E}$.

Lemma 25. *Let* $\mathfrak{R} \in \mathscr{A}2\mathfrak{k}$, *I an ideal of* $\mathfrak{R}$ *and J an ideal of I. If J or I/J satisfies* $x+f(x)=0$ *for some* $f \in F0^2$, *then J is an ideal of* $\mathfrak{R}$.

Proof. Let J satisfy $x+f(x)=0$. Hence by Lemma 23, J satisfies $x=x^m$ for some $m>1$. Let $a \in \mathfrak{R}$, $b \in J$. Then $ab=ab^m=(ab^{m-1})b$. But $ab^{m-1} \in I$. Hence $ab \in J$. Similarly $ba \in J$. Let I/J satisfy $x+f(x)=0$. Hence I/J satisfies $x-x^m=0$ for some $m>1$. If $a \in \mathfrak{R}$, $b \in J$, $c \in I$, then $c-c^m \in J$, $ac, ca \in I$. Thus $ab \in I$, and $ab-(ab)^m \in J$. $(ab)^m=((ab)^{m-1}a)b$. But $(ab)^{m-1}a \in I$. Hence $(ab)^m \in J$ and $ab \in J$. Similarly, $ba \in J$.

Corollary 26. *Let* $\mathscr{U}, \mathscr{V}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$. *If* $\mathscr{U}'=\mathscr{E}$ *or* $\mathscr{V}'=\mathscr{E}$, *then* $(\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W} = =\mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$, $i=2, 3$.

Proof. By Theorem 1, $\mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W}) \subseteqq (\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W}$. Let $\mathfrak{R} \in (\mathscr{U} \cdot_i \mathscr{V}) \cdot_i \mathscr{W}$. Then $\mathfrak{R} \in \mathscr{A}2\mathfrak{k}$, there is an ideal I of $\mathfrak{R}$ and an ideal J of I such that $\mathfrak{R}/I \in \mathscr{W}$, $I \in \mathscr{U} \cdot_i \mathscr{V}$, $J \in \mathscr{U}$, $I/J \in \mathscr{V}$. Since $\mathscr{U}'=\mathscr{E}$ or $\mathscr{V}'=\mathscr{E}$, by Lemma 19, I/J or J satisfies $x+ +f(x)=0$ for some $f \in F0^2$. Hence, by Lemma 25, J is an ideal of $\mathfrak{R}$. Thus $I/J \in \mathscr{V}$ and $\mathfrak{R}/J \in \mathscr{V} \cdot \mathscr{W}$, i.e., $\mathfrak{R} \in \mathscr{U} \cdot_i (\mathscr{V} \cdot_i \mathscr{W})$.

Lemma 27. *Let $\mathfrak{R}\in\mathscr{A}\mathfrak{A}$ and S an ideal of $\mathfrak{R}$ satisfying $x+f(x)=0$ for some $f\in F0^2$. Then $\mathfrak{R}$ is isomorphic to a subdirect product of $\mathfrak{R}/S$ and an algebra satisfying all the identities of S. If $\mathfrak{R}$ is finitely generated, then S is a direct summand of $\mathfrak{R}$.*

Proof. By Lemma 23, S satisfies $x-x^m=0$ for some $m>1$ and S is commutative. In fact S is central in $\mathfrak{R}$. Let $a\in\mathfrak{R}$, $b\in S$. Then $ab=ab^m=(ab)b^{m-1}=$ $=b^{m-1}(ab)=(b^{m-1}a)b=b(b^{m-1}a)=b^m a=ba$. Let $A=\operatorname{Ann} S$, i.e.,

$$A = \{x\colon\ x\in\mathfrak{R},\ xS = 0\}.$$

A is an ideal of $\mathfrak{R}$, $A=\bigcap\{\operatorname{Ann} b\colon\ b\in S\}$, $\operatorname{Ann} b$ is an ideal of $\mathfrak{R}$. $A\cap S=0$, since $b\in A\cap S$ implies $b=b^m=bb^{m-1}=0$. Thus $\mathfrak{R}$ is isomorphic to a subdirect product of $\mathfrak{R}/S$ and $\mathfrak{R}/A$. If $b\in S$, b^{m-1} is a central idempotent and $b^{m-1}\mathfrak{R}=b\mathfrak{R}$. Thus $\mathfrak{R}\cong b\mathfrak{R}\oplus\operatorname{Ann} b$. Hence $\mathfrak{R}/\operatorname{Ann} b\cong b\mathfrak{R}\subseteq S$. But $\mathfrak{R}/A$ is a subdirect product of $\mathfrak{R}/\operatorname{Ann} b\cong b\mathfrak{R}$. Thus $\mathfrak{R}/A$ satisfies all the identities of S. If $\mathfrak{R}$ is finitely generated, then $\mathfrak{R}/A$ is finitely generated. As $A=\operatorname{Ann} S$, there are $b1, \dots, bm\in S$ such that $b1+A, \dots, bm+A$ generate $\mathfrak{R}/A$. Hence $b1, \dots, bm$ generate S. If $ei=(bi)^{m-1}$, then $e1, \dots, em$ are central idempotents, $S=e1R+\dots+emR$. There is an orthogonal set of idempotents $f1, \dots, fr$ such that $S=f1R\oplus\dots\oplus frR$. Thus S has an identity element $e=f1+f2+\dots+fr$, e is a central idempotent $\mathfrak{R}=e\mathfrak{R}\oplus\operatorname{Ann} e=S\oplus\operatorname{Ann} e=$ $=S\oplus A$.

Corollary 28. *Let $\mathscr{U}, \mathscr{V}\in L\mathscr{A}\mathfrak{k}$, $\mathscr{U}'=\mathscr{E}$. Then $\mathscr{U}\vee\mathscr{V}=\mathscr{U}\cdot_i\mathscr{V}$, $i=2, 3$.*

If $\mathfrak{R}\in\mathscr{U}\cdot_i\mathscr{V}$, there is an ideal I of $\mathfrak{R}$ such that $\mathfrak{R}/I\in\mathscr{V}$ and $I\in\mathscr{U}$. The corollary follows from Lemmas 19 and 27.

Corollary 29. *If $\mathscr{U}\in L\mathscr{A}\mathfrak{k}$, $\mathscr{U}'=\mathscr{E}$, then $\mathscr{U}\cdot_i\mathscr{U}=\mathscr{U}$, $i=2, 3$.*

This follows from Corollary 28.

Corollary 30. *Let $\mathscr{V}\in L\mathscr{A}1\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$ and let $\mathscr{V}^{(1)}$ be the variety defined by all one-variable identities of $\mathscr{V}$. Then $\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}=\mathscr{V}^{(1)}$.*

Proof. $\mathscr{V}^{(1)}\in L\mathscr{A}1\mathfrak{k}$ since every member of $\mathscr{V}^{(1)}$ generated by one element belongs to $\mathscr{V}$. $\mathscr{V}^{(1)\prime}=\mathscr{E}$ since $\mathscr{V}$ and also $\mathscr{V}^{(1)}$ satisfy $x+f(x)=0$ for some $f\in F0^2$ (by Lemma 19). Let $\mathfrak{R}=F(1, \mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)})$. By Corollary 24, $\mathfrak{R}$ is a subdirect product of $\mathfrak{G}j\alpha$, $\alpha\in Ij$, $1\leqq j\leqq n$, $G1, \dots, Gn$ are finite fields. Since $\mathfrak{G}j\alpha\in\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}$ and $\mathfrak{G}j\alpha$ is simple $\mathfrak{G}j\alpha\in\mathscr{V}^{(1)}$, i.e., $\mathfrak{G}j\alpha\in\mathscr{V}$. Thus $\mathfrak{R}\in\mathscr{V}$. Hence, $\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}$ satisfies all the one-variable identities of $\mathscr{V}$. Thus $\mathscr{V}^{(1)}\subseteqq\mathscr{V}^{(1)}\cdot_1\mathscr{V}^{(1)}\subseteqq\mathscr{V}^{(1)}$.

Lemma 31. *Let G, H be finite fields, α, β homomorphisms of $\mathfrak{k}$ into G, H, respectively, preserving 1. Then $\mathfrak{H}\beta\in\operatorname{var}\mathfrak{G}\alpha$ iff $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G.*

Proof: If $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G, then $\mathfrak{H}\beta$ is isomorphic to a subalgebra of $\mathfrak{G}\alpha$ since H contains $\mathfrak{k}\beta\cong\mathfrak{k}\alpha$. Conversely, if $\mathfrak{H}\beta\in\operatorname{var}\mathfrak{G}\alpha$, $|G|=p^n$, then H satisfies $x-x^{p^n}=0$ and $ax=0$ for all $a\in\ker\alpha$. Thus H is of order p^m, $m|n$. As $\ker\alpha$ and $\ker\beta$ are maximal ideals of $\mathfrak{k}$, $\mathfrak{H}\beta$ satisfies $ax=0$ for all $a\in\ker\alpha+\ker\beta$, H is non-trivial, $\ker\alpha=\ker\beta$ and H is isomorphic to a subfield of G.

Proposition 32. *The set Ti of all varieties $\mathcal{V}\in L\mathcal{A}i\mathfrak{k}$, $\mathcal{V}'=\mathcal{E}$ is a submonoid of $\langle L\mathcal{A}i\mathfrak{k};\cdot_i\rangle$ and a lattice ideal of $\langle L\mathcal{A}i\mathfrak{k};\wedge,\vee\rangle$. On Ti, the lattice join $\vee$ and the variety multiplication $\cdot_i$ coincide. The lattice $\langle Ti,\wedge,\vee\rangle$ is isomorphic to the lattice of left ideals with a finite number of right components of $\langle\{(\mathfrak{m},p^n):\mathfrak{m}$ is a maximal ideal such that $\mathfrak{k}/\mathfrak{m}$ is a subfield of a finite field of order $p^n\};\leqq\rangle$, $(\mathfrak{m},p^n)\leqq(\mathfrak{m}',q^{n'})$ iff $\mathfrak{m}=\mathfrak{m}'$ and $p=q$, $n|n'$, $i=2,3$.*

Proof. That $\langle Ti,\cdot\rangle$ is a submonoid of $\langle L\mathcal{A}i\mathfrak{k};\cdot_i\rangle$ follows from Corollaries 20 and 26. Also from Corollary 20, Ti is a lattice ideal. By Corollary 28, $\mathcal{U}\cdot_i\mathcal{V}=\mathcal{U}\vee\mathcal{V}$. By Lemma 23, if $\mathcal{V}\in Ti$, $\mathcal{V}\neq\mathcal{E}$, then $\mathcal{V}=\operatorname{var}\{\mathfrak{G}j\alpha:\alpha\in Ij,\ 1\leqq j\leqq n\}$. By Lemma 31, $\mathfrak{H}\beta\in\mathcal{V}$ iff $\mathfrak{H}\beta$ is isomorphic to a subalgebra of $\mathfrak{G}j\alpha$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. Thus $\mathcal{V}\in Ti$ is determined by the set of all pairs $(\mathfrak{m},p^n)$ such that G is a field of p^n elements and $\mathfrak{k}/\mathfrak{m}$ is a subfield of G, $\mathfrak{G}\alpha\in\mathcal{V}$, where α is the natural homomorphism of $\mathfrak{k}$ onto $\mathfrak{k}/\mathfrak{m}\subseteqq G$. The set of all such pairs for a given $\mathcal{V}$ satisfies $(\mathfrak{m},p^n)\leqq(\mathfrak{m}',q^{n'})$, $(\mathfrak{m}',q^{n'})$ is in the set implies $(\mathfrak{m},p^n)$ is in the set. Thus it is a left ideal. Since every $\mathcal{V}$ involves only a finite number of non-isomorphic fields, the set of right components in the set of pairs is finite.

Proof of Theorem 4. Let $\mathcal{V}\in L\mathcal{A}0\mathfrak{k}$ be equationally complete and $\mathcal{V}'\neq\mathcal{E}$. Then $\mathcal{V}=\mathcal{V}'$. By Lemma 17, $V\subseteqq\mathfrak{m}F_0+F_0^2$ for some maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. Hence, $V=\mathfrak{m}F_0+F_0^2$. V is a maximal T-ideal of F_0, $\mathcal{V}$ satisfies $ax=0$ for all $a\in\mathfrak{m}$, and $xy=0$. This is the type of equationally complete varieties $\mathcal{V}\in L\mathcal{A}0\mathfrak{k}$, $\mathcal{V}'\neq\mathcal{E}$. If $\mathcal{V}\in L\mathcal{A}1\mathfrak{k}$, $\mathcal{V}'=\mathcal{E}$, $\mathcal{V}$ is equationally complete, then $\mathcal{V}=\operatorname{var}F(1,\mathcal{V})$, $\operatorname{var}F(1,\mathcal{V})\in L\mathcal{A}3\mathfrak{k}$. By Lemma 23, $\mathcal{V}=\operatorname{var}\{\mathfrak{G}j\alpha:\alpha\in Ij,\ 1\leqq j\leqq n\}$. Hence $\mathcal{V}=\operatorname{var}\mathfrak{G}\alpha$, for some finite field G and a homomorphism α of $\mathfrak{k}$ into G preserving 1. Thus $\mathcal{V}=\operatorname{var}\mathfrak{k}/\mathfrak{m}$ for some maximal ideal of finite index in $\mathfrak{k}$, since $\mathfrak{G}\alpha$ contains a subalgebra isomorphic to $\mathfrak{k}/\ker\alpha$. By Lemma 31, $\operatorname{var}\mathfrak{k}/\mathfrak{m}$ does not contain any non-trivial proper subvarieties. Thus $\mathcal{V}$ is determined by the identities $ax=0$ for all $a\in\mathfrak{m}$, $x-x^{p^r}=0$ where $p^r=|\mathfrak{k}/\mathfrak{m}|$.

6. Varieties of algebras over rings with exactly 2 idempotent ideals

Throughout Section 6, we assume that if $\mathfrak{a}$ is an ideal of $\mathfrak{k}$, and $\mathfrak{a}^2=\mathfrak{a}$, then $\mathfrak{a}=\mathfrak{o}$ or $\mathfrak{a}=\mathfrak{k}$.

Lemma 33. *Let* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$. *Then* $\mathscr{V}=\mathscr{A}i\mathfrak{k}$ *or* $\mathscr{V}'=\mathscr{E}$; $i=0, 1, 2, 3$.

Proof. If $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, then $d(i,\mathscr{V})<\infty$ and $d(i,\mathscr{V})=d(i,\mathscr{V}\cdot_i\mathscr{V})=d(i,\mathscr{V})^2$ (by Lemma 11). Thus, $d(i, V)=1$, i.e., there are non-trivial polynomials of degree 1 in V and $V(Fi)\subseteq V'(Fi)=\mathfrak{a}_1Fi+\mathfrak{a}_2Fi^2+\dots$ where $\mathfrak{a}_1\neq\mathfrak{o}$. Hence $(V\circ V)(Fi)\subseteq\subseteq(V'\circ V')(Fi)\subseteq\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots$ (by Lemma 15). Thus $V(Fi)=(V\circ V)(Fi)\subseteq\subseteq\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots\subseteq V'(Fi)$. But $\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots$ is i-multinilpotent, whence $V'(Fi)=\mathfrak{a}_1^2Fi+\mathfrak{a}_1\mathfrak{a}_2Fi^2+\dots=\mathfrak{a}_1Fi+\mathfrak{a}_2Fi^2+\dots$. Hence $\mathfrak{a}_1^2=\mathfrak{a}_1$. But $\mathfrak{a}_1\neq\mathfrak{o}$. Hence $\mathfrak{a}_1=\mathfrak{k}$, i.e., $V'(Fi)=Fi$, i.e., $\mathscr{V}'=\mathscr{E}$.

Corollary 34. *Let* $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. *Then* $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ *iff* $\mathscr{V}'=\mathscr{E}$, $i=2, 3$.

This follows from Corollary 29 and Lemma 33.

It may be noted that if $\mathfrak{k}$ has an ideal $\mathfrak{a}\neq\mathfrak{o}$, $\mathfrak{a}\neq\mathfrak{k}$, $\mathfrak{a}^2=\mathfrak{a}$, then the variety $\mathscr{V}$ of all $\mathfrak{k}$-algebras satisfying $ax=0$ for all $a\in\mathfrak{a}$ is idempotent, i.e., $\mathscr{V}\cdot_0\mathscr{V}=\mathscr{V}$, $\mathscr{V}'=\mathscr{V}\neq\mathscr{E}$.

Proof of Theorem 5. A set $I\subseteq F0$ is attainable on a variety $\mathscr{V}$ iff the T-ideal of $F0$ generated by I is attainable on $\mathscr{V}$. It was shown by A. I. Mal'cev [13], that if I is attainable on $\mathscr{V}$, then the variety $\mathscr{U}\in L\mathscr{V}$ determined by I satisfies $\mathscr{U}\cdot_V\mathscr{U}=\mathscr{U}$, or equivalently $(U\circ U)(F\mathscr{V})=U(F\mathscr{V})$. If $\mathscr{U}\cdot_i\mathscr{U}=\mathscr{U}$, then $\mathscr{U}=\mathscr{A}i\mathfrak{k}'$ or $\mathscr{U}'=\mathscr{E}$ by Lemma 33. Let $i=1, 2, 3$. Then $\mathscr{U}\cap\mathscr{A}2\mathfrak{k}$ is generated by $\{\mathfrak{G}j\alpha\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$, by Lemma 23, if $\mathscr{U}\neq\mathscr{E}$, $\mathscr{U}\neq\mathscr{A}i\mathfrak{k}$. Let $\mathfrak{m}$ be $\ker\alpha$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. Let $\mathfrak{N}$ be the ideal of $(\mathfrak{k}/\mathfrak{m})[x]$ generated by x. $U(\mathfrak{N})=\bigcap\{Vj\alpha(\mathfrak{N})\colon \alpha\in Ij,\ 1\leqq j\leqq n\}$, $\mathscr{V}j\alpha=\operatorname{var}\mathfrak{G}j\alpha$. $Vj\alpha(\mathfrak{N})\neq\mathfrak{N}$ iff $\mathfrak{m}=\ker\alpha$. Also, $G1,\dots,Gn$ are finitely many and each Gj is a finite field, there is only finitely many $\mathfrak{G}j\alpha$ such that $Vj\alpha(\mathfrak{N})\neq\mathfrak{N}$ for some $\alpha\in Ij$, $1\leqq j\leqq n$. $Vj\alpha(\mathfrak{N})\neq 0$ for any $\alpha\in Ij$, $1\leqq j\leqq n$. Thus $U(\mathfrak{N})$ is a proper non-trivial ideal of $\mathfrak{N}$. Hence, there is a polynomial $h(x)\in\mathfrak{N}$, $h(x)\neq x$, $h(x)\neq 0$, such that $U(\mathfrak{N})=h(x)(\mathfrak{k}/\mathfrak{m})[x]$. By the methods of the proof of A. A. Iskander's [11], Theorem 15, p. 237, replacing the prime field $\mathbf{Z}_p$ by $\mathfrak{k}/\mathfrak{m}$ one can show that $U(U(\mathfrak{N}))\neq U(\mathfrak{N})$. Thus U is not attainable on $\mathfrak{N}$. Hence, if I is attainable on $\mathscr{A}i\mathfrak{k}$, $\mathscr{U}$ is the variety of $L\mathscr{A}i\mathfrak{k}$ determined by I, then $\mathscr{U}=\mathscr{A}i\mathfrak{k}$, i.e., I is equivalent to $x=x$ on $\mathscr{A}i\mathfrak{k}$, or $\mathscr{U}=\mathscr{E}$, i.e., I is equivalent to $x=y$ on $\mathscr{A}i\mathfrak{k}$.

7. Varieties of algebras over Dedekind domains

Throughout Section 7, unless otherwise stated, $\mathfrak{k}$ is a Dedekind domain.

Proposition 35. *The following conditions on a variety $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ are equivalent:*

(1) *$\mathscr{V}$ satisfies $x^n=0$ for some natural number $n>0$.*

(2) *$\mathscr{V}\in N(i, \{\mathfrak{k}/\mathfrak{m}\colon \mathfrak{m}$ is a maximal ideal of $\mathfrak{k}\})$, $i=1, 2, 3$.*

Proof. Since a field does not contain any non-zero nilpotent elements, (1) implies (2). Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathfrak{k}/\mathfrak{m}\notin\mathscr{V}$ for any maximal ideal $\mathfrak{m}$ of $\mathfrak{k}$. As $F(1, \mathscr{V})\in\mathscr{A}3\mathfrak{k}$, the factor algebra $F(1, \mathscr{V})/\mathfrak{N}$, where $\mathfrak{N}$ is the nilradical, is a subdirect product of rings without zero-divisors. Thus, if $\mathfrak{N}\neq F(1, \mathscr{V})$, the algebra $F(1, \mathscr{V})$ has a non-trivial factor algebra $\mathfrak{R}$ without zero-divisors, and it is not difficult to show that $\mathfrak{R}$ can be chosen such that for its "characteristic" $\mathfrak{p}\lhd\mathfrak{k}$ one has either $\mathfrak{R}\cong x(\mathfrak{k}/\mathfrak{p})[x]$ or $\mathfrak{R}\cong x(\mathfrak{k}/\mathfrak{p})[x]/f(x)$ where f is primitive irreducible, and $\mathfrak{R}$ obviously has in both cases field factors, which, in their turn, must have a subfield of the prescribed form.

Proposition 36. *If $\mathfrak{k}$ is a principal ideal ring or a Dedekind domain and $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ is i-multinilpotent, then $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, where $U(Fi)=\mathfrak{a}Fi$, and $\mathscr{W}$ is a nilpotent variety that is i-multinilpotent, $i=0, 1, 2, 3$.*

Proof. $V(Fi)=\mathfrak{a}_1 Fi+\mathfrak{a}_2 Fi^2+\dots$ where $(\mathfrak{a}_n)$ is an ascending chain of ideals of $\mathfrak{k}$. Since $\mathfrak{k}$ is Noetherian there is n such that $\mathfrak{a}=\mathfrak{a}_n=\mathfrak{a}_m$, for all $m>n$. If $\mathfrak{k}$ is a principal ideal ring, $\mathfrak{a}_r=a_r\mathfrak{k}$, $\mathfrak{a}=a\mathfrak{k}$, $\mathfrak{a}_r\subseteq\mathfrak{a}$ implies $a_r=ab_r$, $a, b_r, a_r\in\mathfrak{k}$. Hence $\mathfrak{a}_r=\mathfrak{a}(b_r\mathfrak{k})=\mathfrak{a}\mathfrak{b}_r$ for all $r\leqq n$, $b_n=1$. If $\mathfrak{k}$ is a Dedekind domain, $\mathfrak{a}_r=\mathfrak{m}1^{s1}\dots\mathfrak{m}t^{st}$ and $\mathfrak{a}_r\subseteq\mathfrak{a}=\mathfrak{m}1^{u1}\dots\mathfrak{m}t^{ut}$ implies $s1\geqq u1, \dots, st\geqq ut$. Thus $\mathfrak{a}_r=\mathfrak{b}_r\mathfrak{a}$ where $\mathfrak{b}_r=\mathfrak{m}1^{v1}\dots\mathfrak{m}t^{vt}$, $v1=s1-u1, \dots, vt=st-ut$. Hence,

$$V(Fi) = \mathfrak{a}\mathfrak{b}_1 Fi+\mathfrak{a}\mathfrak{b}_2 Fi^2+\dots+\mathfrak{a}Fi^n =$$
$$= \mathfrak{a}(\mathfrak{b}_1 Fi+\mathfrak{b}_2 Fi^2+\dots+Fi^n) = (\mathfrak{a}F0)(\mathfrak{b}_1 Fi+\mathfrak{b}_2 Fi^2+\dots+Fi^n).$$

Proof of Theorem 3. From Definition 2 and Lemma 17, $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ is i-pseudo-indecomposable iff $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'\neq\mathscr{E}$ and $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U}, \mathscr{W}\in L\mathscr{A}i\mathfrak{k}$ implies $\mathscr{U}'=\mathscr{E}$ or $\mathscr{W}'=\mathscr{E}$, $i=0, 1, 2, 3$. We will write $\mathscr{V}_1\cdot_i\mathscr{V}_2\cdot_i\mathscr{V}_3$ to mean one of the products $(\mathscr{V}_1\cdot_i\mathscr{V}_2)\cdot_i\mathscr{V}_3, \mathscr{V}_1\cdot_i(\mathscr{V}_2\cdot_i\mathscr{V}_3)$. In general, $\mathscr{V}_1\cdot_i\mathscr{V}_2\cdot_i\dots\cdot_i\mathscr{V}_n$ will mean any of the products obtained by the introduction of suitable parentheses.

Lemma 37. *Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$ and $\mathscr{V}=\mathscr{V}_1\cdot_i\mathscr{V}_2\cdot_i\dots\cdot_i\mathscr{V}_n$. Then the number of $\mathscr{V}_j$ such that $d(i, \mathscr{V}_j)>1$ is at most equal to the number of primes (including repetitions) in the prime factorization of $d(i, \mathscr{V})$; the number of $\mathscr{V}_j$ such that $\mathscr{V}_j'\neq\mathscr{E}$ and $d(i, \mathscr{V}_j)=1$ is at most equal to the number of maximal ideals (including repetitions) in the factorization of $\mathfrak{c}(i, \mathscr{V})$ as a product of maximal ideals, $i=0, 1, 2, 3$.*

Proof. By Lemma 11, $d(i, \mathscr{U} \cdot_i \mathscr{W})=d(i, \mathscr{U}).d(i, \mathscr{W})$. By induction on n, $d(i, \mathscr{V})=d(i, \mathscr{V}_1) \dots d(i, \mathscr{V}_n)$. Hence the number of $\mathscr{V}_j$ such that $d(i, \mathscr{V}_j)>1$ cannot exceed the number of primes in the factorization of $d(i, \mathscr{V})$. To prove the rest of the lemma, we show first that for any variety $\mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $d(i, \mathscr{W})=d(i, \mathscr{W}')$. In fact, $d(i, \mathscr{W})=d$ iff $W(Fi) \subseteqq Fi^d$ and $W(Fi) \nsubseteq Fi^{d+1}$. This is true since if $d(i, \mathscr{W})=d$, $W(Fi)$ contains elements of degree d and no elements of degree less than d. Thus $W(Fi) \subseteqq F^d$, $W(Fi) \nsubseteq Fi^{d+1}$ (due to linearization). Conversely, if $W(Fi) \nsubseteq Fi^{d+1}$, then $W(Fi)$ contains elements of degree $\leqq d$; if $W(Fi) \subseteqq Fi^d$, then $W(Fi)$ does not contain any elements of degree less than d (again W is closed under linearization). Thus $d(i, \mathscr{W})=d$. Now, $W'(Fi) \supseteqq W(Fi)$, $W(Fi) \nsubseteq Fi^{d+1}$. So $W'(Fi) \nsubseteq Fi^{d+1}$. Also $Fi^d \supseteqq W(Fi)$, $Fi^d=F0^d(Fi)$. If $\mathscr{U}$ is the subvariety of $\mathscr{A}i\mathfrak{k}$ whose T-ideal in Fi is Fi^d, then $\mathscr{U}$ is i-multinilpotent. Thus $\mathscr{W}' \supseteqq \mathscr{U}$, i.e., $Fi^d=U(Fi) \supseteqq$ $\supseteqq W'(Fi)$. Hence $d(i, \mathscr{W}')=d=d(i, \mathscr{W})$. Let $\mathscr{V}=\mathscr{V}_1 \cdot_i \mathscr{V}_2 \cdot_i \dots \cdot_i \mathscr{V}_n$. Then $\mathscr{V} \supseteqq \mathscr{V}_1' \cdot_i \mathscr{V}_2' \cdot_i \dots \cdot_i \mathscr{V}_n' \supseteqq (\dots((\mathscr{V}_1' \cdot_i \mathscr{V}_2') \cdot_i \mathscr{V}_3') \cdot_i \dots) \cdot_i \mathscr{V}_n'$, (by Theorem 1). $d(i, \mathscr{V})=d(i, \mathscr{V}_1) \dots d(i, \mathscr{V}_n)=d(i, \mathscr{V}_1') \dots d(i, \mathscr{V}_n')$ and

$$\mathscr{V} \supseteqq \mathscr{V}' \supseteqq ((\dots((\mathscr{V}_1' \cdot_i \mathscr{V}_2')' \cdot_i \mathscr{V}_3')' \cdot_i \dots)' \cdot \mathscr{V}_n')'.$$

Hence

$$\begin{aligned} \mathfrak{c}(i, \mathscr{V}) &= \mathfrak{c}(i, d(i, \mathscr{V}), \mathscr{V}) = \mathfrak{c}(i, d(i, \mathscr{V}'), \mathscr{V}) = \\ &= \mathfrak{c}(i, d, \mathscr{V}) \subseteqq \mathfrak{c}(i, d, \mathscr{V}') = \mathfrak{c}(i, \mathscr{V}') \subseteqq \quad \text{(by Lemma 11)} \\ &\subseteqq \mathfrak{c}(i, ((\dots((\mathscr{V}_1' \cdot_i \mathscr{V}_2')' \cdot_i \mathscr{V}_3')' \cdot_i \dots)' \cdot \mathscr{V}_n')') = \\ &= \mathfrak{c}(i, \mathscr{V}_1')\mathfrak{c}(i, \mathscr{V}_2')^{d_1}\mathfrak{c}(i, \mathscr{V}_3')^{d_1 d_2} \dots \mathfrak{c}(i, \mathscr{V}_n')^{d_1 d_2 \dots d_{n-1}}, \end{aligned}$$

by Lemma 15 and by induction on n, where $d_j=d(i, \mathscr{V}_j)=d(i, \mathscr{V}_j')$, $1 \leqq j \leqq r$. If $d(i, \mathscr{V}_j)=1$, $\mathscr{V}_j' \neq \mathscr{E}$, then $\mathfrak{c}(i, \mathscr{V}_j') \neq \mathfrak{k}$, $\mathfrak{c}(i, \mathscr{V}_j') \neq \mathfrak{o}$. Since

$$\mathfrak{c}(i, \mathscr{V}) \subseteqq \mathfrak{c}(i, \mathscr{V}_1')\mathfrak{c}(i, \mathscr{V}_2')^{d_1} \dots \mathfrak{c}(i, \mathscr{V}_n')^{d_1 \dots d_{n-1}} \subseteqq \prod \{\mathfrak{c}(i, \mathscr{V}_j') \colon 1 \leqq j \leqq n\}$$

and $\mathfrak{k}$ is a Dedekind domain, each non-zero proper ideal of $\mathfrak{k}$ is uniquely the product of maximal ideals, possibly non-distinct, of $\mathfrak{k}$. If $\mathfrak{c}(i, \mathscr{V})=\mathfrak{m}1 \dots \mathfrak{m}r$, where $\mathfrak{m}1, \dots, \mathfrak{m}r$ are maximal ideals of $\mathfrak{k}$, possibly equal, each of the ideals $\mathfrak{c}(i, \mathscr{V}_j') \neq \mathfrak{k}$ is a product of some of $\mathfrak{m}1, \cdots, \mathfrak{m}r$. Thus, the number of $\mathscr{V}_j$, such that $\mathscr{V}_j' \neq \mathscr{E}$, $d(i, \mathscr{V}_j)=1$, is at most r.

We return to the proof of the theorem.

If $\mathscr{V} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V} \neq \mathscr{A}i\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$, then either $\mathscr{V}$ is i-pseudo-indecomposable or $\mathscr{V}=\mathscr{U} \cdot_i \mathscr{W}$ for some $\mathscr{U}, \mathscr{W} \in L\mathscr{A}i\mathfrak{k}$, $\mathscr{U} \neq \mathscr{A}i\mathfrak{k}$, $\mathscr{W} \neq \mathscr{A}i\mathfrak{k}$, $\mathscr{U}' \neq \mathscr{E}$, $\mathscr{W}' \neq \mathscr{E}$. Continuing this procedure, by Lemma 37, after a finite number of steps we get $\mathscr{V}=\mathscr{V}_1 \cdot_i \mathscr{V}_2 \cdot_i \dots \cdot_i \mathscr{V}_n$ where $\mathscr{V}_1, \dots, \mathscr{V}_n$ are i-pseudo-indecomposable.

If $\mathscr{U} \in L\mathscr{A}2\mathfrak{k}$, $\mathscr{U}'=\mathscr{E}$, $\mathscr{C}=\mathscr{A}3\mathfrak{k}$, then, by Lemma 23 and Corollary 28, $\mathscr{U} \cdot_2 \mathscr{C}=$ $=\mathscr{U} \vee \mathscr{C}=\mathscr{C}$. Thus $\mathscr{C} \cdot_2 \mathscr{C}=\mathscr{C} \cdot_2 (\mathscr{U} \cdot_2 \mathscr{C})$. By Corollary 26, $(\mathscr{C} \cdot_2 \mathscr{U}) \cdot_2 \mathscr{C}=$ $=\mathscr{C} \cdot_2 (\mathscr{U} \cdot_2 \mathscr{C})=\mathscr{C} \cdot_2 \mathscr{C}$. $\mathscr{C} \cdot_2 \mathscr{U} \neq \mathscr{C}$ if $\mathscr{U} \neq \mathscr{E}$. Thus the decomposition of Theorem 3 is not unique. It is an open question as to whether different factorizations are due only to this reason. Also, pseudo-indecomposables cannot be replaced by indecomposables if $\mathfrak{k}$ contains a maximal ideal of finite index. For then, if α is a non-trivial homomorphism of $\mathfrak{k}$ into a finite field G, $\operatorname{var}\mathfrak{G}\alpha\subseteqq\mathscr{C}$ and $\mathscr{C}\cdot_2\mathscr{C}= =(\mathscr{C}\cdot_2\operatorname{var}\mathfrak{G}\alpha)\cdot_2\mathscr{C}=(\ldots(\mathscr{C}\cdot_2\operatorname{var}\mathfrak{G}\alpha)\cdot_2\ldots\cdot_2\operatorname{var}\mathfrak{H}\alpha)\cdot_2\mathscr{C}$, where $G\subseteqq\ldots\subseteqq H$ are any ascending chain of finite fields.

If $\mathfrak{k}$ is a field of characteristic 0, $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $\mathscr{V}'=\mathscr{E}$, then $x_1+f(x_1)\in V$ for some $f\in F0^2$ (by Lemma 19). By linearization $ax_1\in V$ for some $a\neq 0$, $a\in\mathfrak{k}$. Hence $a^{-1}(ax_1)\in V$, i.e., $V=F0$. Thus, $\mathscr{V}=\mathscr{E}$. Hence, over a field of characteristic 0, i-pseudo-indecomposables are i-indecomposable. This concludes the proof of Theorem 3.

Corollary 38. *Suppose $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, $d(i,\mathscr{V})$ is prime and $\mathfrak{c}(i,\mathscr{V})=\mathfrak{k}$. Then $\mathscr{V}$ is i-pseudo-indecomposable, $i=0,1,2,3$.*

This follows from Lemma 37, since $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$ and $\mathscr{V}'\neq\mathscr{E}$.

Corollary 39. *Let $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$. Then either $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ or $\mathscr{V}$ is a product of a finite number of i-pseudo-indecomposable varieties, $i=2,3$.*

If $\mathscr{V}=\mathscr{A}i\mathfrak{k}$, then $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$. If $\mathscr{V}'=\mathscr{E}$, then $\mathscr{V}\cdot_i\mathscr{V}=\mathscr{V}$ (by Corollary 29). The rest follows from Theorem 3.

Proposition 40. *Suppose $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$ and V contains all words of $G0$ of length n in x_1 for some $n\geqq 1$. Then $\mathscr{V}$ is i-pseudo-indecomposable iff $\mathscr{V}$ is i-indecomposable, $i=0,1,2,3$.*

Proof. If $\mathscr{V}$ is i-indecomposable, then $\mathscr{V}$ is i-pseudo-indecomposable. Let $\mathscr{V}$ be i-pseudo-indecomposable. Then $\mathscr{V}\supseteqq\mathscr{V}'\neq\mathscr{E}$, $\mathscr{V}\neq\mathscr{A}i\mathfrak{k}$. Suppose $\mathscr{V}=\mathscr{U}\cdot_i\mathscr{W}$, $\mathscr{U},\mathscr{W}\in L\mathscr{A}i\mathfrak{k}$. Then $\mathscr{U}'=\mathscr{E}$ or $\mathscr{W}'=\mathscr{E}$, $\mathscr{U},\mathscr{W}\subseteqq\mathscr{V}$. By Lemma 19, $x_1+f(x_1)\in U$ or $x_1+f(x_1)\in W$ for some $f\in F0^2$. Thus $x_1=-f(x_1)$ is an identity in $\mathscr{U}$ or in $\mathscr{W}$. By repeated substitutions, $x_1=-f(-f(\ldots(-f(x_1))\ldots))$, we can get a term of order $\geqq n$ on the right hand side. Hence $x_1=0$ is an identity in $\mathscr{U}$ or in $\mathscr{W}$; i.e., $\mathscr{U}=\mathscr{E}$ or $\mathscr{W}=\mathscr{E}$.

Corollary 41. *If $\mathscr{V}\in L\mathscr{A}i\mathfrak{k}$, V contains all words of $G0$ of length n in x_1 for some $n>0$ and $\mathscr{V}\neq\mathscr{E}$, then $\mathscr{V}$ is a product of a finite number of i-indecomposable varieties, $i=0,1,2,3$.*

This follows from Theorem 3 and Proposition 40.

From Corollary 38 $\mathscr{A}j\mathfrak{k}$ is i-pseudo-indecomposable for $0\leqq i<j\leqq 3$. The variety of all commutative algebras is 0-pseudo-indecomposable. The variety of all Jordan algebras is 0-pseudo-indecomposable. The variety of all Lie algebras is 0-indecomposable and 1-indecomposable. This follows from Corollary 28 and Proposition 40.

8. Changing the domain of operators

We will consider the effect of changing the domain of operators $\mathfrak{k}$ on $\langle L\mathscr{V};\ \cdot_{\mathscr{V}},\ \wedge,\ \vee\rangle$, $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. Let $\mathfrak{k}'$ be a commutative and associative ring with 1. We assume $\mathfrak{k}'$ is non-trivial. Let α be a ring homomorphism of $\mathfrak{k}$ into $\mathfrak{k}'$ preserving 1. Let $\mathfrak{a}=\ker\alpha$. For every $f\in F0=F\mathscr{A}0\mathfrak{k}$, $f\alpha\in F\mathscr{A}0\mathfrak{k}'$ is defined by replacing all the coefficients of elements of $G0$ in f by their images under α. Let $\mathscr{V}\in\mathscr{A}0\mathfrak{k}$. $\alpha\mathscr{V}$ is the subvariety of $\mathscr{A}0\mathfrak{k}'$ defined by $\{f\alpha\colon f\in V\}=V\alpha$. Θ is the equivalence relation on $L\mathscr{A}0\mathfrak{k}$ such that $\mathscr{U}\Theta\mathscr{V}$ iff $\alpha\mathscr{U}=\alpha\mathscr{V}$. Every $\mathfrak{k}'$-algebra $\mathfrak{R}$ can be considered naturally as a $\mathfrak{k}$-algebra: $ax=(a\alpha)x$, $a\in\mathfrak{k}$, $x\in\mathfrak{R}$. With this understanding $\alpha\mathscr{V}=\mathscr{V}\cap\mathscr{A}0\mathfrak{k}'$. For some special cases, cf. J. M. Osborn [17], p. 187 and M. V. Volkov [25], p. 62.

Lemma 42. *Let $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. Then $\mathscr{V}\to\alpha\mathscr{V}$ is a homomorphism of $\langle L\mathscr{V};\ \cdot_{\mathscr{V}},\ \wedge\rangle$ into $\langle L\alpha\mathscr{V};\ \cdot_{\alpha\mathscr{V}},\ \wedge\rangle$ preserving all intersections.*

Proof. Let $\mathscr{V}_t\in L\mathscr{V}$, $t\in I$. $(\sum\{V_t\colon t\in I\})\alpha=\sum\{V_t\alpha\colon t\in I\}$, i.e., $\alpha\bigcap\{\mathscr{V}_t\colon t\in I\}=\bigcap\{\alpha\mathscr{V}_t\colon t\in I\}$. Let $\mathscr{U},\mathscr{W}\in L\mathscr{V}$.

$$\alpha(\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W})=(\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W})\cap\alpha\mathscr{V}=(\mathscr{U}\cap\alpha\mathscr{V})\cdot_{\alpha\mathscr{V}}(\mathscr{W}\cap\alpha\mathscr{V})=\alpha\mathscr{U}\cdot_{\alpha\mathscr{V}}\alpha\mathscr{W}.$$

Till the end of the present paper S is a submonoid of the multiplicative monoid of $\mathfrak{k}$ such that $s\alpha$ is not a zero-divisor in $\mathfrak{k}'$ for any $s\in S$ and $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}\alpha$ relative to $S\alpha$. Every element in $\mathfrak{k}'$ can be written as x/s where $s\in S$, $x\in\mathfrak{k}$, $x/s=y/t$ iff $tx-sy\in\mathfrak{a}=\ker\alpha$. If $\mathfrak{R}\in\mathscr{A}0\mathfrak{k}$, $T(\mathfrak{R})=\{x\colon x\in\mathfrak{R},\ sx\in\mathfrak{a}\mathfrak{R}$ for some $s\in S\}$ and $\alpha\mathfrak{R}$ is the tensor product of $\mathfrak{k}'$ and $\mathfrak{R}$ as $\mathfrak{k}$-algebras. Clearly, $\alpha\mathfrak{R}\in\mathscr{A}0\mathfrak{k}'$. This construction is a covariant functor from the category $\mathscr{A}0\mathfrak{k}$ into the category $\mathscr{A}0\mathfrak{k}'$. In the case under consideration, which unifies the special case where α is a homomorphism of $\mathfrak{k}$ onto $\mathfrak{k}'$, i.e., $S=\{1\}$, and the one where $\mathfrak{k}'$ is the ring of fractions of $\mathfrak{k}$ relative to S, i.e., $\mathfrak{a}=\ker\alpha=\mathfrak{o}$, respectively, $\alpha\mathfrak{R}$ has a simple construction, cf. P. M. Cohn [6], p. 21. The carrier of $\alpha\mathfrak{R}$ is the set $S\times\mathfrak{R}/\sim$ where $(s,x)\sim(t,y)$ iff $sy-tx\in T(\mathfrak{R})$. The equivalence class of (s,x) will be denoted by $(x/s)^{\sim}$. We have

$$(x/s)^{\sim}+(y/t)^{\sim}=((tx+sy)/st)^{\sim},\quad (x/s)^{\sim}(y/t)^{\sim}=(xy/st)^{\sim},\quad (a/s)(y/t)^{\sim}=(ay/st)^{\sim},$$
$$a\in\mathfrak{k},\ s,t\in S,\ x,y\in\mathfrak{R}.$$

Put $x\alpha'=(x/1)^{\sim}$.

Some of the properties of $\alpha\mathfrak{R}$, α' are summarized in the following:

Lemma 43. *Let $\mathfrak{R}\in\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ and let $\mathfrak{R}$ be generated by Y. Then*

(i) $\alpha\mathfrak{R}\in\alpha\mathscr{V}$,

(ii) *α' is a homomorphism of $\mathfrak{k}$-algebras whose kernel is $T(\mathfrak{R})$ and the $\mathfrak{k}$-subalgebra of $\alpha\mathfrak{R}$ generated by $Y\alpha'$ is isomorphic to $\mathfrak{R}/T(\mathfrak{R})$,*

(iii) $\alpha\mathfrak{R}\cong\alpha(\mathfrak{R}/T(\mathfrak{R}))$, *and*

(iv) *if β is a homomorphism of $\mathfrak{k}$-algebras from $\mathfrak{R}$ into $\mathfrak{R}_1\in\mathscr{A}0\mathfrak{k}'$, then there is a unique homomorphism γ of $\mathfrak{k}'$-algebras from $\alpha\mathfrak{R}$ into $\mathfrak{R}_1$ such that $\beta=\alpha'\gamma$.*

Conversely, let $\mathfrak{R}_1 \in \mathscr{A}0\mathfrak{k}'$ *be generated as a* $\mathfrak{k}'$*-algebra by* Y, *and let* $\mathfrak{R}$ *be the* $\mathfrak{k}$*-subalgebra of* $\mathfrak{R}_1$ *generated by* Y. *Then* $y \to y\alpha'$ *can be extended to an isomorphism of* $\mathfrak{R}_1$ *onto* $\alpha\mathfrak{R}$. *If* $\mathfrak{R}_1$ *satisfies* $g=0$ $(g \in F\mathscr{A}0\mathfrak{k}')$ *and* $h\alpha=g$, $h \in F\mathscr{A}0\mathfrak{k}$, *then* $\mathfrak{R}$ *satisfies* $h=0$.

Proof. That $\alpha\mathfrak{R} \in \mathscr{A}0\mathfrak{k}'$ is standard. By the methods of the proof of L. H. ROWEN's [19] Proposition 1.3, p. 393, $\alpha\mathfrak{R}$ satisfies $f\alpha=0$ if $\mathfrak{R}$ satisfies $f=0$. (ii), (iii), (iv) follow from the construction of $\alpha\mathfrak{R}$. To check the converse, let $x \in T(\mathfrak{R})$. Then $sx=\sum \{a_i y_i \colon 1 \leqq i \leqq n\}$, $a_1, \ldots, a_n \in \mathfrak{a}$, $y_1, \ldots, y_n \in \mathfrak{R}$. But $\mathfrak{R} \subseteq \mathfrak{R}_1$, $sx=(s\alpha)x= =\sum \{(a_i\alpha)y_i \colon 1 \leqq i \leqq n\}=0$. Thus $x=(1/s)sx=0$, i.e., $T(\mathfrak{R})=0$. Thus α' is injective from $\mathfrak{R}$ into $\alpha\mathfrak{R}$. If $z \in \mathfrak{R}_1$, then $z=f(y_1, \ldots, y_n)$ where $f \in F\mathscr{A}0\mathfrak{k}'$, $y_1, \ldots, y_n \in Y$. The coefficients in f are of the form $a_1/s_1, \ldots, a_m/s_m$, $a_1, \ldots, a_m \in \mathfrak{k}$, $s_1, \ldots, s_m \in S$; they can be rewritten as $b1/s, \ldots, bm/s$, $b1, \ldots, bm \in \mathfrak{k}$, $s \in S$. Thus $z=(1/s)u$, where $u \in \mathfrak{R}$. The mapping $(1/t)v \to (v/t)^{\sim}$ is well defined from $\mathfrak{R}_1$ onto $\alpha\mathfrak{R}$. $(1/s)u= =(1/t)v$ iff $(u/s)^{\sim}=(v/t)^{\sim}$. This mapping is a homomorphism and it is injective, i.e., it is an isomorphism. If $\mathfrak{R}_1$ satisfies $g=0$ and $h\alpha=g$, then $\mathfrak{R}$ satisfies $h=0$ since in $\mathfrak{R}$, $ax=(a\alpha)x$, $a \in \mathfrak{k}$, $x \in \mathfrak{R}$.

Corollary 44. *If* $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$, *then* $\alpha\mathscr{V}$ *is the class of all isomorphic copies of* $\alpha\mathfrak{R}$, $\mathfrak{R} \in \mathscr{V}$. α *maps* $L\mathscr{A}0\mathfrak{k}$ *onto* $L\mathscr{A}0\mathfrak{k}'$.

Proof. From Lemma 43, $\alpha\mathfrak{R} \in \alpha\mathscr{V}$ if $\mathfrak{R} \in \mathscr{V}$ and $\mathfrak{R}_1 \in \alpha\mathscr{V}$ iff $\mathfrak{R}_1 \cong \alpha\mathfrak{R}$, $\mathfrak{R} \in \mathscr{V}$. If $\mathscr{W} \in L\mathscr{A}0\mathfrak{k}'$, then $F\mathscr{W} \cong \alpha\mathfrak{R}$ where $\mathfrak{R}$ is the $\mathfrak{k}$-subalgebra of $F\mathscr{W}$ generated by X. Let $\mathscr{U}=\operatorname{var}\mathfrak{R}$. Then $\alpha\mathscr{U}=\mathscr{W}$, since by Lemma 43, $\mathfrak{R}$ satisfies $h=0$ for all $h \in F\mathscr{A}0\mathfrak{k}$, $h\alpha \in \mathscr{W}$.

Corollary 45. *For any cardinal number* n, $F(n, \alpha\mathscr{V}) \cong \alpha F(n, \mathscr{V})$.

Proof. Let $\mathscr{V}$ be non-trivial and $\mathfrak{R}_1=F(n, \alpha\mathscr{V})$. By Lemma 43, $\mathfrak{R}_1 \cong \alpha\mathfrak{R}$ where $\mathfrak{R}$ is the $\mathfrak{k}$-subalgebra of $\mathfrak{R}_1$ generated by $X(n)$, $\mathfrak{R} \in \mathscr{V}$. Hence there is a homomorphism β of $F(n, \mathscr{V})$ onto $\mathfrak{R}$ such that $x\beta=x$ for all $x \in X(n)$. Hence, by Lemma 43 there is a homomorphism γ of $\alpha F(n, \mathscr{V})$ onto $\alpha\mathfrak{R}$, i.e., onto $\mathfrak{R}_1$ such that $x\alpha'\gamma=x$ for all $x \in X(n)$. But, there is a homomorphism δ of $\mathfrak{k}'$-algebras from $\mathfrak{R}_1$ onto $\alpha F(n, \mathscr{V}) \in \alpha\mathscr{V}$ such that $x\delta=x\alpha'$ for all $x \in X(n)$. Hence $x\delta\gamma=x$ for all $x \in X$. Thus δ is injective and so δ is an isomorphism. If $\alpha\mathscr{V}$ is trivial, then $F(n, \alpha\mathscr{V})$ is trivial and $\alpha F(n, \mathscr{V}) \in \alpha\mathscr{V}$.

For any variety $\mathscr{V} \in L\mathscr{A}0\mathfrak{k}$, $T(V)=\{f \colon f \in FA0\mathfrak{k}=F0, sf \in V+\mathfrak{a}F0 \text{ for some } s \in S\}$. Clearly $T(V)$ is a T-ideal of $F0$ containing V.

Lemma 46. *Let* $\mathscr{U}, \mathscr{V} \in L\mathscr{A}0\mathfrak{k}$. *Then* $\alpha\mathscr{U}=\alpha\mathscr{V}$ *iff* $T(U)=T(V)$.

Proof. Let $f \in T(U)$. Then $sf \in U+\mathfrak{a}F0$ for some $s \in S$. $\alpha\mathscr{U}$ satisfies $g=0$ iff $\alpha\mathscr{U}$ satisfies $sg=0$. But $sf=u+a_1f_1+\ldots+a_nf_n$, $a_1, \ldots, a_n \in \mathfrak{a}$, $f_1, \ldots, f_n \in F0$. Thus

$(sf)\alpha=u\alpha+(a_1f_1+\ldots+a_nf_n)\alpha=u\alpha+0$. Thus $\alpha\mathscr{U}$ satisfies $f\alpha$ for all $f\in T(U)$. Hence $\alpha T(\mathscr{U})=\alpha\mathscr{U}$. If $T(U)=T(V)$, then $\alpha\mathscr{U}=\alpha T(\mathscr{U})=\alpha T(\mathscr{V})=\alpha\mathscr{V}$. Conversely, if $\alpha\mathscr{U}=\alpha\mathscr{V}$, then $\alpha T(\mathscr{U})=\alpha T(\mathscr{V})$. Hence $F\alpha T(\mathscr{U})=F\alpha T(\mathscr{V})$. The $\mathfrak{k}$-subalgebra of $F\alpha T(\mathscr{U})$ generated by X is isomorphic to $\mathfrak{N}/T(\mathfrak{N})$ (by Lemmas 43 and Corollary 45), where $\mathfrak{N}=FT(\mathscr{U})\cong F0/T(U)$. Let $x\in T(\mathfrak{N})$. Then $sx\in\mathfrak{a}\mathfrak{N}$ for some $s\in S$, i.e., if $x=g+T(U)$, then $sg+T(U)\subseteqq T(U)+\mathfrak{a}F0=T(U)$. Thus $T(\mathfrak{N})=0$, and the $\mathfrak{k}$-subalgebra of $F\alpha T(\mathscr{U})$ generated by X is isomorphic to $F0/T(U)$. Hence $F0/T(U)\cong$ $\cong F0/T(V)$. Since $T(U)$, $T(V)$ are T-ideals of $F0$, $T(U)=T(V)$.

$T(\mathscr{U})$ is the smallest variety among the varieties $\mathscr{W}$ such that $\alpha\mathscr{W}=\alpha\mathscr{U}$, since if $\alpha\mathscr{W}=\alpha\mathscr{U}$, then $W\subseteqq T(W)=T(U)$. In the case $\mathfrak{a}=\mathfrak{o}$, the least variety $\mathscr{W}$ such that $\alpha\mathscr{W}=\alpha\mathscr{U}$ is called by M. V. Volkov [25], p. 66, the S-knotted variety associated to $\mathscr{U}$. Modifying slightly the terminology of M. V. Volkov when $\mathfrak{a}\neq\mathfrak{o}$, define a binary relation λ on $LA0\mathfrak{k}$ by $\mathscr{U}\lambda\mathscr{V}$ iff there is $s\in S$ such that $sU\subseteqq V+\mathfrak{a}F0$, $sV\subseteqq U+\mathfrak{a}F0$. A variety $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ is S-joined if the restrictions of Θ and λ on $L\mathscr{V}$ coincide. M. V. Volkov [25], Lemma 9, p. 67, showed that λ is a congruence on $\langle L\mathscr{A}0\mathfrak{k};\wedge,\vee\rangle$ if $\mathfrak{a}=\mathfrak{o}$. Thus λ is a lattice congruence on the lattice of varieties satisfying $ax=0$ for all $a\in\mathfrak{a}$. However, λ is a congruence on the meet semi-lattice $\langle L\mathscr{A}0\mathfrak{k};\wedge\rangle$. Let $\mathscr{U},\mathscr{V},\mathscr{W}\in L\mathscr{A}0\mathfrak{k}$, $s\in S$, $sU\subseteqq V+\mathfrak{a}F0$, $sV\subseteqq U+\mathfrak{a}F0$. Then $s(U+W)\subseteqq V+W+\mathfrak{a}F0$ and $s(V+W)\subseteqq U+W+\mathfrak{a}F0$. Also, $\lambda\subseteqq\Theta$. The relation between λ and Θ is described by

Proposition 47. *Let $\mathscr{U},\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$. Then $\alpha\mathscr{U}=\alpha\mathscr{V}$ iff there are $\mathscr{U}_i$, $\mathscr{V}_i\in L\mathscr{A}0\mathfrak{k}$, $\mathscr{U}_i\lambda\mathscr{V}_i$, $i\in I$, and $\mathscr{U}=\bigcap\{\mathscr{U}_i\colon i\in I\}$, $\mathscr{V}=\bigcap\{\mathscr{V}_i\colon i\in I\}$.*

Proof. Since $\lambda\subseteqq\Theta$ and α preserves all intersections, we need to show the only if part. Let $\alpha\mathscr{U}=\alpha\mathscr{V}$. By Lemma 46, $T(U)=T(V)$. Let $I=\{(f,g)\colon f\in U$, $g\in V$, $sf-tg\in\mathfrak{a}F0$ for some $s,t\in S\}$. If $i\in I$, $i=(f,g)$, then $\mathscr{U}_i$ is the variety of all algebras satisfying $f=0$ and $\mathscr{V}_i$ is the variety of algebras satisfying $g=0$. Thus $\mathscr{U}\subseteqq\bigcap\{\mathscr{U}_i\colon i\in I\}$ and $\mathscr{V}\subseteqq\bigcap\{\mathscr{V}_i\colon i\in I\}$. Let $f\in U\subseteqq T(U)=T(V)$. Then there is $s\in S$ such that $sf\in V+\mathfrak{a}F0$, i.e., there is $g\in V$ such that $sf-g\in\mathfrak{a}F0$. Thus, $\mathscr{U}\subseteqq\mathscr{U}_i$, $i=(f,g)$. Since $\mathscr{U}=\bigcap\{\mathscr{U}_f\colon f\in U\}$, where $\mathscr{U}_f$ is the variety of all algebras satisfying $f=0$, $\mathscr{U}=\bigcap\{\mathscr{U}_i\colon i\in I\}$ and, similarly, $\mathscr{V}=\bigcap\{\mathscr{V}_i\colon i\in I\}$. If $i=(f,g)$, then $sf-tg\in\mathfrak{a}F0$. Hence, $sU_i\subseteqq tV_i+\mathfrak{a}F0$ and $stU_i\subseteqq t^2V_i+\mathfrak{a}F0\subseteqq V_i+\mathfrak{a}F0$. Similarly $stV_i\subseteqq U_i+\mathfrak{a}F0$, i.e., $\mathscr{U}_i\lambda\mathscr{V}_i$.

It is implicit in M. V. Volkov [25] that the join of S-knotted varieties is S-knotted (in the case $\mathfrak{a}=\mathfrak{o}$). This also follows once we check that $T(U\cap V)=T(U)\cap T(V)$ if $U,V\supseteqq\mathfrak{a}F0$. If $\mathscr{V}$ is a variety satisfying $ax=0$ for all $a\in\mathfrak{a}$, then $\alpha(\mathscr{U}\vee\mathscr{W})=$ $=\alpha\mathscr{U}\vee\alpha\mathscr{W}$ for any $\mathscr{U},\mathscr{W}\in L\mathscr{V}$. This follows from M. V. Volkov [25], p. 63. We give here another proof using T-ideals. The T-ideal αU of the variety $\alpha\mathscr{U}$ is the T-ideal of $F\mathscr{A}0\mathfrak{k}'$ generated by $U\alpha$. Thus $\alpha U=\alpha T(U)$ is generated by $T(U)\alpha$. If $U\supseteqq\mathfrak{a}F0$, then $\alpha U=\alpha T(U)$ is the set of all elements of the form $(f/s)^{\sim}$, $f\in T(U)$,

$s\in S$ where $(f/s)^{\sim}=(g/t)^{\sim}$ iff $tf-sg\in\mathfrak{a}F0$. If $(f/s)^{\sim}\in\alpha T(U\cap W)$; then $f\in T(U\cap W)=T(U)\cap T(W)$, i.e., $(f/s)^{\sim}\in\alpha T(U)\cap\alpha T(W)$. If $(f/s)^{\sim}\in\alpha T(U)\cap$ $\cap\alpha T(W)$, then $f/s=g/t$, where $f\in T(U)$, $g\in T(W)$. Since $sg-tf\in\mathfrak{a}F0$, $sg\in T(U)+$ $+\mathfrak{a}F0=T(U)$. Thus $g\in T(U)$, i.e., $(f/s)^{\sim}\in\alpha(T(U)\cap T(W))$.

We conclude that $\mathscr{U}\to\alpha\mathscr{U}$ is a homomorphism of $\langle L\mathscr{V};\cdot_{\mathscr{V}},\wedge,\vee\rangle$ onto $\langle L\alpha\mathscr{V};\cdot_{\alpha\mathscr{V}},\wedge,\vee\rangle$ for any variety $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ satisfying $ax=0$ for all $a\in\mathfrak{a}$. This follows from Lemma 41, Corollary 44 and $\alpha(\mathscr{U}\vee\mathscr{W})=\alpha\mathscr{U}\vee\alpha\mathscr{W}$ if $\mathscr{U},\mathscr{W}\in L\mathscr{V}$.

A number of characterizations of S-joined varieties, in the case $\mathfrak{a}=\mathfrak{o}$, were given by M. V. Volkov [25]. The same characterizations can be modified to describe the case $\mathfrak{a}\neq\mathfrak{o}$. For instance, $\mathscr{V}$ is S-joined iff for every subvariety $\mathscr{W}$ of $\mathscr{V}$, $T(\mathscr{W})$ is finitely based relative to $\mathscr{W}$; $\mathscr{V}$ is S-joined iff for every subvariety $\mathscr{W}$ of $\mathscr{V}$ there is $s\in S$ such that $sT(W)\subseteqq W+\mathfrak{a}F0$. This is true since if $\mathscr{V}$ is S-joined then $\mathscr{W}\lambda T(\mathscr{W})$ since $\lambda=\Theta$ on $L\mathscr{V}$. Thus there is $s\in S$ such that $sT(W)\subseteqq W+\mathfrak{a}F0$. Conversely, if for every $\mathscr{W}\in L\mathscr{V}$ there is $s\in S$ such that $sT(W)\subseteqq W+\mathfrak{a}F0$, then $\mathscr{W}\lambda T(\mathscr{W})$. If $\mathscr{U},\mathscr{W}\in L\mathscr{V}$, $\alpha\mathscr{U}=\alpha\mathscr{W}$, then $T(\mathscr{U})=T(\mathscr{W})$ and $\mathscr{U}\lambda T(\mathscr{U})$, $\mathscr{W}\lambda T(\mathscr{W})$, i.e., $\mathscr{U}\lambda\mathscr{W}$. The following will show the behavior of S-joined varieties under multiplication of varieties:

Proposition 48. *The S-joined subvarieties of $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$ form a subgroupoid with* 1 *of* $\langle L\mathscr{V};\cdot_{\mathscr{V}}\rangle$.

Proof. Let $\mathscr{U}$, $\mathscr{W}$ be S-joined varieties, $\mathscr{U},\mathscr{W}\in L\mathscr{V}$, and let $\mathscr{K}\subseteqq\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W}$, $\mathscr{K}\in L\mathscr{V}$. We need to show that there is $s\in S$ such that $sT(K)\subseteqq K+\mathfrak{a}F0$. In other words, if $x\in F\mathscr{K}$ and there is $t\in S$ such that $tx\in\mathfrak{a}F\mathscr{K}$, then $sx\in\mathfrak{a}F\mathscr{K}$. Let $\mathfrak{R}=F\mathscr{K}$. Then $\mathfrak{R}\in\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W}$, $\mathfrak{R}/W(\mathfrak{R})\in\mathscr{W}$ and $\mathfrak{R}/W(\mathfrak{R})=F\mathscr{M}$ where $\mathscr{M}\subseteqq\mathscr{W}$, $W(\mathfrak{R})\in\mathscr{U}$. $W(\mathfrak{R})$ generates a variety $\mathscr{M}_1\subseteqq\mathscr{U}$. Let $x\in\mathfrak{R}$, $t\in S$ and $tx\in\mathfrak{a}\mathfrak{R}$. Hence $t\bar{x}\in F\mathscr{M}$ where $\bar{x}=x+W(\mathfrak{R})\in F\mathscr{M}$. Thus there is $s\in S$ not depending on x or t such that $s\bar{x}\in\mathfrak{a}F\mathscr{M}$, i.e., $sx\in W(\mathfrak{R})+\mathfrak{a}\mathfrak{R}$ for all $x\in\mathfrak{R}$ such that there is $t\in S$ and $tx\in\mathfrak{a}\mathfrak{R}$. Thus $tsx\in\mathfrak{a}\mathfrak{R}$ and $tsx=0$ in $W(\mathfrak{R})+\mathfrak{a}R/\mathfrak{a}\mathfrak{R}\cong W(\mathfrak{R})/W(\mathfrak{R})\cap\mathfrak{a}\mathfrak{R}$. x is a polynomial f from $F0$, and $tsx=tsf(x_1,\ldots,x_n)=0$ is an identity in $W(\mathfrak{R})/W(\mathfrak{R})\cap$ $\cap\mathfrak{a}\mathfrak{R}$. Thus $tsf=0$ is an identity in $F\mathscr{M}_1/\mathfrak{a}F\mathscr{M}_1$. Hence, there is $u\in S$ not depending on x or t such that $usx=0$ in $F\mathscr{M}_1/\mathfrak{a}F\mathscr{M}_1$; i.e., $usx=0$ in $W(\mathfrak{R})/W(\mathfrak{R})\cap\mathfrak{a}\mathfrak{R}\cong$ $\cong W(\mathfrak{R})+\mathfrak{a}\mathfrak{R}/\mathfrak{a}\mathfrak{R}$. But $sx\in W(\mathfrak{R})+\mathfrak{a}\mathfrak{R}$. Hence $usx\in\mathfrak{a}\mathfrak{R}$ for any $x\in\mathfrak{R}$ such that $tx\in\mathfrak{a}\mathfrak{R}$ for some $t\in S$, i.e., $usT(K)\subseteqq K+\mathfrak{a}F0$. The variety $\mathscr{E}$ is S-joined.

Corollary 49. *The S-joined varieties of $L\mathscr{V}$, $\mathscr{V}\in L\mathscr{A}0\mathfrak{k}$, form a lattice ideal of* $\langle L\mathscr{V};\wedge,\vee\rangle$.

Since a subvariety of an S-joined variety is S-joined and $\mathscr{U}\vee\mathscr{W}\subseteqq\mathscr{U}\cdot_{\mathscr{V}}\mathscr{W}$ if $\mathscr{U},\mathscr{W}\in L\mathscr{V}$, the corollary follows from Proposition 48.

That the S-joined varieties of $L\mathscr{A}0\mathfrak{k}$ $(\mathfrak{a}=\mathfrak{o})$ form a lattice ideal of $\langle L\mathscr{A}0\mathfrak{k};\wedge,\vee\rangle$ was shown by M. V. Volkov [25], Proposition 8, p. 72.

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Submaximal clones with a prime order automorphism

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1. Introduction

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Let $\mathfrak{L}$ denote the lattice of clones over a finite set A, $|A|\geqq 3$. A clone $C\in\mathfrak{L}$ is called *submaximal* if it is covered by a maximal clone. Although the full list of maximal clones has been known for more than twenty years [14], [15], so far the submaximal clones have not been intensively studied, except for $|A|=3$. In that case all submaximal clones are known. The description was completed by D. LAU [8], making use of some earlier results [10], [3], [21] (also [1]) for the first three types of maximal clones (cf. Theorem 2.1 below). For arbitrary finite A, the first author started to investigate the maximal subclones of $\operatorname{Pol} B$ where $\emptyset\neq B\subset A$ [17]. D. LAU [9] found all maximal subclones of $\operatorname{Pol} B$ when $|B|=1$. Recently the second author determined the maximal subclones of $\operatorname{Pol} s^{\bullet}$ when $|A|$ is prime and s is a cyclic permutation of A [23]. The aim of this paper is to solve the corresponding problem in the general case, i.e., to determine all maximal subclones of $\operatorname{Pol} s^{\bullet}$ where s is a fixed point free permutation of A with $s^p=\mathrm{id}$ (p prime).

In general, the submaximal clones seem to be interesting for the following reasons. The largely unknown lattice $\mathfrak{L}$ has intervals with antichains of cardinality $2^{\aleph_0}$ situated far down from the top. It is not unreasonable to assume that $\mathfrak{L}$ is nicer near the top, and therefore the submaximal clones are good candidates. The problem of determining certain submaximal clones also came up in the second author's study of shortest maximal chains in $\mathfrak{L}$ [24]. Given a maximal clone M one can ask for a primality or completeness criterion for M: under what conditions does the clone $\bar{F}$ generated by some $F\subseteq M$ coincide with M? In case M is finitely

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generated, a full list of clones maximal in M would provide a general criterion, because then $\bar{F}=M$ if and only if F is contained in no clone maximal in M. An application could be a characterization of Sheffer operations for M (i.e., $f\in M$ such that $\overline{\{f\}}=M$). V. B. Kudrjavcev [7] and P. Schofield [22] proved that they exist exactly for maximal clones determined by permutations, equivalences, or unary relations (equivalently, these clones form a unique irredundant cover of the clone of all operations), but the examples provided have many variables. It would be interesting to have simple criteria of the type G. Rousseau [20] gave for $\mathcal{O}$, which, in its turn, could lead to the question: what is the minimum number of functional values whose knowledge can guarantee that an operation is Sheffer ($k+2$ for $\mathcal{O}$ [18])? Finally, the submaximal clones may be of interest on their own, e.g., as a source of examples and counter-examples.

2. Preliminaries and main result

Let A be a finite set, $|A|\geqq 2$. Denote by $\mathcal{O}$ the set of (finitary) operations on A. A *clone* over A is the set of polynomials [5] of some algebra with base set A, i.e., a subset of $\mathcal{O}$ containing the projections and closed with respect to superposition. It is well known that an operation is a polynomial of some algebra $(A; F)$ if and only if it preserves all subalgebras of finite powers of $(A; F)$. This permits one to describe clones by means of "invariant relations" in the following sense: Relations are simply subsets of finite powers of A; the subsets of A^h $(0<h<\aleph_0)$ are called h-ary relations. The set of relations is denoted by $\mathcal{R}$. An operation f is said to *preserve* an h-ary relation ϱ if ϱ is a subalgebra of $(A;f)^h$. For a set of relations $R\subseteqq\mathcal{R}$, let Pol R consist of all operations preserving every relation from R, and for $F\subseteqq\mathcal{O}$ let Inv F consist of all relations preserved by every operation from F. It is well known and easy to check that Pol and Inv determine a Galois connection between the subsets of $\mathcal{O}$ and $\mathcal{R}$, with closure operators $F\mapsto \mathrm{Pol\,Inv}\,F$ on $\mathcal{O}$ and $R\mapsto[R]=\mathrm{Inv\,Pol}\,R$ on $\mathcal{R}$. In view of the above remark the closed sets of operations are exactly the clones. The closed sets of relations are called *relational algebras* [11, 1.1.8]. The set of relational algebras, ordered by inclusion, is a lattice $\mathfrak{L}^*$, which is dually isomorphic to the lattice $\mathfrak{L}$ of clones on A (the mutually inverse dual isomorphisms are $R\mapsto \mathrm{Pol}\,R$ and $F\mapsto \mathrm{Inv}\,F$).

The relational algebras $[R]$ can be described in various ways [2], [4], [11], but for our purposes we shall use the following [11, 2.1]: an h-ary relation ϱ belongs to $[R]$ if and only if there exists a first order formula $\Phi(x_0, \ldots, x_{h-1})$ (with free variables $x_0, \ldots, x_{h-1}$) built up from $\exists$, $\wedge$ and relation symbols from $R\cup\{=\}$ such that

$$\varrho = \{(a_0, \ldots, a_{h-1})\in A^h\colon\ \Phi(a_0, \ldots, a_{h-1})\ \text{ holds true}\}.$$

Simple but useful special cases are, for example, the direct product of relations, the relational product ($\circ$) of two binary relations, the intersection of relations of the same arity, the permutation of the components of a relation, in particular, taking the inverse ($^{-1}$) of a binary relation, or the projection of a relation onto some of its components, e.g., taking the domain (i.e., first projection) or range (second projection) of a binary relation.

It is well known [11, 4.1.3] that the lattice of clones over A is dually atomic and has a finite number of dual atoms, which are termed *maximal clones*. We shall need their explicit description found in [14], [15] (see also [11, 5.2.2], [12]), therefore we recall some definitions.

To every, say n-ary, operation f we can associate the $(n+1)$-ary relation $f^{\bullet}$ consisting of all $(n+1)$-tuples $(a_0, \dots, a_{n-1}, f(a_0, \dots, a_{n-1}))$ with $a_0, \dots, a_{n-1} \in A$. For $F \subseteq \mathcal{O}$ we set $F^{\bullet} = \{f^{\bullet} : f \in F\}$. If $(A; +)$ is an abelian group, the quaternary relation $(x_0 - x_1 + x_2)^{\bullet}$ is referred to as the *affine relation* determined by $(A; +)$. An h-ary relation ϱ is called *central* if $\varrho \neq A^h$, ϱ is totally reflexive (i.e., contains all h-tuples having repeated components), totally symmetric (i.e., invariant under all permutations of components), and the *center* $\{a \in A : \{a\} \times A^{h-1} \subseteqq \varrho\}$ of ϱ is nonempty. A family $T = \{\vartheta_0, \dots, \vartheta_{m-1}\}$ of equivalence relations on A is said to be h-regular, if each ϑ_i $(0 \leqq i < m)$ has $h (\geqq 3)$ blocks, and $\bigcap (B_i : 0 \leqq i < m)$ is nonempty for arbitrary blocks B_i of ϑ_i $(0 \leqq i < m)$. The relation λ_T determined by T consists of all h-tuples whose components meet at most $h-1$ blocks of each ϑ_i $(0 \leqq i < m)$. The relations of the form λ_T will be called *regular* (or *h-regular*, where h is the arity). The equality relation, denoted ω, and the full relation A^2 on A are termed *trivial* equivalence relations.

Theorem 2.1 ([14], [15]). *Let A be a finite set, $|A| \geqq 2$. The maximal clones on A are the clones* Pol ϱ *where ϱ is one of the following relations*:

(O) *a bounded order,*

(P) *a relation $g^{\bullet}$ where g is a fixed point free permutation with $g^p = \mathrm{id}$ (p prime),*

(A) *an affine relation determined by an elementary abelian p-group (p prime),*

(E) *a nontrivial equivalence relation,*

(C) *a central relation,*

(R) *a regular relation.*

These relations will be called *atomic*. In view of the dual isomorphism between the lattices of clones and relational algebras, it is clear that the atomic relations are the generators of the atoms in the lattice of relational algebras.

The aim of this paper is to describe the maximal subclones of Pol $s^{\bullet}$ where $s^{\bullet}$ is an atomic relation of type (P), $s^p = \mathrm{id}$. For the formulation of the main theorem we introduce some notation and definitions. Denote by Θ the equivalence relation consisting of all pairs $(a, b) \in A^2$ with $a = s^i(b)$ for some $0 \leqq i < p$. An h-ary relation ϱ will be termed *Θ-closed* if $(b_0, \ldots, b_{h-1}) \in \varrho$ whenever $(a_0, \ldots, a_{h-1}) \in \varrho$ and $(a_i, b_i) \in \Theta$ for all $0 \leqq i < h$. In other words, ϱ Θ-closed means that ϱ is the full inverse image of a relation on the quotient set A/Θ (consisting of the blocks of Θ) under the natural mapping $A \to A/\Theta$ sending every $a \in A$ into the block containing it. In particular,

(a) an equivalence relation ϱ is Θ-closed if and only if $\Theta \subseteq \varrho$,

(b) a regular relation λ_T with $T = \{\vartheta_0, \ldots, \vartheta_{m-1}\}$ is Θ-closed if and only if $\Theta \subseteq \vartheta_0 \cap \ldots \cap \vartheta_{m-1}$, and

(c) a central relation is Θ-closed if and only if it is the inverse image of a central relation on A/Θ.

An equivalence relation ε will be called *transversal to s* if $s \in \operatorname{Pol} \varepsilon$ and $\varepsilon \cap \Theta = \omega$, i.e., s maps each block of ε onto another block of ε. A unary relation μ is *transversal to s* if $(\mu \times \mu) \cap \Theta \subseteq \omega$, i.e., $s^i(x) \notin \mu$ whenever $x \in \mu$, $1 \leqq i < p$.

In order to determine one type of maximal subclones of $\operatorname{Pol} s^{\bullet}$ we need a result from group theory. For two primes q, r such that $q^n \equiv 1 \pmod r$ and n is the least positive integer with this property, we denote by $\mathfrak{G}(q, r)$ the group of linear functions $ax+b$ on $GF(q^n)$ with $a, b \in GF(q^n)$ and $a^r = 1$. Clearly, $|\mathfrak{G}(q, r)| = q^n r$.

Responding to our inquiry, P. P. Pálfy proved the following fact:

Proposition 2.2. *A finite group has a maximal subgroup of order p (p prime) if and only if it is isomorphic to one of the groups listed below:*

(i) *an abelian group of order pq (q prime),*

(ii) *$\mathfrak{G}(p, q)$ for a prime q with $p \equiv 1 \pmod q$,*

(iii) *$\mathfrak{G}(q, p)$ for a prime $q \neq p$.*

Proof. The sufficiency being obvious, take a finite group $\mathfrak{G}$ which has a maximal subgroup $\mathfrak{H}$ with $|\mathfrak{H}| = p$. To show that $\mathfrak{G}$ is isomorphic to one of the groups (i)—(iii) the only nontrivial case to consider is $|\mathfrak{G}| = pn$ with n composite and $n \not\equiv 0 \pmod p$. Then $\mathfrak{H}$ is not normal, implying by the maximality of $\mathfrak{H}$ that $\mathfrak{H}$ coincides with its normalizer. Hence $\mathfrak{G}$ is a Frobenius group to $\mathfrak{H}$ [6, V.8.1]. Moreover, again by the maximality of $\mathfrak{H}$, the usual permutation representation of $\mathfrak{G}$ [6, V.8.2] is primitive, yielding that the Frobenius kernel of $\mathfrak{G}$ is elementary abelian [6, V.8.19]. Now it is easy to see that every proper subgroup of $\mathfrak{G}$ is abelian, and hence our statement follows from [13, Satz 4].

Now we define permutation groups on A as follows. For a group $\mathfrak{G}$ whose order divides $|A|$, consider a partition of A into $|\mathfrak{G}|$-element blocks $A_0, \ldots, A_{l-1}$ ($l|\mathfrak{G}| = |A|$), and select arbitrary bijections $\varphi_i \colon A_i \to \mathfrak{G}$ ($0 \leqq i < l$). Clearly, the permutations π_g ($g \in \mathfrak{G}$) of A defined by $\pi_g(x) = \varphi_i^{-1}(g \cdot \varphi_i(x))$ for every $0 \leqq i < l$ and $x \in A_i$ form a group, which will be called a *semiregular representation of* $\mathfrak{G}$

on A. (Note that a semiregular representation of $\mathfrak{G}$ on A exists only if $|\mathfrak{G}|$ divides $|A|$.)

After these preparations we are in a position to state our main result:

Theorem 2.3. *Let* A *be a finite set,* $|A| \geqq 2$, *and let* s *be a fixed point free permutation of* A *with* $s^p = \mathrm{id}$ *(p prime). Then the maximal subclones of* $\mathrm{Pol}\, s^{\bullet}$ *are the clones* $\mathrm{Pol}\,\{s^{\bullet}, \varrho\}$ *where* ϱ *is one of the following relations:*

(P_s) *a relation* $g^{\bullet}$ *such that* g *is a permutation of* A *and* $\{s, g\}$ *generates a semiregular representation of a group from Proposition 2.2,*

(A_s) *an affine relation determined by an elementary abelian p-group* $(A; +)$ *such that there exists an element* $c \in A$ *with* $s(x) = x + c$ *for every* $x \in A$,

(E_s) *a nontrivial equivalence relation that is either* Θ*-closed or transversal to* s,

(C_s) *a* Θ*-closed central relation or a nonempty unary relation transversal to* s,

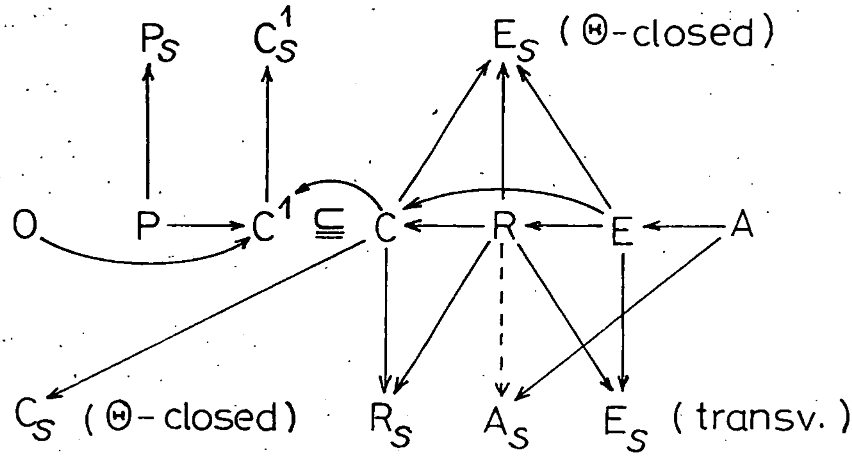
(R_s) *a* Θ*-closed regular relation.*

Corollary 2.4. *An algebra* $(A; F)$ *admitting* s *as an automorphism is polynomially equivalent to* $(A;\ \mathrm{Pol}\, s^{\bullet})$ *if and only if none of the relations* (P_s)—(R_s) *occurs among the subalgebras of finite powers of* $(A; F)$.

The proof of Theorem 2.3 naturally splits into two parts: one has to verify on the one hand that the clones listed in the theorem are indeed maximal in $\mathrm{Pol}\, s^{\bullet}$, and on the other hand, that the list is complete, i.e., all maximal subclones are found. Since $\mathrm{Pol}\, s^{\bullet}$ is finitely generated [11, 4.3.26] and hence the lattice of its subclones is dually atomic, the latter is equivalent to showing that every proper subclone of $\mathrm{Pol}\, s^{\bullet}$ is contained in $\mathrm{Pol}\,\{s^{\bullet}, \varrho\}$ for some relation ϱ listed in Theorem 2.3. In terms of relational algebras this statement can be formulated as follows:

Theorem 2.5. *Let* A *be a finite set,* $|A| \geqq 2$, *and let* s *be a fixed point free permutation of* A *with* $s^p = \mathrm{id}$ *(p prime). Then every relational algebra properly including* $[s^{\bullet}]$ *contains a relation of one of the types* (P_s)—(R_s).

The detailed (and rather lengthy) proof of Theorem 2.5 will be presented in the next section. Here we sketch only the main idea. The first step is to observe that any relational algebra properly including $[s^{\bullet}]$ contains either an atomic relation outside $[s^{\bullet}]$, or a relation $g^{\bullet}$ of type (P_s) such that $g^p = s$ (see Proposition 3.3), which explains also the surprising similarity between Theorems 2.1 and 2.3. Therefore in the rest of the proof it suffices to show for each type of atomic relation $\varrho \notin [s^{\bullet}]$, that $[s^{\bullet}, \varrho]$ contains a relation listed in Theorem 2.3. The proof will be constructive, the steps being illustrated by the arrows in the diagram below, except for one case (dotted arrow) when we use an argument for the operations preserving the relations in question (see Lemma 3.13 and Remark 2 after it).



Here C^1 and C_s^1 denote the unary relations of types C and C_s, respectively.

The maximality of the clones Pol $\{s^\bullet, \varrho\}$ in Pol $s^\bullet$ if ϱ runs over the relations (P_s)—(R_s) will be proved in Section 4. In the language of relational algebras this part of Theorem 2.3 has the following reformulation:

Theorem 2.6. *Let A be a finite set, $|A| \geqq 2$, and let s be a fixed point free permutation of A with $s^p = \mathrm{id}$ (p prime). Then for every relation ϱ of type* (P_s)—(R_s) *the relational algebra* $[s^\bullet, \varrho]$ *covers* $[s^\bullet]$.

3. Proof of Theorem 2.5

First we introduce some notation. For a positive integer n, $\mathbf{n}$ will denote the set $\{0, \ldots, n-1\}$. We can assume without loss of generality that $A = \mathbf{k}$ (hence p divides k) and the cycles of s are $(tp, tp+1, \ldots, tp+p-1)$, $0 \leqq t < k/p$. It will be convenient to write $x \oplus i$ and $x \ominus i$ instead of $s^i(x)$ and $s^{p-i}(x)$, respectively ($x \in \mathbf{k}$, $i \in \mathbf{p}$). In particular, restricted to $\mathbf{p}$, $\oplus$ and $\ominus$ are addition and subtraction modulo p. For an h-ary relation ϱ and $c = (c_0, \ldots, c_{h-1}) \in \mathbf{p}^h$ we will denote by $\varrho \oplus c$ the relation consisting of all h-tuples $(a_0 \oplus c_0, \ldots, a_{h-1} \oplus c_{h-1})$ with $(a_0, \ldots, a_{h-1}) \in \varrho$. The following is obvious:

Lemma 3.1. *For every h-ary relation ϱ and $c \in \mathbf{p}^h$ we have* $[\varrho \oplus c, s^\bullet] = [\varrho, s^\bullet]$.

To check whether a relation belongs to $[s^\bullet]$ or not, we shall often need an explicit description of the members of $[s^\bullet]$. For later applications we formulate a slightly more general statement.

Lemma 3.2. *Let G be a permutation group on $\mathbf{k}$ in which no nonidentity permutation has fixed points. Then, up to a rearrangement of its components, every nonvoid* *member of* $[G^{\bullet}]$ *is a direct product of relations of the form*

$$\{(a, g_1(a), \ldots, g_{h-1}(a))\colon\ a\in\mathbf{k}\} \tag{1}$$

with $h\geqq 1$ *and* $g_1, \ldots, g_{h-1}\in G$.

Proof. Let $\varrho\in[G^{\bullet}]$, $\varrho\neq\emptyset$, say ϱ is n-ary, and consider a formula $\Phi(x_0, \ldots, x_{n-1})$ with bound variables $x_n, \ldots, x_{m-1}$ $(m\geqq n)$, which defines ϱ. Since G consists of permutations, the matrix of Φ is essentially a set of equations of the form $x_j=g(x_i)$ with $0\leqq i, j<m$ and $g\in G$. Let $\sim$ denote the least equivalence relation on $\{x_0, \ldots, x_{m-1}\}$ containing all such pairs (x_i, x_j). Then $\Phi(x_0, \ldots, x_{m-1})$ can be split into the conjunction of its "subformulas" $\Phi_B(x_i\colon\ i<n,\ x_i\in B)$ corresponding to the $\sim$-blocks B. Clearly, up to the order of its components, ϱ is the direct product of the relations determined by Φ_B. Moreover, each Φ_B defines a relation of the form (1), since the assumption on G and $\varrho\neq\emptyset$ imply that for any $i, j\in B$ there is exactly one $g\in G$ with $x_j=g(x_i)$.

In the special case when $G=\{\mathrm{id}\}$, the relations described in Lemma 3.2 are the so called *diagonal relations*. The members of $[\omega]=[\mathrm{id}^{\bullet}]$, i.e., the relations which are empty or diagonal are termed *trivial relations*. Clearly, a relation is trivial if and only if it is preserved by every operation.

Now we can prove that "almost all" relational algebras properly including $[s^{\bullet}]$ contain an atomic relation outside $[s^{\bullet}]$.

Proposition 3.3. *Let R be a relational algebra such that $R\supset[s^{\bullet}]$, and every atomic relation in R belongs to $[s^{\bullet}]$. Then R contains a relation $g^{\bullet}$ for some permutation g with $g^p=s$.*

Proof. We will need the following property of R.

Claim. An h-ary $(h\geqq 1)$ relation $\xi\in R$ is diagonal whenever it contains an h-tuple $(a, \ldots, a)$ for some $a\in\mathbf{k}$. In particular, every nontrivial binary relation $\beta\in R$ is irreflexive (i.e., $\beta\cap\omega=\emptyset$).

To prove the claim assume $(a, \ldots, a)\in\xi$ $(a\in\mathbf{k})$. It is easy to see that this implies $(a, \ldots, a)\in\xi'$ for all $\xi'\in[\xi]$. Therefore $[\xi]$, and hence also $R(\supseteq[\xi])$, contains an atomic relation with the same property, unless ξ is diagonal. This shows that ξ is diagonal, as stated.

Now let $\varrho\in R\setminus[s^{\bullet}]$ be of minimum arity, say t. We prove that $t=2$. Clearly, $t\geqq 2$ since otherwise ϱ would be a nontrivial unary relation, and hence would be atomic. Suppose $t>2$ and let ϱ' denote the projection of ϱ onto its first $t-1$ components. By the minimality of t and $\varrho'\in[\varrho]$ it follows that $\varrho'\in[s^{\bullet}]$. Applying Lemma 3.2 for the permutation group generated by s we get that either $\varrho'=\mathbf{k}^{t-1}$, or ϱ' (and hence also ϱ) has a binary projection, say onto the i-th and j-th components $(0\leqq i<j<t-1)$, which is of the form $(s^l)^{\bullet}$ for some $0\leqq l<p$. In the latter case ϱ

belongs to the relational algebra generated by $s^{\bullet}$ and the $(t-1)$-ary projection of ϱ omitting the j-th component. However, in view of the minimality of t, both of these relations belong to $[s^{\bullet}]$, yielding $\varrho\in[s^{\bullet}]$. This contradiction shows that $\varrho'=\mathbf{k}^{t-1}$. Hence we have $(0, \dots, 0, u)\in\varrho$ for some $u\in\mathbf{k}$. By the minimality of t, the binary relation $\eta\in[\varrho]$ consisting of all pairs $(a, b)\in\mathbf{k}^2$ with $(a, \dots, a, b)\in\varrho$ belongs to $[s^{\bullet}]$. Clearly, $\eta\neq\emptyset$, therefore u can be chosen so that $u\in\mathbf{p}$. Set $d=(0, \dots, 0, \ominus u)$, and form $\varrho\oplus d$. Obviously, $(0, \dots, 0)\in\varrho\oplus d\in R$ and, by Lemma 3.1, $\varrho\oplus d$ is not diagonal. This contradiction proves that $t=2$, i.e., ϱ is binary.

Now form the relation $\sigma=\varrho\circ\varrho^{-1}\in[\varrho]$. Clearly, σ is reflexive $(\omega\subseteq\sigma)$, since the domain and range of ϱ equal $\mathbf{k}$. Hence σ is diagonal. Suppose $\sigma=\mathbf{k}^2$. We prove by induction on $2\leqq n\leqq k$ that for each n-element subset $\{a_0, \dots, a_{n-1}\}$ of $\mathbf{k}$ there is an element $b\in\mathbf{k}$ such that $(a_i, b)\in\varrho$ for all $0\leqq i<n$. By assumption this holds for $n=2$. Suppose it is true for some $2\leqq h<k$, and take the relation

$$\tau = \{(a_0, \dots, a_h)\colon (a_0, b), \dots, (a_h, b)\in\varrho \quad \text{for some} \quad b\in\mathbf{k}\}.$$

By construction $\tau\in[\varrho]$, and by the inductive assumption τ is totally reflexive. Thus τ is diagonal. Hence, in view of $h\geqq 2$, $\tau=\mathbf{k}^{h+1}$. This concludes the proof by induction. In particular, for $n=k$ we obtain that there is an element e such that $(a, e)\in\varrho$ for each $a\in\mathbf{k}$. Then $(e, e)\in\varrho$, contradicting the irreflexivity of ϱ.

Thus $\sigma=\varrho\circ\varrho^{-1}=\omega$. A similar argument for ϱ^{-1} yields $\varrho^{-1}\circ\varrho=\omega$, whence we get that $\varrho=f^{\bullet}$ for a permutation f of $\mathbf{k}$. Let m be the least positive integer with $f^m=\mathrm{id}$. It is easy to see that R contains all powers $(f^l)^{\bullet}=f^{\bullet}\circ\dots\circ f^{\bullet}$ $(0\leqq l<m)$ of $f^{\bullet}$. Thus from the assumptions that $f^{\bullet}\notin[s^{\bullet}]$ and every atomic relation in R belongs to $[s^{\bullet}]$, we get that the nonidentity powers of f are fixed point free, $m>p$ is a power of p, and $(f^{m/p})^{\bullet}\in[s^{\bullet}]$. Therefore some power g of $f^{m/p}$ has the required property $g^p=s$.

From now on we can assume that $R\setminus[s^{\bullet}]$ contains an atomic relation ϱ. Clearly, it suffices to prove the assertion of Theorem 2.5 for the relational algebra $[s^{\bullet}, \varrho]$. The various types of atomic relations will be considered separately.

Proposition 3.4. *For a nontrivial unary relation γ the relational algebra $[s^{\bullet}, \gamma]$ contains a nontrivial Θ-closed unary relation, or a nonempty unary relation transversal to s.*

Proof. Consider a nontrivial unary relation $\mu\in[s^{\bullet}, \gamma]$ of least possible size, and set $\mu_i=\mu\oplus i$ for $0<i<p$. Clearly, for each $0<i<p$ the relation $\mu\cap\mu_i$ belongs to $[s^{\bullet}, \gamma]$, and therefore, by the minimality, it is either μ or $\emptyset$. Since p is prime, μ is Θ-closed whenever $\mu\cap\mu_i=\mu$ for some $0<i<p$. If $\mu\cap\mu_1=\dots=\mu\cap\mu_{p-1}=\emptyset$, then μ is transversal to s.

Proposition 3.5. *For a bounded order $\leqq$ the relational algebra $[s^{\bullet}, \leqq]$ contains a nontrivial unary relation.*

Proof. Let o and e be the least and greatest elements of $\leqq$. Set $\gamma = \{a \in \mathbf{k}: a \leqq (a \oplus 1)\}$. Obviously, $o \in \gamma$ and $e \notin \gamma$, showing that $\gamma \in [s^{\bullet}, \leqq]$ is nontrivial.

Proposition 3.6. *Let f be a fixed point free permutation with $f^q = \mathrm{id}$ (q prime) such that $f^{\bullet} \notin [s^{\bullet}]$. Then the relational algebra $[s^{\bullet}, f^{\bullet}]$ contains either a nontrivial unary relation, or a binary relation $g^{\bullet}$ for some permutation g which together with s generates a semiregular representation of a group from Proposition 2.2.*

Proof. Denote by G the permutation group generated by s and f, and by S its subgroup generated by s. It is easy to see that the set of fixed points of each $g \in G$ belongs to $[s^{\bullet}, f^{\bullet}]$. Thus $[s^{\bullet}, f^{\bullet}]$ contains a nontrivial unary relation unless every nonidentity permutation from G is fixed point free. In that case consider a subgroup H of G properly containing S, and minimal with respect to this property. Clearly, S is a maximal subgroup of H. Thus H is a semiregular representation of a group $\mathfrak{H}$ which has a maximal subgroup of order p, i.e., $\mathfrak{H}$ is one of the groups listed in Proposition 2.2. It is easy to see that all relations $g^{\bullet} \in H^{\bullet} \setminus S^{\bullet} (\subseteq [s^{\bullet}, f^{\bullet}])$ meet the requirements.

When considering central relations and regular relations we will often use the following general result on Θ-closed relations:

Lemma 3.7. *Let Γ be a set of Θ-closed relations and let $\sigma \in [\Gamma]$. If the full relation is the single diagonal relation containing σ, then σ is Θ-closed.*

Proof. Consider a formula $\Phi(x_0, \ldots, x_{h-1})$ with bound variables $x_h, \ldots, x_{m-1}$ defining σ. Since there is no forcible repetition among the coordinates of σ, we may assume that the matrix of Φ contains no condition of the form $x_i = x_j$ (such conditions with at least one bound variable can be easily eliminated). Thus the matrix of Φ consists of conditions $(x_{i_0}, \ldots, x_{i_{t-1}}) \in \varrho$ with $\varrho \in \Gamma$. All $\varrho \in \Gamma$ being Θ-closed, this implies that σ is also Θ-closed.

Proposition 3.8. *For an at least binary central relation γ the relational algebra $[s^{\bullet}, \gamma]$ contains one of the following relations: a nontrivial unary relation, a nontrivial Θ-closed equivalence relation, a Θ-closed central relation, or a Θ-closed regular relation.*

Proof. Let σ be a central relation from $[s^{\bullet}, \gamma]$ of least possible arity, say h. If $h=1$, we are done. Now assume $h \geqq 2$, and define an $(h-1)$-ary relation $\sigma^* \in [s^{\bullet}, \gamma]$ to consist of all $(a_0, \ldots, a_{h-2}) \in \mathbf{k}^{h-1}$ such that $(a_0, \ldots, a_l, a_l \oplus i, a_{l+1}, \ldots, a_{h-2}) \in \sigma$ for all $0 \leqq i < p$ and $0 \leqq l < h-1$. It is easy to see that σ^* inherits total reflexivity and total symmetry from σ. Moreover, σ^* contains every $(h-1)$-tuple with one component in the center of σ. Since σ^* is not a central relation (by the choice of σ), we get that $\sigma^*=\mathbf{k}^{h-1}$.

Denote by τ the h-ary relation consisting of all $(a_0, \ldots, a_{h-1})\in\mathbf{k}^h$ with $(a_0\oplus i_0, \ldots, a_{h-1}\oplus i_{h-1})\in\sigma$ for all $0\leqq i_0, \ldots, i_{h-1}<p$. The construction of τ and the total symmetry of σ guarantee that τ is also totally symmetric. Using $\sigma^*=\mathbf{k}^{h-1}$ it is easy to show that τ is totally reflexive. Finally, τ is a subrelation of σ, and is clearly Θ-closed. Consequently τ is nontrivial, so $[\tau]$ contains an atomic relation. By Lemma 3.7 this atomic relation is Θ-closed, hence it is either an equivalence relation, or a central relation, or a regular relation.

We now turn to the most sophisticated case, when ϱ is a regular relation. Our departure point is

Lemma 3.9. *For an h-regular relation ϱ the relational algebra $[s^\cdot, \varrho]$ contains either a central relation or a regular relation σ such that $s\in\operatorname{Pol}\sigma$.*

Proof. Form the relation $\tau\in[s^\cdot, \varrho]$ consisting of all $(a_0, \ldots, a_{h-1})\in\mathbf{k}^h$ with $(a_0\oplus i, \ldots, a_{h-1}\oplus i)\in\varrho$ for every $0\leqq i<p$. Clearly, $\tau\subseteq\varrho$ is totally reflexive and symmetric. Thus, in view of $h\geqq 3$, the relation τ is nontrivial. Furthermore, obviously, $s\in\operatorname{Pol}\tau$. Now we can make use of the following fact which is implicit in [15], [12].

Claim. Let $l\geqq 2$, and let ξ be an l-ary nontrivial, totally reflexive, totally symmetric relation. Then all less than l-ary relations from $[\xi]$ are trivial, and $[\xi]$ contains a totally reflexive, totally symmetric atomic relation (types (E), (C), or (R)).

By this claim, $[\tau]$ contains a central or regular relation σ of arity at least h. Clearly, $s\in\operatorname{Pol}\tau\subseteq\operatorname{Pol}\sigma$.

In what follows, we need to consider only the regular relations ϱ for which $s\in\operatorname{Pol}\varrho$. Let $\varrho=\lambda_T$ where $T=\{\vartheta_0, \ldots, \vartheta_{m-1}\}$ is an h-regular family of equivalence relations. Denote $\vartheta_0\cap\ldots\cap\vartheta_{m-1}$ by ε_T. It is not hard to see (cf. [19]) that $s\in\operatorname{Pol}\varepsilon_T$, i.e., s maps each block of ε_T onto a block of ε_T.

Lemma 3.10. *Let T be an h-regular family of equivalence relations such that $s\in\operatorname{Pol}\lambda_T$. If $h\neq p$ or $\varepsilon_T\cap\Theta\neq\omega$, then the relational algebra $[s^\cdot, \lambda_T]$ contains a nontrivial Θ-closed equivalence relation, a Θ-closed central relation, or a Θ-closed regular relation.*

Proof. First we show that $h<p$ and $\varepsilon_T\cap\Theta=\omega$ cannot hold simultaneously. Indeed, since T is h-regular, ε_T has exactly h^m blocks. Furthermore, taking into account $\varepsilon_T\cap\Theta=\omega$ we obtain that for each block B of ε_T the blocks B, $s(B), \ldots, s^{p-1}(B)$ are pairwise distinct. Thus the prime p divides h^m, implying $h\geqq p$.

Let σ consist of all h-tuples $(a_0, \ldots, a_{h-1})$ such that $(a_0 \oplus i_0, \ldots, a_{h-1} \oplus i_{h-1}) \in \lambda_T$ for arbitrary $0 \leqq i_0, \ldots, i_{h-1} < p$. Clearly, σ is totally symmetric and Θ-closed. We prove that $\sigma \neq \emptyset$. First let $h > p$. For each block B of Θ, every h-tuple $(b_0, \ldots, b_{h-1}) \in B^h$ belongs to σ because each $(b_0 \oplus i_0, \ldots, b_{h-1} \oplus i_{h-1})$ is in B^h and, in view of $|B| = p < h$, contains a repetition. Now let $\varepsilon_T \cap \Theta \neq \omega$. Since $s \in \operatorname{Pol}\{\varepsilon_T, \Theta\}$, there exists a block B of Θ which is contained in a block of ε_T. Again every $(b_0, \ldots, b_{h-1}) \in B^h$ belongs to σ because $(b_0 \oplus i_0, b_1 \oplus i_1) \in B^2 \subseteq \varepsilon_T$ for all $0 \leqq i_0, i_1 < p$. Taking into account that σ is Θ-closed, totally symmetric, and $\emptyset \neq \sigma \subseteq \lambda_T$, we get that σ is nontrivial. By Lemma 3.7 the set $[\sigma]$ contains a Θ-closed atomic relation, which must be either an equivalence relation, or a central relation, or a regular relation.

In the remaining case of $h = p$ and $\varepsilon_T \cap \Theta = \omega$ we have:

Lemma 3.11. *Let T be a p-regular set of equivalence relations such that $s \in \operatorname{Pol} \lambda_T$ and $\varepsilon_T \cap \Theta = \omega$. Then the elements $\vartheta_0, \ldots, \vartheta_{m-1}$ of T and the blocks B^i_j of ϑ_i $(0 \leqq i < m,\ 0 \leqq j < p)$ can be indexed in such a way that for some integers $0 \leqq l \leqq m/p$ and $0 \leqq q \leqq m - lp$ the following holds:*

$$(2) \qquad s(B^i_j) = \begin{cases} B^{i\oplus 1}_j & \text{if } 0 \leqq i < lp \quad \text{and } 0 \leqq j < p, \\ B^i_j & \text{if } lp \leqq i < m-q \quad \text{and } 0 \leqq j < p, \\ B^i_{j\oplus 1} & \text{if } m-q \leqq i < m \quad \text{and } 0 \leqq j < p. \end{cases}$$

Proof. For the time being, denote by D^i_j $(0 \leqq j < p)$ the blocks of ϑ_i $(0 \leqq i < m)$. By the regularity of T, the blocks of ε_T are the sets $D_c = D^0_{c_0} \cap \ldots \cap D^{m-1}_{c_{m-1}}$ with $c = (c_0, \ldots, c_{m-1}) \in \mathbf{p}^m$. Since $s \in \operatorname{Pol} \varepsilon_T$, s induces a selfmap $\bar{s}$ of $\mathbf{p}^m$ by the equality $s(D_a) = D_{\bar{s}(a)}$, for every $a \in \mathbf{p}^m$. The fact $s \in \operatorname{Pol} \lambda_T$ implies that $\bar{s}$ is a *wreath function*, i.e., there are a permutation μ of $\mathbf{m}$ and permutations ν_i of $\mathbf{p}$ $(0 \leqq i < m)$ such that

$$\bar{s}(c_0, \ldots, c_{m-1}) = \big(\nu_0(c_{\mu(0)}), \ldots, \nu_{m-1}(c_{\mu(m-1)})\big)$$

for every $(c_0, \ldots, c_{m-1}) \in \mathbf{p}^m$ (see [20], [19]). Clearly, $s^p = \mathrm{id}$ implies $\bar{s}^p = \mathrm{id}$, and hence $\mu^p = \mathrm{id}$. Therefore we can assume without loss of generality that

$$\mu^{-1} = (0 \ldots p-1) \ldots \big((l-1)p \ldots lp-1\big)$$

for some $0 \leqq l \leqq m/p$. Thus, for every $0 \leqq i < lp$ and $0 \leqq j < p$, we have

$$(3) \qquad s(D^i_j) = s\big(\bigcup (D_a \colon a \in \mathbf{p}^m,\ a_i = j)\big) = \bigcup (D_{\bar{s}(a)} \colon a \in \mathbf{p}^m,\ a_i = j) = D^{i\oplus 1}_{\nu_{i\oplus 1}(j)},$$

and, similarly, for every $lp \leqq i < m$ and $0 \leqq j < p$,

$$(4) \qquad s(D^i_j) = D^i_{\nu_i(j)}.$$

Consequently, for $n=0, 1, \ldots$ we get

$$s^n(D^i_j) = \begin{cases} D^{i\oplus n}_{v_{i\oplus n}\cdots v_{i\oplus 1}(j)} & \text{if} \quad 0 \leqq i < lp, \quad 0 \leqq j < p, \\ D^i_{v_i^n(j)} & \text{if} \quad lp \leqq i < m, \quad 0 \leqq j < p. \end{cases}$$

The condition $s^p = \mathrm{id}$ obviously implies that

$$v_{pt+p-1} \cdots v_{pt} = \mathrm{id} \quad \text{for every} \quad 0 \leqq t < l, \tag{5}$$

and $v_i^p = \mathrm{id}$ for every $lp \leqq i < m$. In the latter case, obviously, v_i is either a p-cycle, or the identity. Suppose the p-cycles are exactly $v_{m-q}, \ldots, v_{m-1}$ $(0 \leqq q \leqq m-lp)$. Now set $B^{pt\oplus n}_j = D^{pt\oplus n}_{v_{pt\oplus n}\cdots v_{pt\oplus 1}(j)}$ for every $0 \leqq t < l$, $0 \leqq n < p$, $0 \leqq j < p$, $B^i_j = D^i_j$ for every $lp \leqq i < m-q$, $0 \leqq j < p$, and $B^i_j = D^{i}_{v_i^j(0)}$ for every $m-q \leqq i < m$, $0 \leqq j < p$. Using (3)—(5) it is not hard to check that (2) holds.

Lemma 3.12. *Let T be a p-regular set of equivalence relations such that $s \in \mathrm{Pol}\,\lambda_T$, $\varepsilon_T \cap \Theta = \omega$, and (2) is satisfied. Then the equivalence relation $\bigcap(\vartheta_i\colon m-q \leqq i < m)$ is transversal to Θ, and belongs to $[s^\cdot, \lambda_T]$.*

Proof. Observe first that $q \geqq 1$. Indeed, $q=0$ would imply by (2) that $s(B)=B$ holds for the block $B = B^0_0 \cap \ldots \cap B^{m-1}_0$ of ε_T, which is impossible, because $\varepsilon_T \cap \Theta = \omega$ and s has no fixed point. Setting $\vartheta = \bigcap(\vartheta_i\colon m-q \leqq i < m)$ we get from (2) that $s \in \mathrm{Pol}\,\vartheta$, and for every block D of ϑ, $s(D) \cap D = \emptyset$. Thus ϑ is transversal to Θ.

Let σ denote the binary relation consisting of all $(a, b) \in \mathbf{k}^2$ such that

$$(a, a\oplus 1, \ldots, a\oplus(i-1),\ b,\ a\oplus(i+1), \ldots, a\oplus(p-1)) \in \lambda_T \tag{6}$$

for every $0 < i < p$. Clearly, $\sigma \in [s^\cdot, \lambda_T]$. We show that $\sigma^{-1} \circ \sigma = \vartheta$. First consider $(a, b) \in \mathbf{k}^2 \setminus \vartheta$, say, $a \in B^t_u$, $b \in B^t_v$ for some $m-q \leqq t < m$ and $0 \leqq u < v < p$. Then for $i = v-u$ the components of the p-tuple (6) belong to the blocks $B^t_u, B^t_{u\oplus 1}, \ldots$ $\ldots, B^t_{u\oplus(p-1)}$, respectively, hence (6) does not hold, so $(a, b) \notin \sigma$. Thus $\sigma \subseteqq \vartheta$, and hence $\sigma^{-1} \circ \sigma \subseteqq \vartheta$. Conversely, let $b \in B^0_{j_0} \cap \ldots \cap B^{m-1}_{j_{m-1}}$ and $a \in B^0_0 \cap \ldots \cap B^{m-q-1}_0 \cap$ $\cap B^{m-q}_{j_{m-q}} \cap \ldots \cap B^{m-1}_{j_{m-1}}$ for some $0 \leqq j_0, \ldots, j_{m-1} < p$. Then, by (2), $a \oplus i \in B^0_0 \cap \ldots \cap$ $\cap B^{m-q-1}_0$ for every $0 \leqq i < p$. In view of $p \geqq 3$ this shows that (6) holds for $0 < i < p$, i.e., $(a, b) \in \sigma$. Hence $\sigma^{-1} \circ \sigma \supseteqq \vartheta$, completing the proof.

The equivalence relation $\bigcap(\vartheta_i\colon m-q \leqq i < m)$ is trivial if and only if $q = m$ and $\varepsilon_T = \omega$. This case is considered below.

Lemma 3.13. *Let T be a p-regular set of equivalence relations such that $\varepsilon_T = \omega$, $s \in \mathrm{Pol}\,\lambda_T$, and (2) holds with $q = m$. Then $[s^\cdot, \lambda_T]$ contains an affine relation determined by an elementary abelian p-group $(\mathbf{k}; +)$ such that there exists an element $c \in \mathbf{k}$ with $s(x) = x + c$ for every $x \in \mathbf{k}$.*

Proof. In view of $\varepsilon_T=\omega$ the mapping assigning to every $(j_0, \ldots, j_{m-1})\in \mathbf{p}^m$ the (unique) element of $B^0_{j_0}\cap\ldots\cap B^{m-1}_{j_{m-1}}$ is a bijection between $\mathbf{p}^m$ and $\mathbf{k}$. We may identify $\mathbf{k}$ and $\mathbf{p}^m$ via this bijection. The set of equivalence relations corresponding to $T=\{\vartheta_0, \ldots, \vartheta_{m-1}\}$ is $Z=\{\zeta_0, \ldots, \zeta_{m-1}\}$ where ζ_i is defined by $(a, b)\in\zeta_i$ iff the i-th components of a and b coincide $(a, b\in\mathbf{p}^m;\ 0\leqq i<m)$. Furthermore, since (2) holds with $q=m$, the permutation t of $\mathbf{p}^m$ induced by s can be expressed as follows: $t(x)=x\oplus\bar{1}$ for every $x\in\mathbf{p}^m$, where $\bar{1}=(1, \ldots, 1)$, and $\oplus$ denotes the componentwise addition modulo p. We want to show that the affine relation α determined by $(\mathbf{p}^m; \oplus)$ belongs to $[t^\bullet, \lambda_Z]$, or, equivalently, $\mathrm{Pol}\,\{t^\bullet, \lambda_Z\}\subseteq \mathrm{Pol}\,\alpha$.

Consider an n-ary operation $f\in\mathrm{Pol}\,\{t^\bullet, \lambda_Z\}$. To f we associate the following m-tuple $(f_0, \ldots, f_{m-1})$ of nm-ary operations on $\mathbf{p}$: for $x=(x_0, \ldots, x_{n-1})\in(\mathbf{p}^m)^n$, $x_j=(x_{j0}, \ldots, x_{j,m-1})$ $(j=0, \ldots, n-1)$, and $\tilde{x}=(x_{00}, \ldots, x_{0,m-1}, \ldots, x_{n-1,0}, \ldots, x_{n-1,m-1})$ set $f(x)=(f_0(\tilde{x}), \ldots, f_{m-1}(\tilde{x}))$. The operations f_i are surjective on account of $f\in\mathrm{Pol}\,t^\bullet$. The condition $f\in\mathrm{Pol}\,\lambda_Z$ translates into $f_i\in\mathrm{Pol}\,\varkappa$ for all $0\leqq i<m$, where $\varkappa$ denotes the relation on $\mathbf{p}$ consisting of all p-tuples with at least one repetition. Taking into account the well-known fact [11, 2.2.4] that $\mathrm{Pol}\,\varkappa$ consists of all non-surjective or essentially unary operations, we infer that every f_i is essentially unary, i.e., $f_i(\tilde{x})=g_i(x_{u_i, v_i})$ for some $0\leqq u_i<n$, $0\leqq v_i<m$, and some selfmap g_i of $\mathbf{p}$ $(0\leqq i<m)$. In view of $f\in\mathrm{Pol}\,t^\bullet$ we have $g_i(y\oplus 1)=g_i(y)\oplus 1$ for all $y\in\mathbf{p}$ and $0\leqq i<m$, hence there exist $a_i\in\mathbf{p}$ such that $g_i(y)=y\oplus a_i$ for all $y\in\mathbf{p}$. Now it is easy to verify that $f\in\mathrm{Pol}\,\alpha$.

Remarks. 1. The clone $\mathrm{Pol}\,\{t^\bullet, \lambda_Z\}$ consists of the operations

$$a\oplus x_0E_0\oplus\ldots\oplus x_{n-1}E_{n-1}$$

where $a\in\mathbf{p}^m$ and $E_l=(E_l(i,j))$ $(0\leqq l<n)$ are $m\times m$ matrices over $\mathbf{p}$ with all entries 0 or 1 such that each column contains at most one 1, and for every $0\leqq j<m$ there exists exactly one E_l $(0\leqq l<n)$ whose j-th column has a component 1. Indeed, it is straightforward to check that these operations do belong to $\mathrm{Pol}\,\{t^\bullet, \lambda_Z\}$. On the other hand, our argument in the proof of Lemma 3.13 shows that every $f\in\mathrm{Pol}\,\{t^\bullet, \lambda_Z\}$ has the required form with $a=(a_0, \ldots, a_{m-1})$ and the matrices $E_0, \ldots, E_{n-1}$ defined by $E_l(v_j, j)=1$ if $u_j=l$ and $E_l(i,j)=0$ otherwise. For comparison we note that the clone $\mathrm{Pol}\,\{t^\bullet, \alpha\}$ consists of the operations $a\oplus x_0A_0\oplus$ $\oplus\ldots\oplus x_{n-1}A_{n-1}$ where $a\in\mathbf{p}^m$ and A_l $(0\leqq l<n)$ are $m\times m$ matrices over $\mathbf{p}$ satisfying $\bar{1}A_0\oplus\ldots\oplus\bar{1}A_{n-1}=\bar{1}$.

2. An interesting feature of the proof of Lemma 3.13 is that $\alpha\in[t^\bullet, \lambda_Z]$ is shown by means of operations. There seems to be no easy way to construct α from $t^\bullet$ and λ_Z.

Summarizing Lemmas 3.9 through 3.13 we get:

Proposition 3.14. *For a regular relation λ_T the relational algebra $[s^{\bullet}, \lambda_T]$ contains one of the following relations: a central relation, a Θ-closed regular relation, a nontrivial equivalence relation which is either Θ-closed or transversal to s, or an affine relation determined by an elementary abelian p-group* $(\mathbf{k}; +)$ *such that there exists an element $c \in \mathbf{k}$ with $s(x) = x + c$ for all $x \in \mathbf{k}$.*

Proposition 3.15. *For a nontrivial equivalence relation ε the relational algebra $[s^{\bullet}, \varepsilon]$ contains either a central relation, or a regular relation, or a nontrivial equivalence relation which is Θ-closed or transversal to s.*

Proof. Let σ be a maximal nontrivial equivalence relation in $[s^{\bullet}, \varepsilon]$, and put $\sigma_i = \{a\colon (a, a \oplus i) \in \sigma\}$ for $0 < i < p$. Clearly, all σ_i belong to $[s^{\bullet}, \sigma] \subseteq [s^{\bullet}, \varepsilon]$, therefore if $\emptyset \neq \sigma_i \subset \mathbf{k}$ for some $0 < i < p$, we are done. The equality $\sigma_i = \mathbf{k}$ for some $0 < i < p$ implies that σ is Θ-closed. Thus it remains to consider the case when $\sigma_i = \emptyset$ for all $0 < i < p$, i.e., $\sigma \cap \Theta = \omega$. Denote by τ the binary relation consisting of all $(a, b) \in \mathbf{k}^2$ such that (a, c), $(c \ominus 1, b \ominus 1)$, $(a \ominus 1, d \ominus 1,)$ $(d, b) \in \sigma$ for some $c, d \in \mathbf{k}$. Clearly, $\tau \in [s^{\bullet}, \sigma]$ is symmetric, and $\sigma \subseteq \tau$ (set $c = b$, $d = a$). Furthermore, $\tau \cap s^{\bullet} = \emptyset$, since $(a, a \oplus 1) \in \tau$ for some $a \in \mathbf{k}$ would imply the existence of an element $c \in \mathbf{k}$ with $(a, c) \in \sigma$ and $(c \ominus 1, a) \in \sigma$, yielding $(c \ominus 1, c) \in \sigma$ in contradiction to $\sigma \cap \Theta = \omega$. By the claim formulated in the proof of Lemma 3.9 the relational algebra $[\tau]$ contains a nontrivial equivalence relation, a nonunary central relation, or a regular relation. In the latter two cases, the claim of the proposition follows, so suppose $[\tau]$ contains a nontrivial equivalence relation λ. It is easy to show that every binary relation in $[\tau]$ distinct from ω contains τ. Hence $\sigma \subseteq \tau \subseteq \lambda$. Taking into account $\sigma, \lambda \in [s^{\bullet}, \varepsilon]$ and the maximality of σ we get that $\sigma = \lambda = \tau$. Thus for arbitrary $(a, b) \in \sigma$ we have $(a \oplus 1, b \oplus 1) \in \tau = \sigma$ (choose $c = a \oplus 1$, $d = b \oplus 1$), which proves that $s \in \operatorname{Pol} \sigma$, i.e., σ is transversal to s.

Proposition 3.16. *Let α be an affine relation determined by an elementary abelian p-group* $(\mathbf{k}; +)$ *(p prime). Then either there exists an element $c \in \mathbf{k}$ such that $s(x) = x + c$ for all $x \in \mathbf{k}$, or the relational algebra $[s^{\bullet}, \alpha]$ contains a nontrivial equivalence relation.*

Proof. Assume there is no $c \in \mathbf{k}$ with $s(x) = x + c$ for all $x \in \mathbf{k}$, that is, the set $U = \{(a \oplus 1) - a\colon a \in \mathbf{k}\}$ contains at least two elements. Note also that $0 \notin U$ as s is fixed point free. Hence $2 \leqq |U| \leqq k - 1$. Let ν denote the binary relation consisting of all $(a, b) \in \mathbf{k}^2$ such that $(a \oplus 1) - a + b = (b \oplus 1)$, or equivalently, $(a \oplus 1) - - a = (b \oplus 1) - b$. Then $\nu \in [s^{\bullet}, \alpha]$ and ν is an equivalence relation with $|U|$ blocks.

4. Proof of Theorem 2.6

In order to show that for every relation ϱ listed in Theorem 2.3 the relational algebra $[s^\bullet, \varrho]$ indeed covers $[s^\bullet]$, we apply a more or less standard method, the main point being an explicit description of the members of $[s^\bullet, \varrho]$. With this at hand, it is already not hard to show that there is no relational algebra strictly between $[s^\bullet]$ and $[s^\bullet, \varrho]$. In fact, we can accomplish a bit more than that: we determine all relational subalgebras of $[s^\bullet, \varrho]$ (or, equivalently, all clones containing Pol $\{s^\bullet, \varrho\}$).

Making use of Lemma 3.2, type (P_s) is easy to settle.

Proposition 4.1. *Let g be a permutation such that $\{s, g\}$ generates a semiregular representation of a group $\mathfrak{G}$ from Proposition 3.1 on* $\mathbf{k}$. *Then the lattice of relational subalgebras of $[s^\bullet, g^\bullet]$ is isomorphic to the lattice of subgroups of $\mathfrak{G}$.*

Proof. Let G denote the permutation group generated by $\{s, g\}$. It is clear that $[s^\bullet, g^\bullet]=[G^\bullet]$. Furthermore, in view of Lemma 3.2, every relational subalgebra of $[G^\bullet]$ is of the form $[H^\bullet]$ for some subgroup H of G. It follows also that $[H_1^\bullet]\neq[H_2^\bullet]$ for distinct subgroups H_1, H_2 of G.

To describe the relations in $[s^\bullet, \varrho]$ for the remaining four types (A_s)—(R_s), too, we proceed, as in the proof of Lemma 3.2, according to the following scheme: we consider a formula

$$\Phi(x_0, \ldots, x_{h-1}) = \exists x_h \ldots \exists x_{m-1} \Gamma(x_0, \ldots, x_{m-1}) \tag{7}$$

determining a nonempty relation $\sigma\in[s^\bullet, \varrho]$, and then, utilizing the special properties of ϱ, we bring the matrix Γ of Φ to "canonical form", yielding the required description.

Proposition 4.2. *Let α be an affine relation determined by an elementary abelian p-group $(\mathbf{k}; +)$ such that there exists an element $c\in\mathbf{k}$ with $s(x)=x+c$ for all $x\in\mathbf{k}$. Then the relational subalgebras of $[s^\bullet, \alpha]$ form a 4-element Boolean lattice consisting of $[s^\bullet, \alpha]$, $[s^\bullet]$, $[\alpha]$, and $[\omega]$.*

Proof. Denote by L the set of operations $\beta_0 x_0+\ldots+\beta_{l-1}x_{l-1}+\beta_l c$ on $\mathbf{k}$ where $0\leqq\beta_i<p$ $(0\leqq i\leqq l)$ and $\beta_0\oplus\ldots\oplus\beta_{l-1}=1$, and consider a formula (7) determining a nonempty relation $\sigma\in[s^\bullet,\alpha]$. Since s and $x-y+z$ both belong to L, Γ is a solvable system of linear equations $x_j=f(x_{i_0}, \ldots, x_{i_{l-1}})$ with $l\geqq 1, 0\leqq j, i_0, \ldots \ldots, i_{l-1}<m$, and $f\in L$. It is easy to see that every step of the usual elimination process yields equations of this form. Thus we can first eliminate all variables $x_h, \ldots, x_{m-1}$, and then further elimination can express certain unknowns, say $x_t, \ldots, x_{h-1}$, as linear functions (from L) of independent variables. Thus

$$\sigma = \{(a_0, \ldots, a_{t-1}, f_t(a_0, \ldots, a_{t-1}), \ldots, f_{h-1}(a_0, \ldots, a_{t-1}))\colon a_0, \ldots, a_{t-1}\in\mathbf{k}\}$$

with $f_t, \ldots, f_{h-1}\in L$.

Clearly, $[\sigma]=[f_t^\cdot, \ldots, f_{h-1}^\cdot]$. Now recall the well-known and easy fact (cf. [21], [1]) that L has exactly four subclones, namely, the clone of projections, the two clones generated by $x+c$, resp., $x-y+z$, and L itself, which is generated by $\{x+c, x-y+z\}$. Thus, for every $f\in L$ we have $[f^\cdot]=[\omega], [s^\cdot], [\alpha]$, or $[s^\cdot, \alpha]$, completing the proof.

For the rest of the proof it will be convenient to split Γ into a conjunction $\Gamma=\Gamma_1\wedge\Gamma_2$ such that Γ_1 collects all conditions involving ϱ and Γ_2 all conditions involving $=$ or $s^\cdot$. We will denote by $\sim$ the least equivalence relation on $\{x_0, \ldots, x_{m-1}\}$ such that $x_i \sim x_j$ whenever $x_i=x_j$ or $(x_i, x_j)\in s^\cdot$ appears in Γ_2, and $X_0, \ldots, X_{t-1}$ will denote the blocks of $\sim$. It is clear that if we fix one element x_{r_i} in each block X_i $(i=0, \ldots, t-1)$, then for arbitrary variable x_j $(0\leqq j<m)$ with $x_j \sim x_{r_i}$ there is an integer $0\leqq c_j<p$ such that Γ_2 implies

$$(x_{r_i}, x_j)\in(s^{c_j})^\cdot. \tag{8}$$

Since Φ determines a nonempty relation, and $(s^c)^\cdot\cap(s^d)^\cdot=\emptyset$ for all $0\leqq c<d<p$, the exponents c_j in (8) are uniquely determined. Thus Γ_2 is equivalent to the conjunction of the formulas (8) for all $0\leqq i<t$ and $x_j\sim x_{r_i}$, which will be denoted by Γ_2^*. Hence we can as well assume that Φ is given in the form

$$\Phi(x_0, \ldots, x_{h-1}) = \exists x_h \ldots \exists x_{m-1}(\Gamma_1(x_0, \ldots, x_{m-1})\wedge\Gamma_2^*(x_0, \ldots, x_{m-1})). \tag{7'}$$

The Θ-closed relations can be treated together.

Proposition 4.3. *Let ϱ be a Θ-closed equivalence relation, central relation, or regular relation. If ϱ is unary, then the relational subalgebras of* $[s^\cdot, \varrho]$ *are* $[s^\cdot, \varrho]$, $[s^\cdot]$, $[s^\cdot\cap(\varrho\times\varrho)]$, $[\varrho]$, $[\omega]$, *and hence they form a lattice isomorphic to* $\mathfrak{N}_5$. *Otherwise the relational subalgebras of* $[s^\cdot, \varrho]$ *form a 4-element Boolean lattice consisting of* $[s^\cdot, \varrho]$, $[s^\cdot]$, $[\varrho]$, *and* $[\omega]$.

Proof. Consider a formula (7′) determining a nonempty relation $\sigma\in[s^\cdot, \varrho]$. Select the variables $x_{r_i}\in X_i$ $(i=0, \ldots, t-1)$ so that x_{r_i} is free whenever X_i contains a free variable. We can assume without loss of generality that $X_0, \ldots, X_{q-1}$ $(q\leqq t, h)$ are exactly the blocks containing free variables, and $x_{r_i}=x_i$ $(i=0, \ldots, q-1)$. Since ϱ is Θ-closed, every condition $(x_{i_0}, \ldots, x_{i_{l-1}})\in\varrho$ in Γ_1 can be replaced by $(x_{j_0}, \ldots, x_{j_{l-1}})\in\varrho$ where $\{x_{j_0}, \ldots, x_{j_{l-1}}\}\subseteq\{x_{r_0}, \ldots, x_{r_{t-1}}\}$ and $x_{i_0}\sim x_{j_0}, \ldots, x_{i_{l-1}}\sim$ $\sim x_{j_{l-1}}$. Clearly, $\exists x_{r_q}\ldots\exists x_{r_{t-1}}\Gamma_1(x_{r_0}, \ldots, x_{r_{t-1}})$ determines a relation $\tau\in[\varrho]$, and σ is of the form

$$\sigma = \{(a_0, \ldots, a_{q-1}, a_{l_q}\oplus c_q, \ldots, a_{l_{h-1}}\oplus c_{h-1})\colon (a_0, \ldots, a_{q-1})\in\tau\}$$

with $0\leqq l_i<q$ for all $q\leqq i<h$.

Since the relation ϱ is atomic, either τ is trivial, or $[\tau]=[\varrho]$. Thus an easy argument shows that one of the following holds provided ϱ is at least binary: $[\sigma]=$

$=[s^{\bullet}, \varrho], [s^{\bullet}], [\varrho]$, or $[\omega]$. If ϱ is unary, we have one more possibility, namely $[\sigma]=$ $=[s^{\bullet}\cap(\varrho\times\varrho)]$.

It remains to consider equivalence relations and unary relations transversal to s.

Proposition 4.4. *Let ε be a nontrivial equivalence relation transversal to s. Then the relational subalgebras of $[s^{\bullet}, \varepsilon]$ are $[s^{\bullet}, \varepsilon]$, $[s^{\bullet}]$, $[\varepsilon\circ s^{\bullet}]$, $[\varepsilon]$, $[\omega]$, and hence they form a lattice isomorphic to $\mathfrak{N}_5$.*

Proof. Take a formula (7′) determining a nonempty relation $\sigma\in[s^{\bullet}, \varepsilon]$, and define a graph $\mathbf{G}$ on the vertices $0, \ldots, t-1$ as follows: (u, v) is an edge of $\mathbf{G}$ if and only if there are $x\in X_u$ and $y\in X_v$ such that $(x, y)\in\varepsilon$ appears in Γ_1. Since ε is symmetric, $\mathbf{G}$ is undirected. In view of $s\in\operatorname{Pol}\varepsilon$, any condition $(x, y)\in\varepsilon$ in Γ_1 can be replaced by $(x', y')\in\varepsilon$ provided $(x, x')\in(s^i)^{\bullet}$ and $(y, y')\in(s^i)^{\bullet}$ are in Γ_2^* for some $0\leqq i<p$. Thus each vertex l of $\mathbf{G}$ can be labelled by a variable $x_{r_l}\in X_l$ in such a way that

$$\Gamma_1^* = \wedge\{(x_{r_i}, x_{r_j})\in\varepsilon\colon\ (i, j) \text{ is an edge of } \mathbf{G}\}$$

is equivalent to Γ_1. In fact, the labelling can proceed along the paths of $\mathbf{G}$. Circles (loops, multiple edges) do not cause the procedure to fail, because ε is symmetric, transitive, $\varepsilon\cap(s^i)^{\bullet}=\emptyset$ for every $0<i<p$, and by assumption, $\sigma\neq\emptyset$. Clearly, $\Gamma_1^*(x_0, \ldots, x_{m-1})$ determines a relation from $[\varepsilon]$, and hence σ is of the form $\sigma=\tau\oplus c$ with $\tau\in[\varepsilon]$ and $c\in\mathbf{p}^h$.

It is well known and easy to check that, up to the order of its components, τ is a direct product $\tau_0\times\ldots\times\tau_{q-1}$ where each τ_i arises from a relation

$$\{(a_0, \ldots, a_{k-1})\colon\ a_0\varepsilon a_1\varepsilon\ldots\varepsilon a_{k-1}\}$$

by repeating some components. Correspondingly, $\sigma=\sigma_0\times\ldots\times\sigma_{q-1}$ and every binary projection of each σ_i is equal to some $(s^c)^{\bullet}$ or some $(s^{-c})^{\bullet}\circ\varepsilon\circ(s^d)^{\bullet}$ $(0\leqq c, d<p)$. However, taking into account $s\in\operatorname{Pol}\varepsilon$ we get $\varepsilon=(s^{-c})^{\bullet}\circ\varepsilon\circ(s^c)^{\bullet}$ for all $0\leqq c<p$. Consequently, introducing the notation $\varepsilon_j=\varepsilon\circ(s^j)^{\bullet}$ $(0\leqq j<p)$ we have $(s^{-c})^{\bullet}\circ\varepsilon\circ$ $\circ(s^d)^{\bullet}=\varepsilon_{d\ominus c}$ and $\varepsilon_c\circ\varepsilon_d=\varepsilon_{c\oplus d}$ for all $0\leqq c, d<p$. This implies that $\varepsilon\in[\varepsilon_1]=[\varepsilon_2]=\ldots$ $\ldots=[\varepsilon_{p-1}]$. Hence a quick analysis of the various possibilities yields that $[\sigma]=$ $=[s^{\bullet}, \varepsilon](=[s^{\bullet}, \varepsilon_1])$, $[s^{\bullet}]$, $[\varepsilon_1]$, $[\varepsilon]$, or $[\omega]$, completing the proof.

A symmetric, transitive, binary relation will be called a *partial equivalence* (equivalence relation on a subset of the base set). The empty set is also considered a partial equivalence. The lattice of partial equivalences of $\mathbf{p}$ will be denoted by $\mathfrak{Q}_p$, and $\mathfrak{Q}_p^*$ will stand for the lattice arising from $\mathfrak{Q}_p$ by adding a new greatest element, and another element which is comparable only with the least element of $\mathfrak{Q}_p$.

Proposition 4.5. *Let μ be a nontrivial unary relation transversal to s, and let $\mu_i=\mu\oplus i$ $(0\leqq i<p)$, $\pi_{ij}=(\mu_i\times\mu_j)\cap(s^{j\ominus i})^{\bullet}$ $(0\leqq i,j<p)$. Then the relational subalgebras of $[s^{\bullet},\mu]$ are $[s^{\bullet},\mu]$, $[s^{\bullet}]$, and $[\pi_{ij}\colon (i,j)\in\xi]$ with $\xi\in\mathfrak{Q}_p$, and hence they form a lattice isomorphic to $\mathfrak{Q}_p^*$.*

Proof. A similar but simpler argument than in the previous proof yields again that every nonempty relation $\sigma\in[s^{\bullet},\mu]$ is of the form $\sigma=\tau\oplus c$ for some $\tau\in[\mu]$ and $c\in\mathbf{p}^h$. The well-known description of the relations in $[\mu]$ (see, e.g., [11, 2.2.2]) implies that σ is a direct product of relations of the form

$$\sigma'=\{(a\oplus c_0,\ldots,a\oplus c_{t-1})\colon a\in\nu\}$$

where $\nu=\mu$ or $\nu=\mathbf{k}$. Clearly, if $\nu=\mathbf{k}$, then $[\sigma']\subseteq[s^{\bullet}]$. If $\nu=\mu$, then $[\sigma']=$ $=[\pi_{c_0,c_l}\colon 0\leqq l<t]$. Taking into account that $[s^{\bullet},\pi_{ij}]=[s^{\bullet},\mu]$ for all $0\leqq i,j<p$, we get that there are the following three possibilities for a set of relations $R\subseteq[s^{\bullet},\mu]$:

(a) $[R]=[s^{\bullet},\mu]$,

(b) $[R]=[s^{\bullet}]$,

(c) $[R]=[\Pi]$ for some $\Pi\subseteq\{\pi_{ij}\colon 0\leqq i,j<p\}$.

In the last case it is easy to see that $[R]=[\pi_{mn}\colon (m,n)\in\xi]$ where ξ is the least partial equivalence on $\mathbf{p}$ such that $(i,j)\in\xi$ provided $\pi_{ij}\in\Pi$. The straightforward proof of the fact that the relational algebras listed in the proposition are indeed pairwise distinct is left to the reader.

Remarks. 1. The results above show that two clones $\mathrm{Pol}\,\{s^{\bullet},\varrho\}$ and $\mathrm{Pol}\,\{s^{\bullet},\varrho'\}$ where $\varrho\neq\varrho'$ are from the list in Theorem 2.3 coincide if and only if either both of ϱ and ϱ' are of type (P_s) such that the corresponding permutations and s generate the same permutation group, or both of ϱ and ϱ' are unary relations transversal to s such that $\varrho'=s^i(\varrho)$ for some $0<i<p$. This can be verified by comparing the sets of maximal clones containing $\mathrm{Pol}\,\{s^{\bullet},\varrho\}$, resp., $\mathrm{Pol}\,\{s^{\bullet},\varrho'\}$. (Apply Propositions 4.1—4.5 to determine the maximal clones, and make use of the well-known fact [16], [11, 4.3.23] that among the maximal clones, too, there are only some trivial coincidences.)

2. A similar argument shows also that for an atomic relation σ the clone $\mathrm{Pol}\,\{s^{\bullet},\sigma\}$ is maximal in $\mathrm{Pol}\,s^{\bullet}$ if and only if σ falls into one of the types (P_s)—(R_s) in Theorem 2.3.

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On E-disjunctive inverse semigroups

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Throughout the paper, we have the following notations. Let $\Omega=\{\xi, \eta, \zeta, ...\}$ and S be inverse semigroups. Let $E(\Omega)=\{\alpha, \beta, \gamma, \delta, ...\}$ be the set of all idempotents of Ω. A congruence ϱ on S is said to be *idempotent-determined* or *I. D.* if $e \varrho b$ and $e \in E(S)$ imply $b \in E(S)$. Let μ $[\tau]$ be the greatest idempotent-separating [I. D.] congruence on S. Let $N=\bigcup\{G_\gamma: \gamma \in Y\}$ be a Clifford semigroup, that is, a semilattice of groups. S is said to be *E-disjunctive* if $\tau=\iota$. All other definitions and notations follow the conventions of [1].

Green [2] proved that if S is inverse, then $\mu \cap \tau=\iota$. It is easily shown that S/τ is E-disjunctive. Hence every inverse semigroup is isomorphic with a subdirect product of a fundamental inverse semigroup S/μ and an E-disjunctive inverse semigroup S/τ. We shall discuss E-disjunctive inverse semigroups by using [2, 4]. We have immediately

Lemma 1. *Let Ω be a full inverse subsemigroup of an inverse semigroup U. If Ω is E-disjunctive, then U is E-disjunctive.*

An inverse semigroup $S=S[N, \Omega]$ is called a *regular extension* of N by Ω if $N \subseteqq S$ and there exists a homomorphism $\varkappa$ of S onto Ω such that $\bigcup\{(\gamma\varkappa^{-1}): \gamma \in E(\Omega)\}$ is the decomposition of N induced by the finest semilattice congruence on N [4].

Let End(N) be the set of all endomorphisms of N. Fix c in N. Define $\bar{c}$: $N \to N$ by $u \mapsto cuc^{-1}$. Then $\bar{c}$ is the inner endomorphism induced by c. Let Ω, $Y=E(\Omega)$ and N be the same as above and let 1_γ be the identity element of the group G_γ. For each ξ in Ω, define $\bar{\xi} \in$ End(N) such that (i) $\bar{\xi}$ is the inner endomorphism $\bar{1}_\gamma$ if $\xi=\gamma \in Y$, and (ii) $\bar{\xi}$ maps G_γ into $G_{\xi\gamma\xi^{-1}}$ $(\gamma \in Y)$, in particular, it maps $G_{\xi^{-1}\xi}$ onto $G_{\xi\xi^{-1}}$.

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For each pair ξ, η in Ω, define $C(\xi, \eta) \in G_{\xi\eta(\xi\eta)^{-1}}$ satisfying

(1) $$C(\xi\xi^{-1}, \xi) = 1_{\xi\xi^{-1}} = C(\xi, \xi^{-1}\xi) \text{ for all } \xi \text{ in } \Omega, \text{ and } C(\gamma, \delta) = 1_{\gamma\delta}$$
$$\text{for all } \gamma, \delta \text{ in } Y,$$

(2) $$C(\eta, \zeta)^{\bar{\xi}} C(\xi, \eta\zeta) = C(\xi, \eta) C(\xi\eta, \zeta), \quad \text{where} \quad u^{\bar{\xi}} = u\bar{\xi},$$

(3) $$\overline{\xi\eta} = \overline{\eta}\,\overline{\xi}\,\overline{C(\eta, \xi)}.$$

Then $\{\bar{\xi}, C(\xi, \eta)\colon \xi, \eta \in \Omega\}$ is called a *factor set of N belonging to Ω*. Define a product in $N * \Omega = \{(a, \xi)\colon a \in G_{\xi\xi^{-1}}, \xi \in \Omega\}$ by $(a, \xi)(b, \eta) = (ab^{\bar{\xi}} C(\xi, \eta), \xi\eta)$. Then $N * \Omega$ is a regular extension of N by Ω. And the converse is valid [4]. We identify $S[N, \Omega]$ with $N * \Omega$. We obtain the following theorem immediately.

Theorem 2. *Let $S = S[N, \Omega]$ be a regular extension. Then S is E-disjunctive if and only if there exists no pair γ, δ in $E(\Omega)$ with $\delta < \gamma$ satisfying*

$$\{(y, \eta)\colon \xi\gamma\eta \in E(\Omega), \quad y^{\overline{\xi\gamma}} = [C(\xi\gamma, \eta)xC(\xi, \gamma)]^{-1}\} =$$
$$= \{(y, \eta)\colon \xi\delta\eta \in E(\Omega), \, y^{\overline{\xi\delta}} = [C(\xi\delta, \eta)xC(\xi, \delta)]^{-1}\} \quad \textit{for all} \quad (x, \xi) \in S.$$

Let $S = S[N, \Omega]$. S is called an *I-regular extension* if $C(\xi, \eta) = 1_{\xi\eta(\xi\eta)^{-1}}$ for all $\xi, \eta \in \Omega$. Let N be a Clifford semigroup with linking homomorphisms $\{\psi_{\gamma,\delta}\colon \delta < \gamma\}$. Then N is called a *D-Clifford semigroup* if there is no pair δ, γ, $\delta < \gamma$, in Y such that $\psi_{\gamma\beta,\delta\beta}$ is $1-1$ for all β in Y, and N is called a *W-Clifford semigroup* if $\psi_{\gamma,\delta}$ is $1-1$ for all $\delta < \gamma$ in Y. Krgović and Alimpić [3] state the following theorem.

Theorem 3. *Let N be a Clifford semigroup. Then N is E-disjunctive if and only if N is a D-Clifford semigroup.*

Every E-disjunctive inverse semigroup S is found by constructing a regular extension of N by Ω. If N is a D-Clifford semigroup, then N is E-disjunctive by Theorem 3, and thus $S[N, \Omega]$ is E-disjunctive by Lemma 1. We need to describe the case N is not E-disjunctive. Now we get Corollary 4 for the E-disjunctivity of $S[N, \Omega]$ and Ω.

Corollary 4. *Let $S = S[N, \Omega]$ and let $Y = E(\Omega)$. Suppose that $(\bar{\xi}|G_\beta)$ maps G_β onto $G_{\xi\beta\xi^{-1}}$ for all $\xi \in \Omega$, $\beta \in Y$. If Ω is E-disjunctive, then S is E-disjunctive.*

Let N be a W-Clifford semigroup and let S be an I-regular extension. If S is E-disjunctive, then Ω is E-disjunctive.

In what follows let Y be a semilattice and let T_Y be the Munn semigroup. Finally we discuss the E-disjunctivity of T_Y where Y is discrete. 1_A denotes the identity mapping on the set A. $\Delta(\xi)$ $[\nabla(\xi)]$ means the domain [range] of ξ.

We give the preliminary lemmas without proof.

Lemma 5. *Let Ω be a full inverse subsemigroup of T_Y. Let $\xi, \eta \in \Omega$, $e, f \in Y$ and $\nabla(\xi) = Yl$, $\Delta(\eta) = Ym$. Then*

$\xi \mathbf{1}_{(Ye)} \eta$ is idempotent if and only if $(\eta | Ylem) = (\xi^{-1} | Ylem)$;

$\mathbf{1}_{(Ye)} \tau_\Omega \mathbf{1}_{(Yf)}$ if and only if

$$\{\eta \in \Omega \colon (\eta | Ylem) = (\xi^{-1} | Ylem)\} = \tag{4}$$
$$= \{\eta \in \Omega \colon (\eta | Ylfm) = (\xi^{-1} | Ylfm)\} \quad \textit{for all } \xi \textit{ in } \Omega^1.$$

Theorem 6. *Let Y be a semilattice and Ω a full inverse subsemigroup of T_Y. Then Ω is E-disjunctive if and only if there is no pair e, f in Y, $e < f$, such that*

$$(\eta | Yev) = (\zeta | Yev), \quad \Delta(\eta) = Yv = \Delta(\zeta), \quad \eta, \zeta \in \Omega, \quad v \in Y, \tag{5}$$
$$\textit{implies } (\eta | Yfv) = (\zeta | Yfv).$$

Proof. Ω is not E-disjunctive $\Leftrightarrow (\exists e, f$ in Y: $e < f$ and $\mathbf{1}_{(Ye)} \tau_\Omega \mathbf{1}_{(Yf)})$, $\Leftrightarrow (\exists e, f$ in Y: $e < f$ and

$$\left.\begin{array}{l}(\eta_1 | Yelm) = (\zeta_1 | Yelm), \\ \Delta(\eta_1) = Ym, \quad \Delta(\zeta_1) = Yl, \quad \eta_1, \zeta_1 \in \Omega\end{array}\right\} \Rightarrow (\eta_1 | Yflm) = (\zeta_1 | Yflm)). \tag{6}$$

Sufficiency. Suppose Ω is not E-disjunctive. Then there is a pair e, f, $e < f$, satisfying (6). We shall prove that (5) holds for such pair e, f. Assume that the set $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5). Putting $\eta_1 = \eta$ and $\zeta_1 = \zeta$ in (6), we have $m = v = l$. Since $(\eta_1 | Yelm) = (\eta | Yev) = (\zeta | Yev) = (\zeta_1 | Yelm)$, we obtain $(\eta | Yfv) = (\zeta | Yfv)$. Hence (5) holds.

Necessity. Assume that there exists a pair e, f, $e < f$, satisfying (5). If $\{\eta_1, \zeta_1\}$ satisfies the hypothesis in (6), we have $\Delta(\eta) = Yv = \Delta(\zeta)$ by setting $v = lm$, $\eta = (\eta_1 | Yv)$, $\zeta = (\zeta_1 | Yv)$. Since $(\eta | Yev) = (\eta_1 | Yelm) = (\zeta_1 | Yelm) = (\zeta | Yev)$, we find $(\eta_1 | Yflm) = (\zeta_1 | Yflm)$ by (5).

Let $e, f \in Y$, $e < f$. Define $[e, f] = \{g \in Y \colon e \leqq g \leqq f\}$. Y is called *discrete* if $e < f$ implies $[e, f]$ to be finite. Let $Sc(e) = \{f \in Y \colon e < f, \; |[e, f]| = 2\}$ for e in Y. A *tree* Y means a semilattice such that, for all $e, f, g \in Y$, if $e \leqq g$ and $f \leqq g$ then either $e \leqq f$ or $f \leqq e$.

Corollary 7. *Let Y be a discrete tree. Then T_Y is E-disjunctive if and only if there is no element e in Y satisfying*

$$|Sc(e)| = 1, \tag{7}$$

$$Ye \cong Yp \quad \textit{implies} \quad |Sc(p)| \leqq 1. \tag{8}$$

Proof. Sufficiency. Suppose T_Y is not E-disjunctive. Then (5) holds for some e, f, $e < f$, in Y. We may assume that $f \in Sc(e)$. Now suppose that there exists $g \in Sc(e)$ such that $g \neq f$. Define $\eta \colon Yf \cong Yg$ by $x\eta = g$ if $x = f$, and $x\eta = x$ if

$x \leqq e$. Define $\zeta = 1_{(Yf)}$ and $v=f$. Though $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5), we obtain $(\eta|Yfv) \neq (\zeta|Yfv)$, a contradiction. Hence we have $|Sc(e)|=1$.

Next assume that $Ye \cong Yp$ and $q, r \in Sc(p)$; $q \neq r$. Define $v=f$ and ζ: $Yf \cong Yq$. Define η: $Yf \cong Yr$ by $x\eta = r$ if $x=f$, and $x\eta = x\zeta$ if $x \leqq e$. $\{\eta, \zeta, v\}$ satisfies the hypothesis in (5), but we find $(\eta|Yfv) \neq (\zeta|Yfv)$. Hence $|Sc(p)| \leqq 1$.

Necessity. Let $Sc(e)=\{f\}$. We shall prove that $\{e, f\}$ satisfies (5). Suppose that $\{\eta, \zeta, v\}$, $\eta, \zeta \in T_Y$; satisfies the hypothesis in (5). From $Sc(e)=\{f\}$, we have $fv \leqq e$ or $fv=f$. In case $fv \leqq e$ we obtain $ev=fv$, hence $(\eta|Yfv)=(\zeta|Yfv)$. Assume $fv=f$. Then we find $e<f \leqq v$, and thus $ev=e$. Since $(\eta|Ye)=(\zeta|Ye)$, we can put $p=e\eta=e\zeta$. If $(\eta|Yfv) \neq (\zeta|Yfv)$, $f\eta=r$, $f\zeta=q$, then we obtain $r, q \in Sc(p)$, $r \neq q$, contrary to (8). Hence we conclude that $(\eta|Yfv)=(\zeta|Yfv)$.

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On the strong nilstufe of rank two torsion free groups

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1. Introduction

SZELE [7] defined the nilstufe of a group G to be n, n a positive integer, if there exists an associative ring R with additive group G such that $R^n \neq 0$, but for every associative ring R with additive group G the equality $R^{n+1}=0$ holds. If there exists no such positive integer n, we will say that G has nilstufe ∞. FEIGELSTOCK [2] defines the strong nilstufe in a similar manner but allows non-associative rings on G. The nilstufe and strong nilstufe of G will be denoted by $n(G)$ and $N(G)$, respectively.

Unless otherwise stated, all groups in this paper are abelian, rank two torsion-free with addition denoting the group operation. A multiplication on a group G is meant to be the multiplication of a ring R with additive group G.

In this note we study $N(G)$ by classifying G according to the cardinality of the type set, $T(G)$, of G. Here the type set of G means the set of types $t(g)$ of non-zero elements g in G. (See [3], p. 109, for a definition of type.)

By [5] if G is a rank two torsion-free non-nil group (i.e. $N(G)>1$), then the cardinality of $T(G)$ is at most three. In this work we will get the following results for non-nil rank two torsion-free groups:

(i) If the cardinality of $T(G)$ is equal to one then the type must be idempotent and $N(G)=\infty$.

(ii) If the cardinality of $T(G)$ is equal to two then

(a) if G is indecomposable then $N(G)=2$,

(b) if G is decomposable and $T(G)=\{t_1, t_2\}$ such that $t_1<t_2$, $t_1 t_2 > t_2$ and $t_1^2 \neq t_1$, $t_2^2 \neq t_2$ then $N(G)=2$,

(c) in the remaining cases $N(G)=\infty$.

(iii) If the cardinality of $T(G)$ is equal to three then $N(G)=\infty$.

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Let x, y be independent elements of a group G of rank two. Each element w of G has a unique representation $w=ux+vy$, where u, v are rational numbers. Let

$$U = \{u \in Q \mid ux+vy \in G \text{ for some } v \in Q\}, \quad U_0 = \{u_0 \in Q \mid u_0 x \in G\},$$
$$V = \{v \in Q \mid ux+vy \in G \text{ for some } u \in G\}, \quad V_0 = \{v_0 \in Q \mid v_0 y \in G\}.$$

Clearly, U_0, V_0 are subgroups of U, V respectively, which are isomorphic to the pure subgroups $\langle x\rangle^*$ and $\langle y\rangle^*$ of G. ($\langle x\rangle^*$ denotes the pure subgroup of G generated by x.) We call U, U_0, V, V_0 the groups of rank one belonging to the independent set $\{x, y\}$ of G.

Proposition 1 ([1], p. 107). *Let G be a torsion-free abelian group of rank two. If U, U_0, V, V_0 are the groups of rank one belonging to $\{x, y\}$, then $U/U_0 \cong V/V_0$.*

Proposition 2 ([3], p. 114). *Let C be a pure subgroup of the torsion-free group A such that*

(a) *A/C is completely decomposable and homogeneous of type t,*

(b) *all the elements in A but not in C are of type t,*

then C is a direct summand of A.

Proposition 3. *Let A be a torsion-free group of rank two, $T(A)=\{t_1, t_2\}$ and $t_1<t_2$. Let $\{x, y\}$ be an independent set of A such that $t(x)=t_1$, $t(y)=t_2$. Assume U, U_0, V, V_0 are the rank one groups belonging to $\{x, y\}$. If $t(U_0)=t(U)$ then $\langle y\rangle^*$ is a direct summand of A. In particular, if $kU \leqq U_0$ or $kV \leqq V_0$ for some integer $k \neq 0$, then A is decomposable.*

Proof. We have $A/\langle y\rangle^* \cong U$, hence $t(A/\langle y\rangle^*)=t(U)$. Let a be in A but not in $\langle y\rangle^*$; then $t(a)=t_1$. By assumption we have $t(U)=t(U_0)=t_1$, therefore the type of all elements in A but not in $\langle y\rangle^*$ are equal to $t(U)=t(A/\langle y\rangle^*)$. By Proposition 2, $\langle y\rangle^*$ is a direct summand of A. In particular, if $kU \leqq U_0$ or $kV \leqq V_0$ for some integer $k \neq 0$, then because of $U/U_0 \cong V/V_0$ we have that $t(U)=t(U_0)$, and hence A is decomposable.

2. One-element type set

For this case we first assume that the group is indecomposable.

Proposition 4. *If G is an indecomposable and homogeneous group then any non-zero element of $E(G)$, the endomorphism monoid of G, is monic.*

Proof. Let $\varphi \in E(G)$, $0 \neq \operatorname{Ker} \varphi \neq G$. Then $r(G/\operatorname{Ker} \varphi)=1$ since $r(G)=2$ and $\operatorname{Ker} \varphi$ is a pure subgroup of G. We have $G/\operatorname{Ker} \varphi \cong \operatorname{Im} \varphi < G$. Assume $\bar{g}=$

$=g+\mathrm{Ker}\,\varphi\in G/\mathrm{Ker}\,\varphi$ and $g\notin \mathrm{Ker}\,\varphi$. Then

$$t(\bar{g}) = t(G/\mathrm{Ker}\,\varphi) = t(\mathrm{Im}\,\varphi) \leqq t(G) = t(g).$$

On the other hand, $t(\bar{g})\geqq t(g)$, therefore $t(\bar{g})=t(g)$. Hence by Proposition 2 $\mathrm{Ker}\,\varphi$ is a summand of G. But G is indecomposable, so $\mathrm{Ker}\,\varphi=0$, and φ is monic.

Lemma 1. *If G is an indecomposable and homogeneous group then any non-trivial ring over G is without zero divisors.*

Proof. Let $(G, *)$ be a ring over G and let $xy=0$ for some $x, y\in G$, $x\neq 0$, $y\neq 0$. By Proposition 4 any non-trivial element of $E(G)$ is monic. For the left multiplication L_x we have $L_x(y)=xy=0$, which implies that $L_x=0$, so

$$x^2 = L_x(x) = 0. \tag{1}$$

Let $\{x, z\}$ be an independent set of G. Then we have

$$xz = L_x(z) = 0. \tag{2}$$

Furthermore, since the right multiplication R_z is 0 or monic, and $R_z(x)=xz=0$, therefore $R_z=0$. Hence

$$z^2 = R_z(z) = 0. \tag{3}$$

Taking now the left multiplication L_z, by (3) we get that L_z is 0, so

$$zx = L_z(x) = 0. \tag{4}$$

By assumption $\{x, z\}$ is an independent set of G, consequently by (1), (2), (3) and (4) $(G, *)$ is a trivial ring. This shows that any non-trivial ring over G is without zero divisors.

We conclude from this lemma that, if G is an indecomposable and homogeneous group, then $N(G)=1$ or $N(G)=\infty$.

Now we assume G is decomposable.

Proposition 5 (Rédei—Szele [4]). *A ring R with rank one torsion-free additive group G is either an associative domain, or $R^2=0$. R is an integral domain if and only if $t(G)$ is idempotent.*

Proposition 6. *Let $G=H\oplus K$ and $r(H)=r(K)=1$. If $t(H)$ is idempotent then $N(G)=\infty$.*

Proof. If $t(H)$ is idempotent then by Proposition 5, H is an associative integral domain, whence $N(H)=\infty$. We define a ring $(G, *)$ by putting

$$(h, k)*(h', k') = (hh', 0).$$

This proves that $N(G)=\infty$.

Proposition 7. *Let A, B be torsion-free, homogeneous groups of finite ranks. If $t(A)>t(B)$ then* $\mathrm{Hom}\,(A, B)=0$.

Proof. The fact that homomorphisms are type increasing (i.e. type non-decreasing) yields the proposition.

Lemma 2. *Let $G=K\oplus H$ and let $r(K)=r(H)=1$, $t(H)=t(K)$. Then $N(G)>1$ implies that $t(G)$ is idempotent and $N(G)=\infty$.*

Proof. If $t(H)$ is idempotent then by Proposition 6, $N(G)=\infty$. If $t(H)$ is not idempotent then $t(G\otimes G)=t^2(G)>t(G)$, and by Proposition 7, $\mathrm{Hom}\,(G\otimes G, G)=0$. We have

$$\mathrm{mult}\,(G)\cong \mathrm{Hom}\,(G\otimes G, G),$$

therefore G is a nil group and so $N(G)=1$.

3. Two-element type set

Proposition 8. *If R is a finite rank, torsion-free ring without zero divisors, then R^+ is homogeneous.*

Proof. Let $\{x_1, \dots, x_r\}$ be an independent subset of R^+. Let x be in R, $x\neq 0$. First we prove that $xx_1, \dots, xx_r$ are independent. Suppose not. Then there exist integers $a_1, \dots, a_r$ such that $a_1xx_1+\dots+a_rxx_r=0$, i.e. $x(a_1x_1+\dots+a_rx_r)=0$; but R has no zero divisors, therefore $a_1x_1+\dots+a_rx_r=0$, which is a contradiction, since $\{x_1, \dots, x_r\}$ is an independent set.

Hence if $x\neq 0\neq y$ belong to R, then

$$my = m_1xx_1+\dots+m_rxx_r = x(m_1x_1+\dots+m_rx_r)$$

implies that $t(y)\geqq t(x)$, and similarly $nx=n_1yx_1+\dots+n_ryx_r=y(n_1x_1+\dots+n_rx_r)$ implies that $t(x)\geqq t(y)$. Thus $t(x)=t(y)$, consequently R is homogeneous.

Lemma 3. *Let G be a torsion-free indecomposable abelian group of rank two. Let $T(G)=\{t_1, t_2\}$ be such that $t_1<t_2$. If $\{x, y\}$ is an independent set such that $t(x)=t_1$, $t(y)=t_2$, then all non-trivial rings on G satisfy the following multiplication table:*

$$x^2 = by, \quad xy = yx = y^2 = 0, \quad b \text{ is a rational number.}$$

Proof. Let $(G, *)$ be a non-trivial ring over G. Since $t_1<t_2$, in general we have

$$x^2 = ax+by, \quad xy = cy, \quad yx = dy, \quad y^2 = ey.$$

We are going to prove that $a=c=d=e=0$.

Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. We claim $xy=yx$. If not, then $c\neq d$, and for an arbitrary element $g=ux+vy$ of G,

$$gx = ux^2+vyx, \quad xg = ux^2+vxy, \quad gx-xg = v(d-c)y,$$

implying that $(d-c)v\in V_0$ for all $v\in V$. Hence there is an integer $k\neq 0$ such that $kV\leqq V_0$. Now by Proposition 3, $\langle y\rangle^*$ is a direct summand of G, which is a contradiction. Hence $c=d$ and $xy=yx=cy$.

We claim that $a=0$. If not, take two arbitrary elements $g_1=ux+vy$, $g_2=rx+sy$ of G. Then

$$g_1g_2 = urx^2+(su+rv)xy+vsy^2 = aurx+(urb+suc+rvc+vse)y.$$

This implies that $aU^2\leqq U\leqq U^2$, whence $t(U)=t(U^2)$. Consequently,

(1) $\qquad$ if $a\neq 0$ then $t(U)$ is idempotent.

G is not homogeneous, hence by Proposition 8, G should have two non-zero elements $X=rx+sy$, $Y=\alpha x+\beta y$ such that $XY=0$, i.e.

$$XY = (rx+sy)(\alpha x+\beta y) = a\alpha rx+(\alpha rb+s\alpha c+r\beta c+\beta se)y = 0.$$

Since x, y are independent elements, $a\alpha r=0$. By assumption $a\neq 0$, hence we should have one of the following cases:

(i) $\alpha=0,\ r=0,\qquad$ (ii) $\alpha=0,\ r\neq 0,\qquad$ (iii) $\alpha\neq 0,\ r=0.$

In case (i), s and β must be non-zero, as $X\neq 0$, $Y\neq 0$. Hence $0=XY=s\beta y^2=$ $=s\beta ey$, which implies that $e=0$.

In case (ii), $\{X=rx+sy, y\}$ is an independent set of G, and

$$0 = XY = (rx+sy)(\beta y) = \beta(rx+sy)y,$$

since $\alpha=0$. However, $Y\neq 0$, therefore $\beta\neq 0$, so that

(2) $$Xy = (rx+sy)y = 0.$$

Let H, H_0, F, F_0 be the rank one groups belonging to $\{X, y\}$, and let $g=hX+fy$ be an arbitrary element of G where $h\in H, f\in F$. By (2) and by the assumption $y^2=ey$ we have

$$gy = hXy+fy^2 = efy,$$

so we conclude that ef belongs to F_0 for all f in F. If $e\neq 0$ then there is an integer $k\neq 0$ such that $kF\leqq F_0$, so by Proposition 3, $\langle y\rangle^*$ is a direct summand of G, contradicting the indecomposability of G. Hence $e=0$.

Similarly, in case (iii) we also conclude that $e=0$. Therefore,

(3) $\qquad$ if $a\neq 0$ then $e=0$.

Let $g=ux+vy$ be an arbitrary element of G with $u\in U$, $v\in V$. By (3) we have $gy=uxy+vy^2=cuy$, so if c is not zero then $cU\leqq V_0$, hence

(4) $$t(U)\leqq t(V_0).$$

Now, using (1) and (4) we prove that $t(U)=t(U_0)$. By (1), $t(U)$ is idempotent, therefore $h_p^U(1)=0$ or ∞ except for finitely many prime numbers. $U_0\leqq U$ implies that $t(U_0)\leqq t(U)$, so that $h_p^U(1)=0$ implies $h_p^{U_0}(1)=0$ and $h_p^U(1)<\infty$ implies $h_p^{U_0}(1)<\infty$. It remains to prove that $h_p^{U_0}(1)=\infty$ if $h_p^U(1)=\infty$. Let $1/p^n\in U$ and $h_p^U(1)=\infty$. Then by the definition of U there is $K/m\in V$ such that $g=(1/p^n)x+ +(K/m)y\in G$. Let $m=m'p^i$ where $(m',p)=1$. Then

$$g=(1/p^n)x+(K/m'p^i)y,\quad m'g=(m'/p^n)x+(K/p^i)y,\quad (m'g-K(y/p^i))=(m'/p^n)x.$$

By (4), $1/p^i\in V_0$, so that $1/p^n\in U_0$. This is correct for all $n<\infty$, hence $h_p^{U_0}(1)=\infty$, so we conclude that $t(U)\leqq t(U_0)$. But $t(U_0)\leqq t(U)$, therefore $t(U_0)=t(U)$. By Proposition 3, $\langle y\rangle^*$ will be a direct summand of G which is in contradiction with indecomposability. Consequently $c=0$.

By assuming $a\neq 0$ we got $c=0$ and $e=0$, that is $x^2=ax+by$, $xy=yx= =y^2=0$. Thus $\{z=ax+by,y\}$ is an independent set of G, and $z^2=a^2x^2+b^2y^2+ +2abxy=a^2x^2=a^2z$, $zy=yz=y^2=0$. Let W, W_0, T, T_0 be the rank one groups belonging to $\{z,y\}$. Let $g=wz+ty$ be an arbitrary element of G and $w\in W$, $t\in T$. Then $gz=wz^2=a^2wz$.

Since we supposed $a\neq 0$, we have $a^2W\leqq W_0\leqq W$, hence $t(W_0)=t(W)$. Again by Proposition 3, $\langle y\rangle^*$ is a direct summand of G which is a contradiction. All contradictions are due to the assumption $a\neq 0$. Consequently $a=0$.

So far we proved that

$$x^2=by,\quad xy=yx=cy,\quad y^2=ey.$$

Let $g=ux+vy$ be an arbitrary element of G. Then

$$gx=uby+vcy=(ub+cv)y,\quad gy=cuy+evy=(cu+ev)y,$$

hence

$$\begin{aligned} ub+cv&=v_0\\ cu+ev&=v_0'\end{aligned}\quad \text{for some } v_0,\ v_0' \text{ in } V_0.$$

This implies that $(c^2-be)v=v_0''$ for some v_0'' in V_0. If $c^2-be\neq 0$ then there is an integer $k\neq 0$ such that $kU\leqq U_0$, which implies by Proposition 3 that $\langle y\rangle^*$ is a direct summand of G. This is a contradiction. Therefore

(5) $$c^2-be=0.$$

If $b=0$ then $gy=uxy+vy^2=evy$. Again this is a contradiction, hence

(6) $$b\neq 0.$$

By (5) and (6)

$$e = 0 \quad \text{if and only if} \quad c = 0. \tag{7}$$

If $e \neq 0$ and $c \neq 0$ then $\{z_1 = -cx + by, y\}$ is an independent set of G. We get

$$z_1^2 = (-cx+by)^2 = c^2x^2 + b^2y^2 - 2cbxy = c^2by + eb^2y - 2c^2by =$$
$$= b(eb-c^2)y = 0 \quad \text{(by (5))},$$
$$z_1y = yz_1 = -cxy + by^2 = -c^2y + eby = (-c^2+eb)y = 0 \quad \text{(by (5))},$$
$$y^2 = ey.$$

Let M, M_0, N, N_0 be the rank one groups belonging to $\{z_1, y\}$, and let $g = mz_1 + ny$ be an arbitrary element of G where $m \in M$ and $n \in N$. Then $gy = ny^2 = eny$, hence $eN \leqq N_0$. It follows now that there is an integer $k \neq 0$ such that $kN \leqq N_0$, so by Proposition 3, $\langle y \rangle^*$ is a direct summand of G, contradicting the indecomposability of G. Therefore $c=0$ or $e=0$, whence by (7) $c=0$ and $e=0$, completing the proof of Lemma 3.

Remark 1. In case no element of $T(G) = \{t_1, t_2\}$ is idempotent, let $\{x, y\}$ be an independent set of G such that $t(x) = t_1$, $t(y) = t_2$; $t_1 < t_2$ and $t_1 t_2 \neq t_2$. Then $x^2 = by$, $xy = yx = y^2 = 0$ for any ring over G.

Theorem 1. *Let $T(G) = \{t_1, t_2\}$, $t_1 < t_2$, and let $\{x, y\}$ be an independent set of G such that $t(x) = t_1$, $t(y) = t_2$. Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. If G is either indecomposable or neither t_1 nor t_2 is idempotent and $t_1 t_2 > t_2$, then G is a non-nil group if and only if $t(U^2) \leqq t(V_0)$.*

Proof. Suppose G is a non-nil group. By Lemma 3 and Remark 1 we have

$$x^2 = by, \quad xy = yx = y^2 = 0, \quad b \neq 0.$$

Let $g = ux + vy$, $h = rx + sy$ be arbitrary elements of G with $u, r \in U$ and $v, s \in V$. Then $gh = bury$, which implies that $bU^2 \leqq V_0$, that is $t(U^2) \leqq t(V_0)$.

Conversely, if $t(U^2) \leqq t(V_0)$ then there is an integer $b \neq 0$ such that $bU^2 \leqq V_0$. Let $g = ux + vy$, $h = rx + sy$ be arbitrary elements of G, and define a multiplication over G by $gh = bury$. This multiplication is a ring over G, hence G is a non-nil group.

Remark 2. Let G be decomposable and let $T(G) = \{t_1, t_2\}$ be such that $t_1^2 \neq t_1$, $t_2^2 \neq t_2$. In [6] it has been proved that $t_1 t_2 = t_2$ implies $N(G) = \infty$.

Remark 3. Under the hypothesis of Theorem 1, $N(G) = 1$ or 2. $N(G) = 2$ if and only if $t(U^2) \leqq t(V_0)$.

Remark 4. If $G = H \oplus K$ and at least one of $t(H)$ and $t(K)$ is idempotent then by Proposition 6, $N(G) = \infty$.

4. At least three-element type set

Proposition 9. *Let G be a torsion-free group of rank two and let $T(G)=\{t_0, t_1, t_2\}$. Let $x, y \in G$ be such that $t(x)=t_1$ and $t(y)=t_2$. Suppose that $t_0<t_1$, $t_0<t_2$. If t_1, t_2 are incomparable, then for any ring on G we have $x^2=ax$, $y^2=by$, $xy=yx=0$ for some $a, b \in Q$.*

Proof. The set $G(t_1)$ of elements g in G whose types are $\geqq t_1$ form a pure subgroup of G ([3], p. 109). Let $z \in G$ be such that $t(z)=t_0$. Then $z \notin G(t_1)$. Because of the purity of $G(t_1)$, $r[G(t_1)]=1$. Since $t(x^2) \geqq t(x)=t_1$, we have $x^2, x \in G(t_1)$, thus x^2 and x are dependent elements, that is $x^2=ax$ for some $a \in Q$. Similarly we conclude that $y^2=by$ for some $b \in Q$. By the same token $t(yx) \geqq t(x)$ implies that $yx, x \in G(t_1)$, hence $yx=ex$ for some $e \in Q$. Similarly we deduce that $yx=fy$ for some $f \in Q$. If $yx \neq 0$ then $t(yx)=t(x)=t(y)$. This contradicts our hypothesis, therefore $yx=0$. In the same way we conclude that $xy=0$.

If the cardinality of $T(G)$ is greater than three, then by [5] G is nil group, that is, $N(G)=1$.

If the cardinality of $T(G)$ is equal to three then by [5] $T(G)$ has one minimal and two maximal elements; let $\{x, y\}$ be an independent set of G such that $t(x)$ and $t(y)$ are maximal. By Proposition 9, for any ring over G we have $x^2=ax$, $xy=yx=0$, $y^2=by$, where a, b are rational numbers. If G is non-nil, then a or b is non-zero, say $a \neq 0$. Then $x^n=a^{n-1}x$, hence there is no integer n such that $x^n=0$. This implies that $N(G)=\infty$.

Theorem 2. *Let G be a rank two torsion-free group and let $T(G)=\{t_0, t_1, t_2\}$ be such that $t_0<t_1$, $t_0<t_2$. Let $\{x, y\}$ be an independent set such that $t(x)=t_1$, $t(y)=t_2$. Let U, U_0, V, V_0 be the rank one groups belonging to $\{x, y\}$. Then G is a non-nil group if and only if either $t(U_0)=t(U)$ and $t(U_0)$ is idempotent or $t(V_0)=t(V)$ and $t(V_0)$ is idempotent.*

Proof. Let G be a non-nil group. Then $x^2=ax$, $xy=yx=0$, $y^2=by$ and a or b is non-zero. We assume $a \neq 0$. Let $g=ux+vy$ be an arbitrary element of G where $u \in U$ and $v \in V$. Then $gx=ux^2+vyx=aux$. This implies that $au \in U_0$ for all $u \in U$, so it follows that $aU \subseteq U_0$. However, $U_0 \subseteq U$, therefore $t(U)=t(U_0)$. Furthermore, since $a \neq 0$, we have $x^2 \neq 0$, hence $t(U_0)$ is idempotent.

Conversely, if $t(U)=t(U_0)$ and $t(U_0)$ is idempotent then there is an integer m such that $mU \subseteq U_0$. Let $g=ux+vy$, $h=rx+sy$ be arbitrary elements of G, and define a multiplication over G by $gh=m^2urx$. This multiplication is a ring over G, therefore G is a non-nil group.

5. Concluding remarks

(a) Lemma 1 and Lemma 2 imply that if G is a homogeneous torsion-free group of rank two then $n(G)=N(G)$.

(b) Let G be a torsion-free group of rank two and let $T(G)=\{t_1, t_2\}$ be such that $t_1<t_2$. Then Lemma 3 shows that if G is indecomposable then $n(G)=N(G)$. However, in the decomposable case it has been shown in [6] by an example that, in general, $n(G)$ and $N(G)$ are not equal.

(c) If G is a torsion-free group of rank two with $|T(G)|\geqq 3$ then $n(G)=N(G)$.

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Amalgamated free products of n-groups

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1. Introduction

The aim of this paper is to give a description for the amalgamated free products of n-groups. The present paper is the second in a series of papers devoted to the study of constructions of some special limits in the category $\mathbf{Gr}_n$ of n-groups (in [7] two constructions of free products were given). Both these papers are based on the results of [6]. The relation of the functors Φ and Ψ to inductive and projective limits given in [6] is used to investigate the above-mentioned inductive limits of n-groups.

In these constructions the notion of a free covering group (cf. [10]) plays a central rôle. The paper [3] (cf. also [5]) contains a generalization of this notion, namely a free covering $(k+1)$-group of an $(n+1)$-group (where $n=sk$). The assignment of free covering $(k+1)$-groups to $(n+1)$-groups is functorial. This leads to the functor Φ_s: $\mathbf{Gr}_{n+1}\to\mathbf{Gr}_{k+1}$ (in fact a class of functors depending on which construction of the free covering $(k+1)$-group we consider), which is left adjoint to the forgetful functor Ψ_s: $\mathbf{Gr}_{k+1}\to\mathbf{Gr}_{n+1}$ (cf. [3], [6]). Here (contrary to [7]) the meaning of Φ_s is the same as in [6]. In Proposition 1, by $\Phi_q(\tilde{H})$ and $\Phi_q(\tilde{G}_t)$ we mean the respective free covering $(k+1)$-groups of the $(qk+1)$-groups $\tilde{H}$ and $\tilde{G}_t$, disregarding their constructions.

In a category with initial objects any free product with an amalgamated initial object is isomorphic to the corresponding free product. The construction of free products of n-groups was given in [7]. However, the construction of amalgamated free products given here exploits the non-emptiness of amalgamated sub-$(n+1)$-groups. Therefore we always assume that a polyadic group is nonempty (like in [7]).

The terminology and notation of this paper is the same as in [5], [6], [7]. We recall only that $(\mathfrak{C}_{s,k+1},\varphi)$ denotes the cyclic $(k+1)$-group of order s (cf. [10], [3]) and the letters f and g denote the $(n+1)$-group and $(k+1)$-group operations, respec-

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tively, in the $(n+1)$-groups and $(k+1)$-groups under consideration. The symbol $f_{(s)}$ is understood as in [7] (in particular $f_{(0)}(x)=x$). Throughout the paper we assume $n=sk$, $s=mq$, $n>1$.

Let us introduce some new abbreviations of the notation. In place of $f(\ldots, x_1, \ldots, x_r, \ldots, x_1, \ldots, x_r, \ldots)$ with $x_1, \ldots, x_r$ repeated t times we write $f(\ldots, [\![x_1, \ldots, x_r]\!]^t, \ldots)$. In particular, instead of $f(\ldots, x, \ldots, x, \ldots)$ with x repeated t times we write briefly $f(\ldots, [\![x]\!]^t, \ldots)$.

2. Preliminaries

We start with recalling the construction of an amalgamated free product in the category $\mathbf{Gr}_2$ of groups (cf. [2], [11]).

Consider a nonempty family of monomorphisms $\{\gamma_t\colon B\to A_t\}_{t\in T}$ in $\mathbf{Gr}_2$, where the groups A_t are mutually disjoint. Let e denote the neutral element of A_t for each $t\in T$, and $B_t=\gamma_t(B)$. Form a set D_t consisting of exactly one representative for any left coset xB_t of the subgroup B_t and such that $eB_t\cap D_t=\{e\}$. Thus every element $a\in A_t$ can be expressed uniquely in the form $a=\hat{a}\gamma_t(b)$ where $\hat{a}\in D_t$, $b\in B$. The sequences of the form $\hat{a}_1\ldots\hat{a}_r b$ where $\hat{a}_i\in D_{t_i}-B_{t_i}$, $b\in B$, $t_i\neq t_{i+1}$, $r=0, 1, 2, \ldots$, will be called words. In the set A of words define a binary operation which from any two words forms a word in the following manner. Juxtapose the words to get a "long word" and perform all the necessary cancellations. The set A with this operation is the free product of A_t with amalgamated subgroup B. Henceforth by the amalgamated free product of groups we always mean the group described above.

Lemma 1. *Let $[L'; \{\gamma_t\colon \Phi_n(G_t)\to L'\}_{t\in T}]$ be the free product of a nonempty family of groups $\{\Phi_n(G_t)\}_{t\in T}$ with amalgamated subgroup $\Phi_n(H)$ (i.e., the inductive limit of the family of monomorphisms $\{\Phi_n(\varepsilon_t)\colon \Phi_n(H)\to\Phi_n(G_t)\}_{t\in T}$) where $\langle\Phi_n(H), \tau_H, \zeta_H\rangle$ and $\{\langle\Phi_n(G_t), \tau_t, \zeta_t\rangle\}_{t\in T}$ are the free covering groups of the $(n+1)$-groups H and $\{G_t\}_{t\in T}$, respectively. Then the morphism $\zeta\colon L'\to\mathfrak{C}_{n,2}$ defined by $\zeta(\hat{a}_1\ldots\hat{a}_r b)=\varphi_{(r)}(\zeta_{t_1}(a_1), \ldots, \zeta_{t_r}(a_r), \zeta_H(b))$ (where $\hat{a}_1\ldots\hat{a}_r b\in L'$, $b\in\Phi_n(H)$, $\hat{a}_i\in\Phi_n(G_{t_i})$ for $i=1, \ldots, r$) is an epimorphism and a pair $\langle L', \tau\rangle$, where τ is the inclusion of $\zeta^{-1}(0)$ into L', is the free covering group of the $(n+1)$-group $L=\zeta^{-1}(0)$. Furthermore, $[L; \{\alpha_t\colon G_t\to L\}_{t\in T}]$, where $\tau\alpha_t=\Psi_n(\gamma_t)\tau_t$, is the free product of the $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. The proof of this lemma is analogous to that of Lemma 1 of [7].

The following two lemmas concern the decomposition of an $(n+1)$-group $\mathfrak{G}=(G,f)$ into left cosets of a nonempty sub-$(n+1)$-group H (cf. [1], [10]). As usual, for the construction of the free covering group one fixes an element $c \in G$ (cf. [3], [5]). Since the element c is arbitrary, we may assume $c \in H$.

Lemma 2. *Let H be a sub-$(n+1)$-group of an $(n+1)$-group $\mathfrak{G}=(G,f)$ and $a \in G$. Then every element of the form (a, l) (with $l=0, 1, \dots, n-1$) in the group $\mathfrak{G}^{*n}=(G^{*n}, f^*)$ belongs to $(a, 0)H^{*n}$.*

Proof. Let $a \in G$ and $l=0, 1, \dots, n-1$. Then

$$f^*((a,0),(c,l-1)) = \left(f(a, c, \llbracket c \rrbracket^{l-1}, \bar{c}, \llbracket c \rrbracket^{n-1-l}), l\right) = (a, l) \in (a,0)H^{*n},$$

since $(c, l-1) \in H^{*n}$.

Lemma 3. *Two elements (a_1, l_1) and (a_2, l_2) of $\mathfrak{G}^{*n}$ belong to the same left coset of the subgroup H^{*n} if and only if there exists an element $b \in H$ such that $a_1= =f(a_2, b, \llbracket c \rrbracket^{n-1})$.*

Proof. Let $(a_1, l_1) \in (a_2, l_2)H^{*n}$. In view of Lemma 2 $(a_1, 0) \in (a_2, 0)H^{*n}$, whence $(a_1, 0)=f^*((a_2, 0), (b, n-1))=\left(f(a_2, b, \llbracket c \rrbracket^{n-1}, \bar{c}, \llbracket c \rrbracket^{n-1}), 0\right)=\left(f(a_2, b, \llbracket c \rrbracket^{n-1}), 0\right)$ for some $b \in H$. Thus $a_1=f(a_2, b, \llbracket c \rrbracket^{n-1})$.

Conversely, let $a_1=f(a_2, b, \llbracket c \rrbracket^{n-1})$. Thus $(a_1, 0)=f^*((a_2, 0), (b, n-1))$, whence $(a_1, 0) \in (a_2, 0)H^{*n}$. Then, by Lemma 2, $(a_1, l_1) \in (a_2, l_2)H^{*n}$.

3. A construction of amalgamated free products

Consider a nonempty family of monomorphisms $\{\varepsilon_t \colon H \to G_t\}_{t \in T}$ where H and G_t are nonempty $(n+1)$-groups. Choose an arbitrary but fixed element $c \in H$. Let $c_t=\varepsilon_t(c)$. Decompose every G_t into left cosets of the sub-$(n+1)$-group $H_t=\varepsilon_t(H)$ (i.e., elements a' and a'' belong to the same coset if and only if there exists an element $b_t \in H_t$ such that $a'=f(a'', b_t, \llbracket c_t \rrbracket^{n-1})$) and choose one element in every coset distinct from H_t. The representative of the coset aH_t (where $a \in G_t - H_t$) is denoted by $\hat{a}$. Therefore $a=f(\hat{a}, b_t, \llbracket c_t \rrbracket^{n-1})$ for some $b_t \in H_t$. By a word we shall mean a sequence of the form $\hat{a}_1 \dots \hat{a}_r b \overset{l}{c}$, where $r=0, 1, \dots$ and for $i=1, \dots, r$ we have $a_i \in G_{t_i} - H_{t_i}$, $b \in H$, $t_i \neq t_{i+1}$, $l=0, 1, \dots, n-1$, $r+l \equiv 0 \pmod{n}$. Now we define an $(n+1)$-ary operation f on the set L of all words. Given $n+1$ words, form by juxtaposition a "long word" and perform the following cancellations: If in the "long word" there appear neighbouring expressions of the form

1. $b_1 \overset{l_1}{c}$ and $b_2 \overset{l_2}{c}$, where $b_1, b_2 \in H$, then we replace them by $b \overset{\varphi(l_1, l_2)}{c}$, where $b=f_{(\cdot)}(b_1, \llbracket c \rrbracket^{l_1}, b_2, \llbracket c \rrbracket^{l_2}, \bar{c}, \llbracket c \rrbracket^{n-1-\varphi(l_1, l_2)})$. If $b=\bar{c}$ and $\varphi(l_1, l_2)=n-1$, then we cancel the resulting expression $\bar{c} \overset{n-1}{c}$, unless it remains at the end of the "long word".

2. $\hat{a}_1$ and $\hat{a}_2$, where $\hat{a}_1, \hat{a}_2 \in G_t - H_t$, then depending on the element $a = f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2})$ we replace them by

(a) $\hat{a}b\overset{0}{c}$, where $b \in H$ is the solution of the equation $a = f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$, if $a \notin H_t$;

(b) $a'\overset{1}{c}$, where $\varepsilon_t(a') = a$, if $a \in H_t$.

3. $b_1\overset{l}{c}$ and $\hat{a}_1$, where $\hat{a}_1 \in G_t - H_t$, then we replace them by $\hat{a}b\overset{l}{c}$, where $a = f_{(\cdot)}(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$ and b is the solution of the equation $a = f_{(\cdot)}(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$.

After a finite number of steps the "long word" becomes a word. Note that the resulting word does not depend on the order of the cancellations performed.

Define the family of morphisms $\{\alpha_t \colon G_t \to L\}_{t \in T}$ by the formula:

1. $\alpha_t(a) = \hat{a}b\overset{n-1}{c}$, where b is the solution of the equation $a = f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$, if $a \in G_t - H_t$;

2. $\alpha_t(a) = a'\overset{0}{c}$, where $\varepsilon_t(a') = a$, if $a \in H_t$.

Theorem 1. *The $(n+1)$-groupoid L is an $(n+1)$-group. The $(n+1)$-group L together with the family of morphisms $\{\alpha_t \colon G_t \to L\}_{t \in T}$ is the free product of the $(n+1)$-groups $\{G_t\}_{t \in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. We use the same notation as in Lemma 1. Let $\langle \Phi_n(H), \tau_H, \zeta_H \rangle$ and $\{\langle \Phi_n(G_t), \tau_t, \zeta_t \rangle\}_{t \in T}$ be the free covering groups of the $(n+1)$-groups H and $\{G_t\}_{t \in T}$, respectively, with distinguished elements $c \in H$ and $c_t = \varepsilon_t(c) \in G_t$.

As was mentioned above, the elements of the free product L' of the groups $\Phi_n(G_t)$ with an amalgamated subgroup $\Phi_n(H)$ are words of the form $\hat{a}_1^* \ldots \hat{a}_r^* b^*$, where $\hat{a}_i^* \in \Phi_n(G_{t_i}) - \Phi_n(H_{t_i})$, $b^* \in \Phi_n(H)$, $t_i \neq t_{i+1}$, $r = 0, 1, 2, \ldots$. According to Lemma 2, the elements $\hat{a}_i^*$ can be chosen to be of the form $\hat{a}_i^* = (\hat{a}_i, 0)$. On the other hand, by Lemma 3, the decomposition of $\Phi_n(G_t)$ into left cosets of the subgroup $\Phi_n(H_t)$ coincides for elements of the form $(a, 0)$ with the decomposition of the $(n+1)$-group G_t into left cosets of the sub-$(n+1)$-group H_t. Therefore every element of L' is of the form $w = (\hat{a}_1, 0) \ldots (\hat{a}_r, 0)(b, l)$ where $b \in H$, $l = 0, \ldots, n-1$ and $\hat{a}_i \in G_{t_i} - H_{t_i}$; $t_i \neq t_{i+1}$ for $i = 1, \ldots, r$. According to Lemma 1 the morphism $\zeta \colon L' \to \mathfrak{C}_{n,2}$ defined by $\zeta(w) = \varphi_{(r)}(\xi_{t_1}(\hat{a}_1, 0), \ldots, \zeta_{t_r}(\hat{a}_r, 0), \zeta_H(b, l))$ is an epimorphism. Let $L = \zeta^{-1}(0)$ and let $\tau \colon L \to L'$ be the inclusion of L into L'. Then $w \in L$ if and only if $r + l \equiv 0 \pmod{n}$ (since $\zeta(w) = \varphi_{(r)}(0, \ldots, 0, l) \equiv r + l \pmod{n}$). The $(n+1)$-group operation f on L is simply the long product obtained from the group operation f^* on L'. To simplify words of the form $(a, 0)$ in the $(n+1)$-group L we write simply a. Then $\tau(a) = (a, 0) \in L$. Let $w = (\hat{a}_1, 0) \ldots (\hat{a}_r, 0)(b, l) \in L$. Then (cf. [5])

$$w = f^*_{(\cdot)}((\hat{a}_1, 0), \ldots, (\hat{a}_r, 0), (b, 0), [\![(c, 0)]\!]^l) =$$
$$= f^*_{(\cdot)}(\tau(\hat{a}_1), \ldots, \tau(\hat{a}_r), \tau(b), [\![\tau(c)]\!]^l) = \tau(\hat{a}_1 \ldots \hat{a}_r b\overset{l}{c}).$$

Thus it is convenient to define L as the set of all sequences of the form $\hat{a}_1 \ldots \hat{a}_r b\overset{l}{c}$ where $b \in H$, $r=0,1,2,\ldots$, $l=0,\ldots,n-1$, $r+l\equiv 0 \pmod n$ and $a_i \in G_{t_i}-H_{t_i}$, $t_i \neq t_{i+1}$ for $i=1,\ldots,r$. The $(n+1)$-ary operation f on L is given by juxtaposition of $n+1$ words and performing all possible cancellations:

1. If there appear neighbouring expressions $b_1\overset{l_1}{c}$ and $b_2\overset{l_2}{c}$, where $b_1, b_2 \in H$, then

$$\tau(\ldots b_1\overset{l_1}{c}b_2\overset{l_2}{c}\ldots) = f^*_{(\cdot)}(\ldots, \tau(b_1), [\![\tau(c)]\!]^{l_1}, \tau(b_2), [\![\tau(c)]\!]^{l_2}, \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1,l_2)}), \varphi(l_1,l_2)), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, \tau(f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1,l_2)})), [\![\tau(c)]\!]^{\varphi(l_1,l_2)}, \ldots) =$$
$$= \tau(\ldots f(b_1, [\![c]\!]^{l_1}, b_2, [\![c]\!]^{l_2}, \bar{c}, [\![c]\!]^{n-1-\varphi(l_1,l_2)})\overset{\varphi(l_1,l_2)}{c}\ldots).$$

If we obtain the expression $\bar{c}\overset{n-1}{c}$ not at the end of the "long word", then as in the proof of Theorem 2 of [7] one can show that it may be cancelled.

2. If there appear neighbouring expressions $\hat{a}_1$ and $\hat{a}_2$, where $\hat{a}_1, \hat{a}_2 \in G_t - H_t$, then

$$\tau(\ldots \hat{a}_1\hat{a}_2\ldots) = f^*_{(\cdot)}(\ldots, \tau(\hat{a}_1), \tau(\hat{a}_2), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2}), 1), \ldots) = f^*_{(\cdot)}(\ldots, (a, 1), \ldots)$$

where $a=f(\hat{a}_1, \hat{a}_2, \bar{c}_t, [\![c_t]\!]^{n-2}) \in G_t$. Consider two cases:

(a) Let $a \notin H_t$. Then $(a,1)=f^*((\hat{a},0), (\varepsilon_t(b), 0))=(f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2}), 1)$, thus $a=f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$ and therefore $b \in H$ given by the equality $(a,1)=f^*((\hat{a},0), (\varepsilon_t(b),0))$ is the solution of the equation $a=f(\hat{a}, \varepsilon_t(b), \bar{c}_t, [\![c_t]\!]^{n-2})$. Hence

$$\tau(\ldots \hat{a}_1\hat{a}_2\ldots) = f^*_{(\cdot)}(\ldots, (\hat{a},0), (b,0), \ldots) = f^*_{(\cdot)}(\ldots, \tau(\hat{a}), \tau(b), \ldots) = \tau(\ldots \hat{a}b\overset{0}{c}\ldots).$$

(b) Let $a \in H_t$. Then

$$\tau(\ldots \hat{a}_1\hat{a}_2\ldots) = f^*_{(\cdot)}(\ldots, (a,1), \ldots) = f^*_{(\cdot)}(\ldots, \tau(a), \tau(c), \ldots) = \tau(\ldots a'\overset{1}{c}\ldots)$$

where $\varepsilon_t(a')=a$.

3. If there appear neighbouring expressions $b_1\overset{l}{c}$ and $\hat{a}_1$, where $\hat{a}_1 \in G_t - H_t$, then

$$\tau(\ldots b_1\overset{l}{c}\hat{a}_1\ldots) = f^*_{(\cdot)}(\ldots, (\varepsilon_t(b_1), l), (\hat{a}_1, 0), \ldots) =$$
$$= f^*_{(\cdot)}(\ldots, (f(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)}), \varphi(l,0)), \ldots) = f^*_{(\cdot)}(\ldots, (a, \varphi(l,0)), \ldots)$$

where $a=f_{(\cdot)}(\varepsilon_t(b_1), [\![c_t]\!]^l, \hat{a}_1, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$. Then

$$(a, \varphi(l,0)) = f^*((\hat{a},0), (\varepsilon_t(b), l)) = (f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)}), \varphi(l,0));$$

thus $a=f_{(\cdot)}(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$ and therefore $b\in H$ given by the equality $(a, \varphi(l, 0))=f^*((\hat{a}, 0), (\varepsilon_t(b), l))$ is the solution of the equation $a= =f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^l, \bar{c}_t, [\![c_t]\!]^{n-1-\varphi(l,0)})$. Hence

$$\tau(\ldots b_1 \overset{l}{c} \hat{a}_1 \ldots) = f^*_{(\cdot)}(\ldots, (\hat{a}, 0), (\varepsilon_t(b), l), \ldots) =$$

$$= f^*_{(\cdot)}(\ldots, \tau(\hat{a}), \tau(b), [\![\tau(c)]\!]^l, \ldots) = \tau(\ldots \hat{a} b \overset{l}{c} \ldots).$$

The uniqueness of the resulting word is implied by the uniqueness of the form of a word in the amalgamated free product of groups.

According to Lemma 1, $\tau\alpha_t(a)=\Psi_n(\gamma_t)(a, 0)$. Consider two cases:

1. Let $a\in G_t-H_t$. Then $(a, 0)=f^*((\hat{a}, 0), (\varepsilon_t(b), n-1))=(f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1}), 0)$; thus $a=f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$ and therefore $b\in H$ given by the equality $(a, 0)= =f^*((\hat{a}, 0), (\varepsilon_t(b), n-1))$ is the solution of the equation $a=f(\hat{a}, \varepsilon_t(b), [\![c_t]\!]^{n-1})$. Hence $\gamma_t(a, 0) = (a, 0)(b, n-1)$, so $\tau\alpha_t(a) = (\hat{a}, 0)(b, n-1) = \tau(\hat{a} b \overset{n-1}{c})$.

2. Let $a\in H_t$. Then $\gamma_t(a, 0)=(a, 0)$, whence $\tau\alpha_t(a)=\tau(a, 0)=a'\overset{0}{c}$ where $\varepsilon_t(a')=a$.

This completes the proof of Theorem 1.

4. Some properties of amalgamated free products

In view of Theorem 3 of [6], if every $(n+1)$-group G_t and also the $(n+1)$-group H are derived from $(k+1)$-groups, then the amalgamated free product is also derived from a $(k+1)$-group. The converse is also true except for the following two cases: when T has only one element or when at most one of the monomorphisms is not an isomorphism. Then the amalgamated free product is isomorphic either to G_t or to H. It may happen that in this case the amalgamated free product (being isomorphic to one of the $(n+1)$-groups G_t) is derived from a $(k+1)$-group; none the less the $(n+1)$-group H (as a sub-$(n+1)$-group of that $(n+1)$-group G_t) need not be derived from any $(k+1)$-group. For this reason we have to make some additional assumptions.

Theorem 2. *Let L be the free product of $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H, where more than one monomorphism ε_t: $H\to G_t$ is not an isomorphism. Then the $(n+1)$-group L is derived from a $(k+1)$-group if and only if every $(n+1)$-group G_t and the $(n+1)$-group H are also derived from $(k+1)$-groups.*

Proof. We use the notation of Theorem 1. Let L be an $(n+1)$-group derived from a certain $(k+1)$-group and let the word $w=\hat{a}_1\ldots\hat{a}_r b \overset{l}{c}$ be skew to the element

$\overset{0}{cc}=\alpha(c)\in\alpha(H)$ (where $\alpha=\alpha_t\varepsilon_t$) in that $(k+1)$-group. In view of Corollary 2 of [9] the element w is s-skew to $\overset{0}{cc}$ in the $(n+1)$-group L.

Suppose that $r\neq 0$. Then $a_1\in G_{t_1}$ for some $t_1\in T$. Take any element of the form $\hat{a}\bar{c}\overset{n-1}{c}$ where $a\in G_t-H_t$ and $t\neq t_1$. From the definition of an s-skew element (cf. [9]) it follows that $w[\![\overset{0}{cc}]\!]^{k-1}[\![\hat{a}\bar{c}\overset{n-1}{c}]\!]^{n-k-1}=\hat{a}\bar{c}\overset{n-1}{c}\,w[\![\overset{0}{cc}]\!]^{k-1}[\![\hat{a}\bar{c}\overset{n-1}{c}]\!]^{n-k}$. After performing all the necessary cancellations the reduced word on the left side of the equality starts with $\hat{a}_1$, the reduced word on the right side starts with $\hat{a}$. This contradicts the uniqueness of the form of a reduced word, since $\hat{a}_1\neq\hat{a}$ $(t_1\neq t)$.

Thus the word w is of the form $w=\overset{0}{bc}$. Since $w=\overset{0}{bc}=\alpha(b)\in\alpha(H)$, by Proposition 3 of [9] the sub-$(n+1)$-group $\alpha(H)$ of the $(n+1)$-group L is also a sub-$(k+1)$-group of the creating $(k+1)$-group of L. Hence the $(n+1)$-group H (isomorphic to the $(n+1)$-group $\alpha(H)$) is also derived from a $(k+1)$-group. On the other hand $\alpha(H)\subset\gamma_t(G_t)$; so every $(n+1)$-group $\gamma_t(G_t)$ is derived from a $(k+1)$-group.

Conversely, let $(n+1)$-groups $\{G_t\}_{t\in T}$ and H be derived from $(k+1)$-groups. Then, by Theorem 3 of [6], the $(n+1)$-group L is also derived from a $(k+1)$-group, which completes the proof of Theorem 2.

In [6] we proved a general theorem on the inductive limits of covering $(k+1)$-groups of $(n+1)$-groups. This theorem applied to the case of the free product yields Theorem 4 of [7].

In a category complete with respect to inductive limits the free product is a particular case of the free product with an amalgamated subobject (taking an initial object for the subobject). This is the case for the category $\mathbf{Gr}_2$ and also for the categories $\mathbf{Gr}_n$ with $n>2$. Therefore in $\mathbf{Gr}_2$ the construction of a free product is a particular case of the construction of the free product with an amalgamated subgroup (in this case a one-element group). Note that the situation is quite different when we pass to $\mathbf{Gr}_n$ for $n>2$. In the construction of a free product with an amalgamated sub-n-group presented here it is important that this sub-n-group is nonempty. Hence the construction is not a generalization of the construction of the free product. In particular, Theorem 4 of [7] is not applicable to the description of an amalgamated free product of covering $(k+1)$-groups of $(n+1)$-groups.

Proposition 1. *Let $\{\varepsilon_t\colon H\to G_t\}_{t\in T}$ and $\{\varepsilon_t'\colon H'\to G_t'\}_{t\in T}$ be nonempty families of monomorphisms, where $\langle H',\lambda_H,\zeta_H\rangle$ and $\{\langle G_t',\lambda_t,\zeta_t\rangle\}_{t\in T}$ are covering $(k+1)$-groups of indices q_H and $\{q_t\}_{t\in T}$ of the $(n+1)$-groups H and $\{G_t\}_{t\in T}$, respectively, and in addition $\Psi_s(\varepsilon_t')\lambda_H=\lambda_t\varepsilon_t$ for each $t\in T$. Then for each $t\in T$ we have $q_t=q_H$ and the free product of the $(k+1)$-groups $\{G_t'\}_{t\in T}$ with an amalgamated sub-$(k+1)$-group H' is a covering $(k+1)$-group of index q_H of the free product of the $(n+1)$-groups $\{G_t\}_{t\in T}$ with an amalgamated sub-$(n+1)$-group H.*

Proof. The commutativity of the diagram

$$\begin{tikzcd} \Psi_s(H') \arrow[r, "\Psi_s(\varepsilon_t')"] & \Psi_s(G_t') \\ H \arrow[u, "\lambda_H"] \arrow[r, "\varepsilon_t"] & G_t \arrow[u, "\lambda_t"] \end{tikzcd}$$

together with Theorem 4 of [5] implies the existence of morphisms $\xi_t\colon \mathfrak{C}_{q_H,k+1} \to \mathfrak{C}_{q_t,k+1}$ such that $\xi_t\zeta_H = \zeta_t\varepsilon_t'$. Since the morphisms $\varepsilon_t'\colon H' \to G_t'$ are (by assumption) monomorphisms, the morphisms ξ_t are isomorphisms (cf. Corollary 4 of [8]). Hence $q_H = q_t$. For simplicity we shall write q instead of q_H. In view of Corollary 3 of [5], the $(n+1)$-groups H and $\{G_t\}_{t\in T}$ are derived from $(qk+1)$-groups $\tilde{H}$ and $\tilde{G}_t$, respectively, where in addition (see the remark on the definition of the functor Φ in the Introduction) $H' = \Phi_q(\tilde{H})$, $G_t' = \Phi_q(\tilde{G}_t)$, $\lambda_H = \Psi_m(\tau_{\tilde{H}})$, $\lambda_t = \Psi_m(\tau_t)$ (here $s = mq$). Let $[\tilde{L}; \{\tilde{\alpha}_t\colon \tilde{G}_t \to \tilde{L}\}_{t\in T}]$, $[L; \{\alpha_t\colon G_t \to L\}_{t\in T}]$, $[L'; \{\alpha_t'\colon G_t' \to L'\}_{t\in T}]$ be the amalgamated free products. The functors Ψ_m and Φ_q preserve and reflect amalgamated free products (cf. [6]). Hence $L = \Psi_m(\tilde{L})$, $L' = \Phi_q(\tilde{L})$. Let $\langle L', \tau_{\tilde{L}}\rangle$ be the free covering $(k+1)$-group of the $(qk+1)$-group $\tilde{L}$. Thus, by Corollary 4 of [5], $\langle L', \lambda_L\rangle$ (where $\lambda_L = \Psi_m(\tau_{\tilde{L}})$) is a covering $(k+1)$-group of index q of the $(n+1)$-group L, which completes the proof of Proposition 1.

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On some special limits of n-groups

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1. Introduction

In [7], [8], [9] a systematic study of the category of n-groups has been started. The present paper follows the lines of these papers, especially [7]. We improve the results in [7] (Theorems 1 and 3) on the preservation of projective and inductive limits by the functors Φ and Ψ, respectively, under certain additional conditions on the diagram scheme in consideration. In the present paper we weaken the conditions that turned out in [7] to be sufficient for the preservation of limits so that they become necessary and sufficient. In this way the relation of Φ to projective limits and Ψ to inductive limits becomes clear.

2. Preliminaries

The terminology of this paper is the same as in [5]—[9], where we also discussed relevant notions. Recall briefly some of the most significant notions and notation introduced there.

We assume throughout the paper that $n=sk$ (allowing $k=1$). However, it is sensible (contrary to [7]) to make the assumption that $n>k$. For the case $n=k$ (i.e., $s=1$) some statements become trivial and others become false.

As in [10], n-groups will sometimes be called *polyadic groups,* especially when the arity of the operation is not crucial. Similarly, a sequence $a_1, \ldots, a_m$ of elements of an $(n+1)$-group G is called (following Post) a *polyad* (or shortly an *m-ad*). For convenience such sequences are denoted by $\langle a_1, \ldots, a_m\rangle$. To simplify the notation, in place of $\langle a_1, \ldots, a_{m-r}, \underbrace{b, \ldots, b}_{r}\rangle$ we shall write briefly $\langle a_1, \ldots, a_{m-r}, \overset{r}{b}\rangle$.

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Post has introduced an equivalence relation Θ on the set of all m-ads (for fixed m) of a given $(n+1)$-group (G,f). The relation Θ is defined as follows:

$$\langle a_1, \dots, a_m\rangle \,\Theta\, \langle b_1, \dots, b_m\rangle$$

if and only if for a certain $i=1, \dots, n+1-m$, and for some elements $c_1, \dots, c_{n+1-m} \in G$ we have the equality

$$f(c_1, \dots, c_i, a_1, \dots, a_m, c_{i+1}, \dots, c_{n+1-m}) = f(c_1, \dots, c_i, b_1, \dots, b_m, c_{i+1}, \dots, c_{n+1-m}).$$

One can prove (cf. [10]) that $\langle a_1, \dots, a_m\rangle \,\Theta\, \langle b_1, \dots, b_m\rangle$ implies that for every $i=1, \dots, n+1-m$ and for every sequence $x_1, \dots, x_{n+1-m} \in G$ the following equality holds:

$$f(x_1, \dots, x_i, a_1, \dots, a_m, x_{i+1}, \dots, x_{n+1-m}) =$$
$$= f(x_1, \dots, x_i, b_1, \dots, b_m, x_{i+1}, \dots, x_{n+1-m}).$$

The notion of polyads equivalent with respect to Θ will appear in Lemmas 10—15 and we will make use of the above mentioned theorem in the proof of Theorem 2.

In the paper we deal only with categorical properties of polyadic groups; however, in some proofs (especially in the proof of Theorem 2) we essentially turn to the inner view point, i.e., we consider polyadic groups as sets together with certain operations. This causes some inconsistency in notation. Usually we denote a polyadic group simply by one letter (say G), but whenever the group operation (say f) appears in an explicit form, we write (G,f). To avoid numerous repetitions, we assume that f and g always denote $(n+1)$-group and $(k+1)$-group operations, respectively, and we write (G,f) and (G,g) only to avoid a possible confusion.

The identity morphism is denoted by $e_A\colon A \to A$ or briefly by e, if it is not misleading.

For a $(k+1)$-semigroup (G,g) one can define a new $(sk+1)$-ary operation $g_{(s)}$ by

$$g_{(s)}(x_1, \dots, x_{n+1}) =$$
$$= \underbrace{g(g(\dots g(}_{s}g(x_1, \dots, x_{k+1}), x_{k+2}, \dots, x_{2k+1}), \dots), x_{(s-1)k+2}, \dots, x_{n+1}).$$

If (G,g) is a $(k+1)$-group, then the $(sk+1)$-group $(G, g_{(s)})$ is an $(n+1)$-group, too (cf. [2]). This $(n+1)$-group is said to be a *derived* $(n+1)$-*group of the* $(k+1)$-*group* (G,g) (cf. [2], [5]) and is denoted by $\Psi_s(G,g)$ or shortly by $\Psi_s(G)$ (cf. [3], [7]).

In this way one can obtain a forgetful functor $\Psi_s\colon \mathbf{Gr}_{k+1} \to \mathbf{Gr}_{n+1}$ (in this paper, as in [3], [7], $\mathbf{Gr}_n$ denotes the category of n-groups). The functor Ψ_s has a left adjoint. This is the functor Φ_s: $\mathbf{Gr}_{n+1} \to \mathbf{Gr}_{k+1}$ assigning to each $(n+1)$-group its free covering $(k+1)$-group (cf. [3], [5], [7]).

The notion of a free covering $(k+1)$-group of an $(n+1)$-group, introduced in [3] and investigated in [5], [7], is a generalization of the well-known notion of a free covering group which was introduced by Post in [10].

3. Some lemmas

This section is of auxiliary character. The facts presented can be treated as known and can be found in any basic course on category theory (e.g., [1], [11]) or easily inferred from statements given there. Most of these facts belong to the "folklore" of category theory, and therefore they are given without references. In this section we collect all the auxiliary categorical lemmas that be will applied in later sections.

As is known, each theorem of category theory can be given a dual formulation. To avoid repetitions, we do not formulate the dual versions to the given statements. When referring to the dual version of a lemma given in this section we indicate it by adding an asterisk to the number of the lemma.

In this paper the term functor always means a covariant functor. We use interchangeably the following terms: a small category and a diagram scheme, a functor from a small category and a diagram. The terms diagram scheme and diagram are used especially in dealing with limits. The symbol $\mathscr{D}$ always denotes a small category and the symbol F a functor from that category $\mathscr{D}$ (i.e., F denotes a diagram).

We assume in the lemmas (except Lemma 2) that the categories $\mathscr{K}$, $\mathscr{K}_1$ and $\mathscr{K}_2$ are complete with respect to projective limits of all diagrams, including the empty diagram scheme. As a consequence, these categories possess final objects. Since Lemma 2 is formulated for inductive limits, we assume in it of course the completeness of $\mathscr{K}$ with respect to inductive limits. This convention has to be understood so that the assumptions on the categories $\mathscr{K}$, $\mathscr{K}_1$ and $\mathscr{K}_2$ in the dual versions of the lemmas are also dual.

Lemma 1. *Let a faithful functor Λ: $\mathscr{K}_1 \to \mathscr{K}_2$ have the following property: if $\Lambda(\gamma) = \Lambda(\beta)\delta$ where β: $B \to C$, γ: $A \to C$, δ: $\Lambda(A) \to \Lambda(B)$, then the morphism δ is of the form $\delta = \Lambda(\alpha)$ for some α: $A \to B$. Then the functor Λ reflects projective limits.*

Proof. Let $[G; \{\alpha_D: G \to F(D)\}_{D\in\mathscr{D}}]$ and $[\Lambda(L); \{\Lambda(\pi_D): \Lambda(L) \to \Lambda F(D)\}_{D\in\mathscr{D}}]$ be the projective limits of F: $\mathscr{D} \to \mathscr{K}_1$ and ΛF: $\mathscr{D} \to \mathscr{K}_2$, respectively. From the faithfulness of Λ it follows that the family $\{\pi_D: L \to F(D)\}_{D\in\mathscr{D}}$ is compatible with

F. Therefore there exists a morphism δ such that $\alpha_D\delta=\pi_D$ for $D\in\mathscr{D}$. The family $\{\Lambda(\alpha_D)\colon \Lambda(G)\to\Lambda F(D)\}_{D\in\mathscr{D}}$ is compatible with ΛF, and so there exists a morphism $\eta\colon \Lambda(G)\to\Lambda(L)$ such that $\Lambda(\pi_D)\eta=\Lambda(\alpha_D)$. From the equalities $\Lambda(\pi_D)\eta\Lambda(\delta)=\Lambda(\alpha_D\delta)=\Lambda(\pi_D)$ for $D\in\mathscr{D}$ it follows that $\eta\Lambda(\delta)=e_{\Lambda(L)}$. Hence η is a retraction. On the other hand, from the assumption on Λ it follows that η is of the form $\eta=\Lambda(\mu)$ where $\mu\colon G\to L$. Thus $\Lambda(\alpha_D\delta\mu)=\Lambda(\pi_D)\eta=\Lambda(\alpha_D)$, which, by the faithfulness of Λ, implies that $\alpha_D\delta\mu=\alpha_D$ for $D\in\mathscr{D}$ and hence $\delta\mu=e_G$. Therefore μ is a co-retraction. The functor Λ, being faithful, preserves co-retractions and so $\eta=\Lambda(\mu)$ is a co-retraction. But η is also a retraction, and so η is an isomorphism. As is easy to check, Λ reflects isomorphisms, whence μ is an isomorphism (since $\eta=\Lambda(\mu)$). Therefore $[L;\ \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ is the projective limit of F, which is what was to be proved.

Let a category $\mathscr{K}$ have an initial object U which satisfies an additional condition: for every object $X\in\mathscr{K}$ distinct from U we have $\operatorname{Mor}(X,U)=\emptyset$. It is worth adding that not every category with initial objects has initial objects with this property. For instance, this condition is not satisfied in $\mathbf{Gr}_n$ for $n=2$; however, for $n>2$ (and also in the category of sets) it is satisfied.

Consider a diagram $F\colon \mathscr{D}\to\mathscr{K}$. Let $\mathscr{D}_0$ be the full subcategory of $\mathscr{D}$ consisting of all objects D such that $F(D)\neq U$, and let F_0 be the restriction of F to $\mathscr{D}_0$. Then the following lemma is true.

Lemma 2. *$[L;\ \{\gamma_D\colon F(D)\to L\}_{D\in\mathscr{D}}]$ is the inductive limit of F if and only if $[L;\ \{\gamma_D\colon F_0(D)\to L\}_{D\in\mathscr{D}_0}]$ is the inductive limit of F_0.*

Proof. Let $[L;\ \{\gamma_D\colon F(D)\to L\}_{D\in\mathscr{D}}]$ be the inductive limit of F. The family $\{\gamma_D\colon F_0(D)\to L\}_{D\in\mathscr{D}_0}$ is compatible with F_0. Take an arbitrary family

$$\{\alpha_D\colon F_0(D)\to G\}_{D\in\mathscr{D}_0}$$

with $G\in\mathscr{K}$, which is compatible with F_0. That family can be extended to a family $\{\alpha_D\colon F(D)\to G\}_{D\in\mathscr{D}}$ by choosing as morphism $\alpha_D\colon F(D)\to G$ for $D\notin\mathscr{D}_0$, the only morphism from the initial object $F(D)$ (in the category $\mathscr{K}$) into the object G. It is easy to verify that the extended family of morphisms is compatible with F. Thus there exists a unique morphism $\delta\colon L\to G$ with $\delta\gamma_D=\alpha_D$ for $D\in\mathscr{D}$. Hence, in particular, $\delta\gamma_D=\alpha_D$ for $D\in\mathscr{D}_0$, which proves that $[L;\ \{\gamma_D\colon F_0(D)\to L\}_{D\in\mathscr{D}_0}]$ is the inductive limit of F_0.

Conversely, if $[L;\ \{\gamma_D\colon F_0(D)\to L\}_{D\in\mathscr{D}_0}]$ is the inductive limit of F_0, then the family $\{\gamma_D\}_{D\in\mathscr{D}_0}$ can be extended in a natural way to a family $\{\gamma_D\}_{D\in\mathscr{D}}$. So $[L;\ \{\gamma_D\}_{D\in\mathscr{D}}]$ is already the inductive limit of F. This completes the proof of Lemma 2.

In our further considerations the notions of discrete and connected categories prove to be very useful. A category $\mathscr{K}$ is said to be connected if for every pair of objects $X, Y \in \mathscr{K}$ there exists a finite sequence of objects $A_0, \ldots, A_m \in \mathscr{K}$ such that $A_0 = X$, $A_m = Y$ and $\operatorname{Mor}(A_i, A_{i+1}) \cup \operatorname{Mor}(A_{i+1}, A_i) \neq \emptyset$ for $i = 0, \ldots, m-1$. A full subcategory $\mathscr{K}'$ of a category $\mathscr{K}$ is said to be discrete if for any pair of distinct objects $X, Y \in \mathscr{K}'$ there exists no connected subcategory of $\mathscr{K}$ containing X and Y.

Using the Kuratowski—Zorn Lemma one can prove the following two lemmas.

Lemma 3. *For each object D of a small category $\mathscr{D}$, $\mathscr{D}$ has a maximal connected full subcategory $\mathscr{D}_D$, i.e., a connected subcategory $\mathscr{D}_D$ such that $D \in \mathscr{D}_D$ and for any pair of objects $A, B \in \mathscr{D}$ with $A \in \mathscr{D}_D$, $B \notin \mathscr{D}_D$ we have* $\operatorname{Mor}(A, B) \cup \operatorname{Mor}(B, A) = \emptyset$.

Lemma 4. *Every small category $\mathscr{D}$ has a maximal discrete full subcategory $\mathscr{D}_d$, i.e., a discrete subcategory $\mathscr{D}_d$ such that for each object $X \in \mathscr{D}$ there exists an object $A \in \mathscr{D}_d$ with $X \in \mathscr{D}_A$.*

Consider a diagram $F\colon \mathscr{D} \to \mathscr{K}$ with the following special property: every pair $\alpha, \beta \in \operatorname{Mor}(X, Y)$ with $X, Y \in \mathscr{D}$ satisfies the equality $F(\alpha) = F(\beta)$; furthermore, $F(\alpha)$ is an isomorphism. Let $\mathscr{D}_d$ be a maximal discrete full subcategory of $\mathscr{D}$ and let $F_d\colon \mathscr{D}_d \to \mathscr{K}$ be the restriction of F to the full subcategory $\mathscr{D}_d$. For such a diagram F we have the following lemma.

Lemma 5. *If $[L; \{\alpha_D\colon L \to F(D)\}_{D \in \mathscr{D}}]$ is the projective limit of F, then $[L; \{\alpha_D\colon L \to F_D(D)\}_{D \in \mathscr{D}_d}]$ is the projective limit of F_d.*

Proof. Let $[L; \{\alpha_D\}_{D \in \mathscr{D}}]$ be the projective limit of F. The family

$$\{\alpha_D\colon L \to F_d(D)\}_{D \in \mathscr{D}_d}$$

is compatible with F_d. Take any family $\{\beta_D\colon G \to F_d(D)\}_{D \in \mathscr{D}_d}$, where $D \in \mathscr{D}_d$, which is compatible with F_d. To show that this family can be extended to a family $\{\beta_D\colon G \to F(D)\}_{D \in \mathscr{D}}$ take an arbitrary object $X \in \mathscr{D}$. The definition of $\mathscr{D}_d$ implies the existence of an object $A \in \mathscr{D}_d$ with $X \in \mathscr{D}_A$. Then there exists a sequence of objects $A_0, \ldots, A_m \in \mathscr{D}$ such that $A_0 = A$, $A_m = X$ and $\operatorname{Mor}(A_i, A_{i+1}) \cup \operatorname{Mor}(A_{i+1}, A_i) \neq \emptyset$ for $i = 0, \ldots, m-1$. Let $\alpha_i \in \operatorname{Mor}(A_i, A_{i+1}) \cup \operatorname{Mor}(A_{i+1}, A_i)$ $(i = 0, \ldots, m-1)$. We define morphisms $\mu_i\colon F(A_i) \to F(A_{i+1})$ by putting $\mu_i = F(\alpha_i)$ for $\alpha_i \in \operatorname{Mor}(A_i, A_{i+1})$ and $\mu_i = F^{-1}(\alpha_i)$ for $\alpha_i \in \operatorname{Mor}(A_{i+1}, A_i)$. Let

$$\mu = \mu_{m-1}\mu_{m-2}\ldots\mu_0\colon F(A) \to F(X).$$

It is easy to check that the morphism μ does not depend on the choice of the objects connecting A to X. So we can define β_X as a composition of μ and β_A, i.e., $\beta_X = \mu\beta_A\colon G \to F(X)$. It is evident that the morphism β_X is uniquely determined, independently of the choice of the objects $A_0, \ldots, A_m$. In this way we get the family

$\{\beta_D\colon G\to F(D)\}_{D\in\mathscr{D}}$. From the construction of β_X it follows that this family is compatible with every diagram F_A (here F_A denotes the diagram F restricted to the subcategory $\mathscr{D}_A$) for $A\in\mathscr{D}_d$. Note that $\mathrm{Mor}(X, Y)=\emptyset$ whenever $X\in\mathscr{D}_A$, $Y\in\mathscr{D}_B$, $A, B\in\mathscr{D}_d$ and $A\neq B$. Hence it follows that the family $\{\beta_D\}_{D\in\mathscr{D}}$ is compatible with F. Then there exists a unique morphism $\delta\colon G\to L$ with $\alpha_D\delta=\beta_D$ for $D\in\mathscr{D}$. This shows that $[L; \{\alpha_D\colon L\to F_d(D)\}_{D\in\mathscr{D}_d}]$ is the projective limit of F, which is what was to be proved.

Consider any small category $\mathscr{D}$. This category can be embedded in a small category $\mathscr{D}_e$ which is obtained by adding to $\mathscr{D}$ one (final) object E and a family of morphisms $\{\varepsilon_D\colon D\to E\}_{D\in\mathscr{D}}$, one morphism to each object $D\in\mathscr{D}_e$. The resulting category $\mathscr{D}_e$ is obviously connected. A diagram $F\colon\mathscr{D}\to\mathscr{K}$ can always be extended to $F_e\colon\mathscr{D}_e\to\mathscr{K}$ by defining $F_e(E)$ to be the final object in the category $\mathscr{K}$ and $F_e(\varepsilon_D)$ to be the morphism induced by the final object $F_e(E)$.

Lemma 6. *$[L; \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ is the projective limit of F if and only if $[L; \{\pi_D\colon L\to F_e(D)\}_{D\in\mathscr{D}_e}]$ (with $\pi_E\colon L\to F_e(E)$ the morphism induced by $F_e(E)$) is the projective limit of F_e.*

Lemma 7. *Let a functor $\Lambda\colon\mathscr{K}_1\to\mathscr{K}_2$ preserve projective limits of all diagrams of connected diagram schemes. If $[L; \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ is the projective limit of $F\colon\mathscr{D}\to\mathscr{K}_1$ where $\mathscr{D}$ is any, not necessarily connected, diagram scheme, then $[\Lambda(L); \{\Lambda(\pi_D)\colon\Lambda(L)\to\Lambda F_e(D)\}_{D\in\mathscr{D}_e}]$ is the projective limit of the extended diagram $\Lambda F_e\colon\mathscr{D}_e\to\mathscr{K}_2$.*

Proof. Let $[L; \{\pi_D\}_{D\in\mathscr{D}}]$ be the projective limit of F. According to Lemma 6, $[L; \{\pi_D\colon L\to F_e(D)\}_{D\in\mathscr{D}_e}]$ is the projective limit of $F_e\colon\mathscr{D}_e\to\mathscr{K}_1$. The category $\mathscr{D}_e$ is connected, whence $[\Lambda(L); \{\Lambda(\pi_D)\colon\Lambda(L)\to\Lambda F_e(D)\}_{D\in\mathscr{D}_e}]$ is the projective limit of ΛF_e.

4. The relation of the functor Φ to projective limits

We devote this section to the study of the relation of Φ to projective limits. We start with a lemma.

Lemma 8. *If a composition of morphisms $\gamma\colon\Phi_s(A)\to\Phi_s(D)$ and $\Phi_s(\beta)\colon\Phi_s(D)\to\Phi_s(B)$ with $A, B, D\in\mathrm{Gr}_{n+1}$ is of the form $\Phi_s(\beta)\gamma=\Phi_s(\alpha)$ for some $\alpha\colon A\to B$, then γ is also of the form $\gamma=\Phi_s(\delta)$ where $\delta\colon A\to D$.*

Proof. In view of Theorem 4 of [5] we have the equalities $\zeta_B\Phi_s(\alpha)=\zeta_A$ and $\zeta_B\Phi_s(\beta)=\zeta_D$. Then $\zeta_D\gamma=\zeta_B\Phi_s(\beta)\gamma=\zeta_B\Phi_s(\alpha)=\zeta_A$. Hence, by Theorem 4 of [5], there exists a morphism $\delta\colon A\to D$ such that $\Phi_s(\delta)=\gamma$, which is what was to be proved.

Note that from the faithfulness of Φ it follows that the morphism δ does not depend on the choice of β provided the morphism $\Phi_s(\beta)$ remains the same.

Proposition 1. *If* $[\Phi_s(L);\ \{\Phi_s(\pi_D)\colon \Phi_s(L)\to\Phi_s F(D)\}_{D\in\mathscr{D}}]$ *is the projective limit of* $\Phi_s F\colon \mathscr{D}\to\mathbf{Gr}_{k+1}$, *then* $[L;\ \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ *is the projective limit of* $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$.

Proof. From Lemma 8 it follows that Φ_s satisfies the assumption of Lemma 1. Then, by Lemma 1, Φ_s reflects projective limits, which is what was to be proved.

The theorem converse to Proposition 1 is not true in general. This was already indicated in [7], where an example was shown to demonstrate that Φ does not preserve the Cartesian product. On the other hand, in [7] a sufficient condition was given which, when imposed upon a diagram scheme $\mathscr{D}$, made Φ preserve the projective limits of diagrams $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$.

Now we show that this condition fails to be necessary. Moreover, we characterize the categories $\mathscr{D}$ for which Φ preserves projective limits. Theorem 1 of [7] is a particular case of the theorem given below.

Theorem 1. *Let $\mathscr{D}$ be a nonempty diagram scheme. The functor Φ_s preserves the projective limits of all diagrams $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$ if and only if $\mathscr{D}$ is connected.*

Proof. Let $\mathscr{D}$ be connected and let

$$[L;\ \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}] \text{ and } [L';\ \{\gamma_D\colon L'\to \Phi_s F(D)\}_{D\in\mathscr{D}}]$$

be the projective limits of $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$ and $\Phi_s F\colon \mathscr{D}\to\mathbf{Gr}_{k+1}$, respectively. The family $\{\Phi_s(\pi_D)\}_{D\in\mathscr{D}}$ is compatible with $\Phi_s F$, and so there exists a morphism $\mu\colon \Phi_s(L)\to L'$ with $\gamma_D\mu=\Phi_s(\pi_D)$ for $D\in\mathscr{D}$. Fix some (arbitrary) object $U\in\mathscr{D}$. Then $\gamma_U\mu=\Phi_s(\pi_U)$. By Corollary 6 of [5] the object $[L';\ \{\gamma_D\}_{D\in\mathscr{D}}]$ (determined up to isomorphism) can be chosen in such a way that $L'=\Phi_s(G)$, $\gamma_U=\Phi_s(\eta_U)$, $\mu=\Phi_s(\delta)$, where $G\in\mathbf{Gr}_{n+1}$, $\eta_U\colon G\to F(U)$, $\delta\colon L\to G$. We show that every morphism γ_D is of the form $\gamma_D=\Phi_s(\eta_D)$ for an appropriately chosen $\eta_D\colon G\to F(D)$. To verify this, take an object $A\in\mathscr{D}$. The connectivity of $\mathscr{D}$ implies the existence of a finite sequence of objects $A_0,\dots,A_l\in\mathscr{D}$ such that $A_0=U$, $A_l=A$, $\operatorname{Mor}(A_i,A_{i+1})\cup\operatorname{Mor}(A_{i+1},A_i)\neq\emptyset$ for $i=0,\dots,l-1$. The morphisms η_{A_i} will be constructed by induction, step by step, starting with η_{A_1}. If $\operatorname{Mor}(U,A_1)\neq\emptyset$ (i.e., there exists a morphism $\alpha\colon U\to A_1$), we put $\eta_{A_1}=F(\alpha)\eta_U$. By the compatibility of the family $\{\gamma_D\}_{D\in\mathscr{D}}$ with $\Phi_s F$ and by the faithfulness of Φ_s it follows that $F(\alpha)\eta_U$ does not depend on the choice of α (note that the set $\operatorname{Mor}(F(U),F(A_1))$ may consist of a lot of morphisms!). So the morphism $\eta_{A_1}\colon G\to F(A_1)$ is well-defined. If, on the other hand, $\operatorname{Mor}(A_1,U)\neq\emptyset$ (i.e., there exists an $\alpha\colon A_1\to U$), then by Lemma 8 there exists a morphism $\eta_{A_1}\colon G\to F(A_1)$ which is well-defined (independently of the choice of α). In this way we get the morphism η_{A_1}. Further on, to obtain $\eta_{A_{i+1}}$ from η_{A_i} we proceed as in the first step, depending on which one of the sets $\mathrm{Mor}(A_i, A_{i+1})$ or $\mathrm{Mor}(A_{i+1}, A_i)$ is nonempty. After performing l such steps we obtain $\eta_A\colon G\to F(A)$. As is easy to verify the family $\{\eta_D\}_{D\in\mathscr{D}}$ is compatible with F. So there exists a morphism $\varrho\colon G\to L$ with $\pi_D\varrho=\eta_D$ for $D\in\mathscr{D}$ (since by assumption $[L; \{\pi_D\}_{D\in\mathscr{D}}]$ is the projective limit of F). The equalities $\pi_D\varrho\delta=\eta_D\delta=\pi_D$ hold for every $D\in\mathscr{D}$, whence $\varrho\delta=e_L$. Similarly, from the equalities $\gamma_D\Phi_s(\delta\varrho)=$ $=\Phi_s(\pi_D\varrho)=\Phi_s(\eta_D)$ it follows that $\Phi_s(\delta\varrho)=e_{L'}$, whence $\delta\varrho=e_G$. Then δ (thus also $\Phi_s(\delta)$) is an isomorphism, which proves that $[\Phi_s(L); \{\Phi_s(\pi_D)\}_{D\in\mathscr{D}}]$ is the projective limit of $\Phi_s F$.

Conversely, let Φ_s preserve the projective limits of all diagrams $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$ for a fixed category $\mathscr{D}$. Consider the functor $F\colon\mathscr{D}\to\mathbf{Gr}_{n+1}$ defined as follows: for $D\in\mathscr{D}$ let $F(D)$ be a one-element $(n+1)$-group and for $\alpha\colon X\to Y$ let $F(\alpha)$ be the unique morphism from $F(X)$ onto $F(Y)$. Let $[L; \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ be the projective limit of F. Since all objects $F(D)$ for $D\in\mathscr{D}$ are final in $\mathbf{Gr}_{n+1}$, the object L is also a final object in $\mathbf{Gr}_{n+1}$, i.e. a one-element $(n+1)$-group. Thus $\Phi_s(L)=\mathfrak{C}_{s,k+1}$ (cf. [3], [7]). By assumption $[\Phi_s(L); \{\Phi_s(\pi_D)\colon\Phi_s(L)\to\Phi_s F(D)\}_{D\in\mathscr{D}}]$ is the projective limit of $\Phi_s F$. However, we can see that for any $\alpha\colon X\to Y$ with $X, Y\in\mathscr{D}$, the morphism $\Phi_s F(\alpha)$ is the only isomorphism of the cyclic $(k+1)$-group $\Phi_s F(A)=\mathfrak{C}_{s,k+1}$ onto the cyclic $(k+1)$-group $\Phi_s F(B)=\mathfrak{C}_{s,k+1}$ with the property $\Phi_s F(\alpha)(0)=0$. Therefore, in view of Lemma 5, $[\Phi_s(L); \{\Phi_s(\pi_D)\colon \Phi_s(L)\to\Phi_s F_d(D)\}_{D\in\mathscr{D}_d}]$ (where F_d is the restriction of F to $\mathscr{D}_d$) is the projective limit of $\Phi_s F_d\colon\mathscr{D}_d\to\mathbf{Gr}_{k+1}$. Since $\mathscr{D}_d$ is discrete, $[\Phi_s(L); \{\Phi_s(\pi_D)\}_{D\in\mathscr{D}_d}]$ is simply the Cartesian product of the family of $(k+1)$-groups $\{\Phi_s F(D)\}_{D\in\mathscr{D}_d}$, i.e., the Cartesian power of the cyclic $(k+1)$-group $\mathfrak{C}_{s,k+1}$. On the other hand, the $(k+1)$-group $\Phi_s(L)$ is the cyclic $(k+1)$-group $\mathfrak{C}_{s,k+1}$, whence the family $\{\Phi_s F(D)\}_{D\in\mathscr{D}_d}$ consists of one element, which means that $\mathscr{D}_d$ consists only of one object. Hence $\mathscr{D}$ is a connected category. This completes the proof of Theorem 1.

From Proposition 1 and Theorem 1 we immediately infer the following

Corollary 1. *Let $\mathscr{D}$ be a connected diagram scheme. Then* $[L; \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ *is the projective limit of* $F\colon\mathscr{D}\to\mathbf{Gr}_{n+1}$ *if and only if*

$$[\Phi_s(L); \{\Phi_s(\pi_D)\colon \Phi_s(L)\to\Phi_s F(D)\}_{D\in\mathscr{D}}]$$

is the projective limit of $\Phi_s F\colon\mathscr{D}\to\mathbf{Gr}_{k+1}$.

The question arises what are the free covering $(k+1)$-groups of projective limits of diagrams $F\colon\mathscr{D}\to\mathbf{Gr}_{n+1}$ in the case when the diagram scheme is not connected. Note that a partial answer was given in Lemma 7. In our case of the category of n-groups a more specific answer can be given.

Take an arbitrary diagram scheme $\mathscr{D}$ and a diagram $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$. Let $\mathscr{D}_e$ and F_e have the same meaning as in Section 3. As is easy to see, $\Phi_s F(E)$ is nothing else than the cyclic $(k+1)$-group $\mathfrak{C}_{s,k+1}$ (cf. [3], [5]) and $\Phi_s F(\varepsilon_D)\colon \Phi_s F(D)\to\mathfrak{C}_{s,k+1}$ are simply the morphisms $\zeta_D\colon \Phi_s F(D)\to\mathfrak{C}_{s,k+1}$ (cf. [3], [5]). Every diagram $\Phi_s F$ can be extended to $\Phi_s F_e\colon \mathscr{D}_e\to\mathbf{Gr}_{k+1}$ by adding the object $\mathfrak{C}_{s,k+1}$ and the family of morphisms $\{\zeta_D\colon \Phi_s F(D)\to\mathfrak{C}_{s,k+1}\}_{D\in\mathscr{D}}$. Hence we obtain

Proposition 2. *If* $[L;\ \{\pi_D\colon L\to F(D)\}_{D\in\mathscr{D}}]$ *is the projective limit of* $F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$, *then* $[\Phi_s(L);\ \{\Phi_s(\pi_D)\colon \Phi_s(L)\to\Phi_s F_e(D)\}_{D\in\mathscr{D}_e}]$ *(where* $\Phi_s(\pi_E)=\zeta\colon \Phi_s(L)\to\mathfrak{C}_{s,k+1}$*) is the projective limit of the extended diagram* $\Phi_s F_e\colon \mathscr{D}_e\to\mathbf{Gr}_{k+1}$.

In this way free covering $(k+1)$-groups of projective limits are always projective limits, but perhaps of an extended diagram.

5. The relation of the functor Ψ to inductive limits

As in the dual case of Φ and projective limits, the functor Ψ reflects inductive limits. To show this fact, we need the following lemma.

Lemma 9. *If a composition of morphisms* $\Psi_s(\alpha)\colon \Psi_s(A)\to\Psi_s(D)$ *and* $\gamma\colon \Psi_s(D)\to\Psi_s(B)$ *with* $A, B, D\in\mathbf{Gr}_{k+1}$ *is of the form* $\gamma\Psi_s(\alpha)=\Psi_s(\beta)$ *for some* $\beta\colon A\to B$, *then* γ *is of the form* $\gamma=\Psi_s(\delta)$ *where* $\delta\colon D\to B$.

Proof. Take any element $c_0\in A$ and let d_0 be the skew element to c_0 in the $(k+1)$-group A. Let $d=\alpha(d_0)$, $c=\alpha(c_0)$. It is easy to check that d is skew to c in the $(k+1)$-group D. On the other hand, the element $\gamma(d)=\gamma\alpha(d_0)=\beta(d_0)$ is skew to $\gamma(c)=\gamma\alpha(c_0)=\beta(c_0)$ (since $\beta\colon A\to B$). Hence, by Corollary 3 of [6], γ is a homomorphism of $(k+1)$-groups, which is what was to be proved.

Proposition 3. *If* $[\Psi_s(L);\ \{\Psi_s(\gamma_D)\to\Psi_s(L)\}_{D\in\mathscr{D}}]$ *is the inductive limit of* $\Psi_s F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$, *then* $[L;\ \{\gamma_D\colon F(D)\to L\}_{D\in\mathscr{D}}]$ *is the inductive limit of* $F\colon \mathscr{D}\to\mathbf{Gr}_{k+1}$.

Proof. Lemma 9 shows that Ψ_s satisfies the assumption of Lemma 1*. Thus Ψ_s reflects inductive limits.

Theorem 1 describes the preservation of projective limits by Φ. Theorem 2 (dual to Theorem 1), formulated below, gives a condition characterizing those diagram schemes for which Ψ preserves inductive limits. The proof of Theorem 2 proceeds via complicated calculations. To stress the main idea of the proof a part of those calculations is presented in a sequence of five lemmas. All those lemmas have some common assumptions. To avoid repetition, we formulate these assumptions before starting the lemmas.

Given a diagram $F\colon \mathscr{D}\to \mathbf{Gr}_{k+1}$, let $[L'; \{\gamma_D\colon \Psi_s F(D)\to L'\}_{D\in\mathscr{D}}]$ be the inductive limit of $\Psi_s F$. As was mentioned in Section 2, we denote by g the $(k+1)$-group operation in all $(k+1)$-groups (i.e., $F(D)$, L, G), while by f the $(n+1)$-group operation in all $(n+1)$-groups (i.e., $\Psi_s F(D)$, $\Psi_s(L)$, $\Psi_s(G)$, L'). To avoid confusion we assume that the symbol $\bar{x}$ always denotes the element skew to x in the corresponding $(n+1)$-group (but not in a $(k+1)$-group). The equivalence of polyads is understood in the sense of [10].

Lemma 10. *If for some objects $A, B\in\mathscr{D}$ we have $\operatorname{Mor}(A,B)\cup\operatorname{Mor}(B,A)\neq\emptyset$, then for an arbitrary element $a\in F(A)$ there exists an element $b\in F(B)$ such that the k-ads $\langle\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}\rangle$ and $\langle\gamma_B(g_{(s-1)}(\bar{b}, \overset{n-k}{b})), \overset{k-1}{\gamma_B(b)}\rangle$ are equivalent.*

Proof. Let $\alpha\colon A\to B$ and $b=F(\alpha)(a)$. Take elements $x_1, \ldots, x_{n+1-k}\in L'$. Then

$$\begin{aligned}
&f(x_1, \ldots, x_{n+1-k}, \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}) =\\
&\quad = f(\ldots, x_{n+1-k}, \gamma_B\Psi_s F(\alpha)(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_B\Psi_s F(\alpha)(a)}) =\\
&\quad = f(\ldots, x_{n+1-k}, \gamma_B(F(\alpha)(g_{(s-1)}(\bar{a}, \overset{n-k}{a}))), \overset{k-1}{\gamma_B(F(\alpha)(a))}) =\\
&\quad = f(\ldots, x_{n+1-k}, \gamma_B(g_{(s-1)}(F(\alpha)(\bar{a}), \overset{n-k}{F(\alpha)(a)})), \overset{k-1}{\gamma_B(b)}) =\\
&\quad = f(x_1, \ldots, x_{n+1-k}, \gamma_B(g_{(s-1)}(\bar{b}, \overset{n-k}{b})), \overset{k-1}{\gamma_B(b)}).
\end{aligned}$$

Next, let $\beta\colon B\to A$. Take an arbitrary element $b\in F(B)$. Then

$$\begin{aligned}
&f(x_1, \ldots, x_{n+1-k}, \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}) =\\
&= f_{(2)}(\ldots, x_{n+1-k}, \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_B(\bar{b}), \overset{n-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, f(\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A\Psi_s F(\beta)(\bar{b}), \overset{n-k}{\gamma_A\Psi_s F(\beta)(b)}), \overset{k-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, f(\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(F(\beta)(\bar{b})), \overset{n-k}{\gamma_A(F(\beta)(b))}), \overset{k-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, \gamma_A(g_{(s)}(g_{(s-1)}(\bar{a}, \overset{n-k}{a}), \overset{k-1}{a}, F(\beta)(\bar{b}), \overset{n-k}{F(\beta)(b)})), \overset{k-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, \gamma_A(g_{(s-1)}(g_{(s)}(\bar{a}, \overset{n-k}{a}, \overset{k-1}{a}, F(\beta)(\bar{b})), \overset{n-k}{F(\beta)(b)})), \overset{k-1}{\gamma_B(b)})) =\\
&= f(\ldots, x_{n+1-k}, \gamma_A(g_{(s-1)}(f(\bar{a}, \overset{n-1}{a}, F(\beta)(\bar{b})), \overset{n-k}{F(\beta)(b)})), \overset{k-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, \gamma_A(F(\beta)(g_{(s-1)}(\bar{b}, \overset{n-k}{b}))), \overset{k-1}{\gamma_B(b)}) =\\
&= f(\ldots, x_{n+1-k}, \gamma_A\Psi_s F(\beta)(g_{(s-1)}(\bar{b}, \overset{n-k}{b})), \overset{k-1}{\gamma_B(b)}) =\\
&= f(x_1, \ldots, x_{n+1-k}, \gamma_B(g_{(s-1)}(\bar{b}, \overset{n-k}{b})), \overset{k-1}{\gamma_B(b)}).
\end{aligned}$$

Lemma 11. *If a category $\mathscr{D}$ is connected, then for every pair of objects $A, B\in\mathscr{D}$ and for any element $a\in F(A)$ there exists an element $b\in F(B)$ such that the k-ads $\langle\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}\rangle$ and $\langle\gamma_B(g_{(s-1)}(\bar{b}, \overset{n-k}{b})), \overset{k-1}{\gamma_B(b)}\rangle$ are equivalent.*

Proof. The category $\mathscr{D}$ is connected by assumption, so for any pair of objects $A, B\in\mathscr{D}$ there exists a sequence of objects $A_0, \ldots, A_r\in\mathscr{D}$ such that $A_0=A$, $A_r=B$ and $\mathrm{Mor}(A_i, A_{i+1})\cup\mathrm{Mor}(A_{i+1}, A_i)\neq\emptyset$ for $i=0, \ldots, r-1$. Applying Lemma 10 r times, we infer the equivalence of the polyads in question.

Lemma 12. *For any elements $a, y\in F(A)$ the $(k+1)$-ads*

$$\langle\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(y)\rangle \quad \textit{and} \quad \langle\gamma_A(y), \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}\rangle$$

are equivalent.

Proof. Let $x_1, \ldots, x_{n-k}\in L'$. Then

$$
\begin{aligned}
& f(x_1, \ldots, x_{n-k}, \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(y)) = \\
&= f_{(2)}(\ldots, x_{n-k}, \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(y), \gamma_A(\bar{a}), \overset{n-1}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, f(\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(y), \gamma_A(\bar{a}), \overset{n-k-1}{\gamma_A(a)}), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s)}(g_{(s-1)}(\bar{a}, \overset{n-k}{a}), \overset{k-1}{a}, y, \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s-1)}(g_{(s)}(\bar{a}, \overset{n-k}{a}, \overset{k-1}{a}, y), \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s-1)}(f(\bar{a}, \overset{n-1}{a}, y), \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s-1)}(f(y, \bar{a}, \overset{n-1}{a}), \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s-1)}(g_{(s)}(y, \bar{a}, \overset{n-1}{a}), \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(g_{(s)}(y, g_{(s-1)}(\bar{a}, \overset{n-k}{a}), \overset{k-1}{a}, \bar{a}, \overset{n-k-1}{a})), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, f(\gamma_A(y), \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(\bar{a}), \overset{n-1-k}{\gamma_A(a)}), \overset{k}{\gamma_A(a)}) = \\
&= f(\ldots, x_{n-k}, \gamma_A(y), f(\gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}, \gamma_A(\bar{a}), \overset{n-k-1}{\gamma_A(a)}, \gamma_A(a)), \overset{k-1}{\gamma_A(a)}) = \\
&= f(x_1, \ldots, x_{n-k}, \gamma_A(y), \gamma_A(g_{(s-1)}(\bar{a}, \overset{n-k}{a})), \overset{k-1}{\gamma_A(a)}).
\end{aligned}
$$

Lemma 13. *If for some objects $A, B\in\mathscr{D}$ we have $\mathrm{Mor}(A, B)\cup\mathrm{Mor}(B, A)\neq\emptyset$, then for an arbitrary element $a\in F(A)$ there exists an element $b\in F(B)$ such that the $(n-k)$-ads $\langle\overset{s-1}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}\rangle$ and $\langle\overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)}{\gamma_B(b)}\rangle$ are equivalent (here a_s and*

b_s denote the skew elements to a and b in the $(k+1)$-groups $F(A)$ and $F(B)$, respectively).

Proof. Let $\alpha\colon A\to B$ and $b=F(\alpha)(a)$ (hence also $b_s=F(\alpha)(a_s)$). Take elements $x_1, \ldots, x_{k+1}\in L'$. Then

$$f(x_1, \ldots, x_{k+1}, \overset{s-1}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}) = f(\ldots, x_{k+1}, \overset{s-1}{\gamma_B\Psi_s F(\alpha)(a_s)}, \overset{(k-1)(s-1)}{\gamma_B\Psi_s F(\alpha)(a)}) =$$

$$= f(\ldots, x_{k+1}, \overset{s-1}{\gamma_B(F(\alpha)(a_s))}, \overset{(k-1)(s-1)}{\gamma_B(F(\alpha)(a))}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)}{\gamma_B(b)}).$$

Next, let $\beta\colon B\to A$. Take an arbitrary element $b\in F(B)$. Let b_s be the skew element to b in the $(k+1)$-group $F(B)$. Then

$$f(x_1, \ldots, x_{k+1}, \overset{s-1}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}) =$$

$$= f(\ldots, x_{k+1}, \gamma_A(g_{(s)}(\overset{s-1}{F(\beta)(b_s)}, \overset{(k-1)(s-1)}{F(\beta)(b)},$$
$$F(\beta)(b_s), \overset{k-1}{F(\beta)(b)}, a_s)), \overset{s-2}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}) =$$

$$= f(\ldots, x_{k+1}, f(\overset{s-1}{\gamma_A(F(\beta)(b_s))}, \overset{(k-1)(s-1)}{\gamma_A(F(\beta)(b))},$$
$$\gamma_A(F(\beta)(b_s)), \overset{k-1}{\gamma_A(F(\beta)(b))}, \gamma_A(a_s)), \overset{s-2}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}) =$$

$$= f(\ldots, x_{k+1}, f(\overset{s-1}{\gamma_A\Psi_s F(\beta)(b_s)}, \overset{(k-1)(s-1)}{\gamma_A\Psi_s F(\beta)(b)},$$
$$\gamma_A\Psi_s F(b_s), \overset{k-1}{\gamma_A\Psi_s F(b)}, \gamma_A(a_s)), \overset{s-2}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}) =$$

$$= f(\ldots, x_{k+1}, \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)-1}{\gamma_B(b)}, f(\gamma_A(F(\beta)(b)),$$
$$\gamma_A(F(\beta)(b_s)), \overset{k-1}{\gamma_A(F(\beta)(b))}, \gamma_A(a_s), \overset{s-2}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)})) =$$

$$= f(\ldots, x_{k+1}, \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)-1}{\gamma_B(b)}, \gamma_A(g_{(s)}(F(\beta)(b),$$
$$F(\beta)(b_s), \overset{k-1}{F(\beta)(b)}, \overset{s-1}{a_s}, \overset{(k-1)(s-1)}{a}))) =$$

$$= f(\ldots, x_{k+1}, \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)-1}{\gamma_B(b)}, \gamma_A(F(\beta)(b))) =$$

$$= f(x_1, \ldots, x_{k+1}, \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)}{\gamma_B(b)}).$$

Lemma 14. *If a category $\mathcal{D}$ is connected, then for every pair of objects $A, B\in\mathcal{D}$ and for an arbitrary element $a\in F(A)$ there exists an element $b\in F(B)$ such that the $(n-k)$-ads $\langle \overset{s-1}{\gamma_A(a_s)}, \overset{(k-1)(s-1)}{\gamma_A(a)}\rangle$ and $\langle \overset{s-1}{\gamma_B(b_s)}, \overset{(k-1)(s-1)}{\gamma_B(b)}\rangle$ are equivalent (here a_s and b_s denote the skew element to a and b in the $(k+1)$-groups $F(A)$ and $F(B)$, respectively).*

Proof. The proof of this lemma is analogous to that of Lemma 11.

Lemma 15. *If a, b are any elements of an object $F(D)$ with $D\in\mathcal{D}$, then the k-ads $\langle\gamma_D(g_{(s-1)}(\bar a, \overset{n-k}{a})), \overset{k-1}{\gamma_D(a)}\rangle$ and $\langle\gamma_D(g_{(s-1)}(\bar b, \overset{n-k}{b})), \overset{k-1}{\gamma_D(b)}\rangle$ are equivalent.*

Proof. Let $x_1, \ldots, x_{n+1-k}\in L'$. Then

$$f(x_1, \ldots, x_{n+1-k}, \gamma_D(g_{(s-1)}(\bar a, \overset{n-k}{a})), \overset{k-1}{\gamma_D(a)}) =$$

$$= f_{(2)}(\ldots, x_{n+1-k}, \gamma_D(g_{(s-1)}(\bar a, \overset{n-k}{a})), \overset{k-1}{\gamma_D(a)}, \overset{n-1}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, f(\gamma_D(g_{(s-1)}(\bar a, \overset{n-k}{a})), \overset{k-1}{\gamma_D(a)}, \overset{n+1-k}{\gamma_D(b)}), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, \gamma_D(g_{(s)}(g_{(s-1)}(\bar a, \overset{n-k}{a}), \overset{k-1}{a}, \overset{n+1-k}{b})), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, \gamma_D(g_{(s-1)}(f(\bar a, \overset{n-1}{a}, b), \overset{n-k}{b})), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, \gamma_D(g_{(s-1)}(f(\bar b, \overset{n-1}{b}, b), \overset{n-k}{b})), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, \gamma_D(g_{(s)}(g_{(s-1)}(\bar b, \overset{n-k}{b}), \overset{k-1}{b}, \overset{n+1-k}{b})), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(\ldots, x_{n+1-k}, f(\gamma_D(g_{(s-1)}(\bar b, \overset{n-k}{b})), \overset{k-1}{\gamma_D(b)}, \overset{n+1-k}{\gamma_D(b)}), \overset{k-2}{\gamma_D(b)}, \gamma_D(\bar b)) =$$

$$= f(x_1, \ldots, x_{n+1-k}, \gamma_D(g_{(s-1)}(\bar b, \overset{n-k}{b})), \overset{k-1}{\gamma_D(b)}).$$

Theorem 2. *Given a diagram scheme $\mathcal{D}$, assume that $\mathcal{D}$ is nonempty or $k>1$. The functor Ψ_s preserves the inductive limits of all diagrams $F\colon \mathcal{D}\to\mathbf{Gr}_{k+1}$ if and only if the full subcategory $\mathcal{D}_0$ of $\mathcal{D}$, which consists of those objects D for which $\Psi_s F(D)$ is not an initial object in $\mathbf{Gr}_{n+1}$, is connected.*

Proof. Assume that the nonempty category $\mathcal{D}_0$ is connected. Let

$$[L; \{\sigma_D\colon F(D)\to L\}_{D\in\mathcal{D}}] \quad\text{and}\quad [L'; \{\gamma_D\colon \Psi_s F(D)\to L'\}_{D\in\mathcal{D}}]$$

be the inductive limits of $F\colon \mathcal{D}\to\mathbf{Gr}_{k+1}$ and $\Psi_s F\colon \mathcal{D}\to\mathbf{Gr}_{n+1}$, respectively. Note that for $k=1$ the diagram scheme $\mathcal{D}_0$ is equal to $\mathcal{D}$ (since $\Psi_n F(D)$ is not an empty $(n+1)$-group). On the other hand, for $k>1$ the full subcategory of $\mathcal{D}$ consisting of those objects for which $F(D)$ (but not $\Psi_s F(D)$ as in the definition of $\mathcal{D}_0$) is a nonempty $(k+1)$-group, equals $\mathcal{D}_0$ (since $F(D)$ is nonempty iff $\Psi_s F(D)$ is nonempty). In view of Lemma 2, $[L; \{\sigma_D\colon F_0(D)\to L\}_{D\in\mathcal{D}_0}]$ and $[L'; \{\gamma_D\colon \Psi_s F_0(D)\to L'\}_{D\in\mathcal{D}_0}]$ (where F_0 is the restriction of F to $\mathcal{D}_0$) are also the inductive limits of F_0 and $\Psi_s F_0$.

Take an arbitrary (but fixed) object $C \in \mathcal{D}_0$ and choose an element $c_0 \in C$. Let $c_s \in C$ be the skew element to c_0 in the $(k+1)$-group $F(C)$. We prove that the element $d = \gamma_C(c_s)$ is an s-skew element to the element $c = \gamma_C(c_0)$ in the $(n+1)$-group L'. Indeed, for any element $x \in L'$ we have

$$f(\overset{s}{d}, \overset{(k-1)s}{c}, x) = f(\overset{s}{\gamma_C(c_s)}, \overset{(k-1)s}{\gamma_C(c_0)}, f(\gamma_C(\bar{c}_0), \overset{n-1}{\gamma_C(c_0)}, x)) =$$

$$= f(f(\overset{s}{\gamma_C(c_s)}, \overset{(k-1)s}{\gamma_C(c_0)}, \gamma_C(\bar{c}_0)), \overset{n-1}{\gamma_C(c_0)}, x) =$$

$$= f(\gamma_C(g_{(s)}(\overset{s}{c_s}, \overset{(k-1)s}{c_0}, \bar{c}_0)), \overset{n-1}{\gamma_C(c_0)}, x) = f(\gamma_C(\bar{c}_0), \overset{n-1}{\gamma_C(c_0)}, x) = x,$$

which shows that the elements d and c satisfy condition 1° of the definition of an s-skew element (cf. [6]).

Next, take elements $x_1, \ldots, x_{n+1-k} \in L'$ and fix $i = 1, \ldots, n+1-k$. Then

$$f(x_1, \ldots, x_i, d, \overset{k-1}{c}, x_{i+1}, \ldots, x_{n+1-k}) =$$

$$= f(\ldots, x_i, \gamma_C(c_s), \overset{k-1}{\gamma_C(c_0)}, f(\gamma_C(\bar{c}_0), \overset{n-1}{\gamma_C(c_0)}, x_{i+1}), x_{i+2}, \ldots) =$$

$$= f(\ldots, x_i, f(\gamma_C(c_s), \overset{k-1}{\gamma_C(c_0)}, \gamma_C(\bar{c}_0), \overset{n-k}{\gamma_C(c_0)}), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f(\ldots, x_i, \gamma_C(g_{(s)}(c_s, \overset{k-1}{c_0}, \bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f(\ldots, x_i, \gamma_C(g_{(s)}(\overset{k-1}{c_0}, c_s, \bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f(\ldots, x_i, f(\overset{k-1}{\gamma_C(c_0)}, \gamma_C(c_s), \gamma_C(\bar{c}_0), \overset{n-k}{\gamma_C(c_0)}), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f(\ldots, x_i, \overset{k-1}{c}, d, f(\gamma_C(\bar{c}_0), \overset{n-k}{\gamma_C(c_0)}, \overset{k-1}{\gamma_C(c_0)}, x_{i+1}), x_{i+2}, \ldots) =$$

$$= f(\ldots, x_i, \overset{k-1}{c}, d, x_{i+1}, \ldots, x_{n+1-k}).$$

Moreover, by the definition of the $(n+1)$-group L' (as an inductive limit of $(n+1)$-groups) it follows that the elements of L' are generated by the set $\bigcup_{D \in \mathcal{D}} \gamma_D(F(D))$. Hence in particular $x_i = f_{(\cdot)}(\gamma_{D_1}(y_1), \ldots, \gamma_{D_r}(y_r))$, where $r \equiv 1 \pmod{n}$, $y_j \in F(D_j)$ for $j = 1, \ldots, r$, and $x_1 = f_{(\cdot)}(\gamma_{A_1}(z_1), \ldots, \gamma_{A_t}(z_t))$, where $t \equiv 1 \pmod{n}$, $z_j \in F(A_j)$ for $j = 1, \ldots, t$.

To explain the sequence of transformations, we will write the numbers of the lemmas we refer to, below the sign of equality. The elements chosen according to Lemmas 10 and 11 will be denoted by d_i in the $(k+1)$-groups $F(D_i)$ and by a_i in the

$(k+1)$-groups $F(A_i)$. Then

$$f(x_1, \ldots, x_i, d, \overset{k-1}{c}, x_{i+1}, \ldots, x_{n+1-k}) =$$

$$= f(\ldots, x_i, \gamma_C(c_s), \overset{k-1}{\gamma_C(c_0)}, f(\gamma_C(\bar{c}_0), \overset{n-1}{\gamma_C(c_0)}, x_{i+1}), x_{i+2}, \ldots) =$$

$$= f(\ldots, x_i, f(\gamma_C(c_s), \overset{k-1}{\gamma_C(c_0)}, \gamma_C(\bar{c}_0), \overset{n-k}{\gamma_C(c_0)}), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f(\ldots, x_{i-1}, f_{(\cdot)}(\gamma_{D_1}(y_1), \ldots, \gamma_{D_r}(y_r)), \gamma_C(g_{(s)}(c_s, \overset{k-1}{c_0}, \bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$= f_{(\cdot)}(\ldots, x_{i-1}, \gamma_{D_1}(y_1), \ldots, \gamma_{D_r}(y_r), \gamma_C(g_{(s-1)}(\bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_{i+1}, \ldots) =$$

$$\underset{(11)}{=} f_{(\cdot)}(\ldots, x_{i-1}, \gamma_{D_1}(y_1), \ldots, \gamma_{D_{r-1}}(y_{r-1}), \gamma_{D_r}(y_r), \gamma_{D_r}(g_{(s-1)}(\bar{d}_r, \overset{n-k}{d_r})), \overset{k-1}{\gamma_{D_r}(d_r)}, x_{i+1}, \ldots) =$$

$$\underset{(12)}{=} f_{(\cdot)}(\ldots, x_{i-1}, \gamma_{D_1}(y_1), \ldots, \gamma_{D_{r-1}}(y_{r-1}), \gamma_{D_r}(g_{(s-1)}(\bar{d}_r, \overset{n-k}{d_r})), \overset{k-1}{\gamma_{D_r}(d_r)}, \gamma_{D_r}(y_r), x_{i+1}, \ldots) =$$

$$\underset{(11)}{=} f_{(\cdot)}(\ldots, x_{i-1}, \gamma_{D_1}(y_1), \ldots, \gamma_{D_{r-1}}(y_{r-1}), \gamma_{D_{r-1}}(g_{(s-1)}(\bar{d}_{r-1}, \overset{n-k}{d_{r-1}})),$$

$$\overset{k-1}{\gamma_{D_{r-1}}(d_{r-1})}, \gamma_{D_r}(y_r), x_{i+1}, \ldots) =$$

$$\underset{(12)}{=} f_{(\cdot)}(\ldots, x_{i-1}, \gamma_{D_1}(y_1), \ldots, \gamma_{D_{r-2}}(y_{r-2}), \gamma_{D_{r-1}}(g_{(s-1)}(\bar{d}_{r-1}, \overset{n-k}{d_{r-1}})),$$

$$\overset{k-1}{\gamma_{D_{r-1}}(d_{r-1})}, \gamma_{D_{r-1}}(y_{r-1}), \gamma_{D_r}(y_r), x_{i+1}, \ldots) = \ldots =$$

$$= f_{(\cdot)}(\gamma_{A_1}(g_{(s-1)}(\bar{a}_1, \overset{n-k}{a_1})), \overset{k-1}{\gamma_{A_1}(a_1)}, \gamma_{A_1}(z_1), \ldots, \gamma_{A_t}(z_t), x_2, \ldots) =$$

$$\underset{(11),(15)}{=} f(\gamma_C(g_{(s-1)}(\bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_1, \ldots) =$$

$$= f(\gamma_C(g_{(s)}(c_s, \overset{k-1}{c_0}, \bar{c}_0, \overset{n-k}{c_0})), \overset{k-1}{\gamma_C(c_0)}, x_1, \ldots) =$$

$$= f(f(\gamma_C(c_s), \overset{k-1}{\gamma_C(c_0)}, \gamma_C(\bar{c}_0), \overset{n-k}{\gamma_C(c_0)}), \overset{k-1}{\gamma_C(c_0)}, x_1, \ldots) = f(d, \overset{k-1}{c}, x_1, \ldots),$$

which proves that the elements d and c satisfy condition 2° of the definition of an s-skew element (cf. [6]). Therefore the element d is s-skew to c in the $(n+1)$-group L'. Thus, by Proposition 1 of [6] (cf. also Theorem 5 of [5]), the $(n+1)$-group (L', f) is derived from some $(k+1)$-group (G, g), i.e., $\Psi_s(G) = L'$. Furthermore, the $(k+1)$-group operation g in G is given by

$$g(x_1, \ldots, x_{k+1}) = f(x_1, \ldots, x_{k+1}, \overset{s-1}{d}, \overset{(k-1)(s-1)}{c}).$$

By Corollary 2 of [6] the element d is skew to c in that $(k+1)$-group G. Let $x_1, \ldots, x_{k+1} \in F(D)$ for any $D \in \mathscr{D}$. Then

$$g(\gamma_D(x_1), \ldots, \gamma_D(x_{k+1})) = f\big(\gamma_D(x_1), \ldots, \gamma_D(x_{k+1}), \overset{s-1}{\gamma_C(c_s)}, \overset{(k-1)(s-1)}{\gamma_C(c_0)}\big) =$$

$$\underset{(14)}{=} f\big(\gamma_D(x_1), \ldots, \gamma_D(x_{k+1}), \overset{s-1}{\gamma_D(d_s)}, \overset{(k-1)(s-1)}{\gamma_D(d_0)}\big) = \gamma_D(g(x_1, \ldots, x_{k+1})),$$

where d_0 is some element of $F(D)$ and d_s is the skew element to d_0 in this $(k+1)$-group $F(D)$. This shows that γ_D is of the form $\gamma_D = \Psi_s(\beta_D)$ with $\beta_D\colon F(D) \to G$ for $D \in \mathscr{D}$. The faithfulness of Ψ_s implies the compatibility of the family $\{\beta_D\colon F_0(D) \to G\}_{D \in \mathscr{D}_0}$ with F_0. So there exists a unique morphism $\delta\colon L \to G$ such that $\delta\sigma_D = \beta_D$ for $D \in \mathscr{D}_0$. The family $\{\Psi_s(\sigma_D)\}_{D \in \mathscr{D}}$ is compatible with $\Psi_s F$, which implies the existence of a unique morphism $\omega\colon \Psi_s(G) \to \Psi_s(L)$ with $\omega\gamma_D = \Psi_s(\sigma_D)$ for $D \in \mathscr{D}$ (since $L' = \Psi_s(G)$ is the inductive limit of $\Psi_s F$). Then $\Psi_s(\delta)\omega\gamma_D = \Psi_s(\delta\sigma_D) = \Psi_s(\beta_D) = \gamma_D$ for $D \in \mathscr{D}_0$, which shows that $\Psi_s(\delta)\omega = e_{L'}$. Hence $\Psi_s(\delta)$ is an epimorphism, and so δ is an epimorphism, too. It is easy to verify that the element $\omega(d)$ is skew to $\omega(c)$ in the $(k+1)$-group L. As was proved above, d is skew to c in the $(k+1)$-group G. Therefore by Corollary 3 of [6] ω is of the form $\omega = \Psi_s(\nu)$ where $\nu\colon G \to L$. Hence $\Psi_s(\nu\delta\sigma_D) = \omega\Psi_s(\beta_D) = \Psi_s(\sigma_D)$ for $D \in \mathscr{D}_0$. By the faithfulness of Ψ_s we obtain $\nu\delta\sigma_D = \sigma_D$ for $D \in \mathscr{D}_0$. Then $\nu\delta = e_L$, whence δ is a monomorphism. The morphism $\Psi_s(\delta)$, being an epimorphism and a monomorphism, is an isomorphism. Therefore

$$[\Psi_s(L);\ \{\Psi_s(\sigma_D)\colon \Psi_s F_0(D) \to \Psi_s(L)\}_{D \in \mathscr{D}_0}]$$

is the inductive limit of $\Psi_s F_0$, and so by Lemma 2

$$[\Psi_s(L);\ \{\Psi_s(\sigma_D)\colon \Psi_s F(D) \to \Psi_s(L)\}_{D \in \mathscr{D}}]$$

is the inductive limit of $\Psi_s F$. The functor Ψ_s preserves the inductive limit of F.

Conversely, let Ψ_s preserve the inductive limits of all diagrams $F\colon \mathscr{D} \to \mathbf{Gr}_{k+1}$ where $\mathscr{D}$ is nonempty. Consider the functor $F\colon \mathscr{D} \to \mathbf{Gr}_{k+1}$ defined as follows: for $D \in \mathscr{D}$ the object $F(D)$ is a one-element $(k+1)$-group and for $\alpha\colon X \to Y$ the morphism $F(\alpha)$ is the (unique) isomorphism of $F(X)$ onto $F(Y)$. By the definition of F it follows that in this case $\mathscr{D}_0 = \mathscr{D}$. Let $[L; \{\gamma_D\colon F(D) \to L\}_{D \in \mathscr{D}}]$ be the inductive limit of F. By assumption, Ψ_s preserves inductive limits, therefore

$$[\Psi_s(L);\ \{\Psi_s(\gamma_D)\colon \Psi_s F(D) \to \Psi_s(L)\}_{D \in \mathscr{D}}]$$

is the inductive limit of $\Psi_s F$. Note that for any morphism $\alpha\colon X \to Y$ with $X, Y \in \mathscr{D}$, the morphism $\Psi_s F(\alpha)$ is the (unique) isomorphism of the one-element $(n+1)$-group $\Psi_s F(X)$ onto the one-element $(n+1)$-group $\Psi_s F(Y)$. Therefore, in view of Lemma 5*, $[\Psi_s(L);\ \{\Psi_s(\gamma_D)\colon \Psi_s F_d(D) \to \Psi_s(L)\}_{D \in \mathscr{D}_d}]$ (where F_d is the restriction of F to $\mathscr{D}_d$) is the inductive limit of $\Psi_s F_d$. The category $\mathscr{D}_d$ is discrete, and hence $[\Psi_s(L);\ \{\Psi_s(\gamma_D)\}_{D \in \mathscr{D}_d}]$ is simply the free product of the family of $(n+1)$-groups

$\{\Psi_s F_d(D)\}_{D\in\mathscr{D}_d}$. According to Theorem 3 of [8] the free product of at least two nonempty $(n+1)$-groups is not an $(n+1)$-group derived from a $(k+1)$-group; so the family of $(n+1)$-groups $\{\Psi_s F_d(D)\}_{D\in\mathscr{D}_d}$ is a one-element family (since $\Psi_s(L)$ is obviously derived from the $(k+1)$-group L). Thus $\mathscr{D}_d$ consists of one object only, whence $\mathscr{D}$ is a connected category.

If $\mathscr{D}$ is an empty category and $k>1$, then L (as the inductive limit of the empty diagram F) is the empty $(k+1)$-group. Hence $\Psi_s(L)$ is the inductive limit of $\Psi_s F$. The empty category is obviously connected. This completes the proof of Theorem 2.

Corollary 2. *Let $\mathscr{D}$ be a nonempty connected diagram scheme. Then $[L; \{\sigma_D\colon F(D)\to L\}_{D\in\mathscr{D}}]$ is the inductive limit of $F\colon \mathscr{D}\to \mathbf{Gr}_{k+1}$ if and only if $[\Psi_s(L); \{\Psi_s(\sigma_D)\colon \Psi_s F(D)\to\Psi_s(L)\}_{D\in\mathscr{D}}]$ is the inductive limit of $\Psi_s F\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$.*

Note that in the case $k=1$ we always have $\mathscr{D}_0=\mathscr{D}$ (since $(n+1)$-groups derived from groups are always nonempty). But for $k>1$ the $(n+1)$-group derived from the empty $(k+1)$-group is empty. That case has to be excluded. This is the reason for considering the category $\mathscr{D}_0$ instead of $\mathscr{D}$. This, however, is only a minor restriction since, as mentioned in Lemma 2, in considering inductive limits of $(n+1)$-groups the empty $(n+1)$-group is inessential.

As in the dual case of Φ (Section 4), the question arises what are the $(n+1)$-groups derived from the inductive limits of diagrams $F\colon \mathscr{D}\to\mathbf{Gr}_{k+1}$ in the case when $\mathscr{D}$ is not connected. As in the case of Φ, a partial answer is offered by Lemma 7* for $k=1$, but here too (i.e. in the case of $\mathbf{Gr}_2$) more details can be given.

Take any diagram scheme $\mathscr{D}$ and a diagram $F\colon \mathscr{D}\to\mathbf{Gr}_2$. Let $\mathscr{D}_i$ denote the category obtained from $\mathscr{D}$ by adding an initial object I and F_i the functor F extended to that category $\mathscr{D}_i$. The object $F_i(I)$ is obviously a trivial (i.e., one-element) group. For $D\in\mathscr{D}$ let $\mu_D\colon F_i(I)\to F_i(D)$ denote the embedding of the trivial group into any group $F(D)$. Every $(n+1)$-group $\Psi_s F_i(D)$, being derived from a group, contains an invariant element of order one (cf. [2], [10]). The embedding of that element (treated as a one-element group) is just the morphism $\Psi_n(\mu_D)\colon \Psi_n F_i(I)\to\Psi_n F_i(D)$. Thus every diagram $\Psi_n F$ can be extended to $\Psi_n F_i$ by adding the one-element $(n+1)$-group $\Psi_n F_i(I)$ and the family of morphisms $\{\Psi_n(\mu_D)\colon \Psi_n F_i(I)\to\Psi_n F_i(D)\}_{D\in\mathscr{D}}$. Hence we obtain

Proposition 4. *If $[L; \{\gamma_D\colon F(D)\to L\}_{D\in\mathscr{D}}]$ is the inductive limit of $F\colon\mathscr{D}\to\mathbf{Gr}_2$, then $[\Psi_n(L); \{\Psi_n(\gamma_D)\colon \Psi_n F_i(D)\to\Psi_n(L)\}_{D\in\mathscr{D}_i}]$ is the inductive limit of the extended diagram $\Psi_n F_i\colon \mathscr{D}\to\mathbf{Gr}_{n+1}$.*

In particular, for the case when $\mathscr{D}$ is a discrete category we get

Corollary 3. *An $(n+1)$-group derived from a free product of groups is the free product of $(n+1)$-groups with an amalgamated one-element sub-$(n+1)$-group.*

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On homomorphic images of normal complexes in varieties of semigroups

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Following LJAPIN [2] we call a subset T of a semigroup S a normal complex if T satisfies the condition:

$$xt_1y\in T \Leftrightarrow xt_2y\in T \quad \text{for all} \quad x, y\in S^1 \quad \text{and} \quad t_1, t_2\in T.$$

A subset of a semigroup is a normal complex if and only if it is a congruence class for some congruence relation. Under a homomorphism of S, the image of a normal complex need not be a normal complex of the image of S. In connection with the investigation of M-radicals introduced by HOEHNKE [1], the question arises, which subsemigroups T of a semigroup S are homomorphic images of normal complexes under homomorphisms of semigroups from the given class onto S (STRECKER [4]). In the class of all semigroups it is easily seen that there are a semigroup S', a normal complex T' of S' and a surjection φ of S' onto S with $\varphi(T')=T$, where T is an arbitrary subsemigroup of S. In the present paper we consider an arbitrary semigroup variety V and describe, which subsemigroups are homomorphic images of normal complexes of semigroups from V. The result we obtain generalizes the corresponding one on monoids (see [3]).

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Let V be a variety of semigroups. A subsemigroup T of a semigroup $S\in V$ is called V-normal if there are a semigroup $S'\in V$ with a normal complex T' and a surjection φ: $S'\to S$ such that $\varphi(T')=T$.

Theorem. *Let V be a variety of semigroups. If V consists of completely simple semigroups, then the V-normal subsemigroups are exactly the normal subsemigroups; otherwise every subsemigroup of semigroups in V is V-normal.*

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Remark. For some varieties, every subsemigroup is normal; this is the case e.g. if V is the variety of zero semigroups or a variety of rectangular bands, but not if V contains the variety of semilattices.

The Theorem will be proved in three steps.

Proposition 1. *If V contains the two element zero semigroup (hence all zero semigroups) or the two element semilattice (hence all semilattices), then every subsemigroup T of a semigroup $S \in V$ is V-normal.*

Proof. Denote by $F_V(T)$ and $F_V(S)$ the free semigroups in V generated by the underlying sets E_T of T and E_S of S, respectively.

Suppose that V contains the zero semigroups and denote by $F_Z(S)$ the free zero semigroup generated by S. No equality of the form $wtw'=t'$ can hold in $F_V(S)$, where $w, w' \in F_V(S)$ or empty, but not both empty, $t, t' \in E_T$. For, if $wtw'=t'$ then let φ be the natural homomorphism of $F_V(S)$ onto $F_Z(S)$. It follows $\varphi(w)\varphi(t)\varphi(w')=\varphi(t')$. Here the left hand side is equal to zero, the right hand side is not, a contradiction. Therefore E_T is a normal complex of $F_V(S)$, and the natural homomorphism from $F_V(S)$ onto S maps E_T onto T.

Suppose now that V contains the semilattices and denote by $F_W(T)$ and $F_W(S)$ the free semilattices generated by E_T and E_S, respectively. $F_W(T)$ is a normal complex in $F_W(S)$. Consider the natural homomorphism φ of $F_V(S)$ onto $F_W(S)$. For any word $w \in F_V(S)$, $\varphi(w)$ contains all the letters from w. Therefore $\varphi^{-1}(F_W(T))= =F_V(T)$, hence the latter is a normal complex in $F_V(S)$. Now the natural homomorphism from $F_V(S)$ onto S maps $F_V(T)$ onto T.

Proposition 2. *Let V be a variety of semigroups containing neither zero semigroups nor semilattices. Then every semigroup in V is completely simple.*

Proof. Since the free cyclic semigroup in V admits no non-trivial zero semigroup as a homomorphic image, it must be a (finite) group. Thus all semigroups in V are unions of groups. Such a semigroup is a semilattice of completely simple semigroups, but V contains no semilattices either, hence every semigroup in V is completely simple.

Proposition 3. *The homomorphic images of normal complexes of completely simple semigroups are normal complexes of the images.*

Proof. (i) In a completely simple semigroup S, if $xy=e$ and e is an idempotent then $ex=x$ and $ye=y$, for $xS \supseteq eS$, xS is a minimal right ideal, therefore $x \in eS$.

(ii) Let $T \subset S$ be a normal complex of S, $x, y \in S$, $t, u \in T$. By u^{-1} we denote the inverse of u in the maximal subgroup of S containing u. Let φ be a homomorphism of S onto $\bar{S}$. $\bar{S}$ is also completely simple. From $\varphi(x)\varphi(t)\varphi(y)=\varphi(u)\in\bar{e}\bar{S}\bar{e}$, $\bar{e}$ idempotent, it follows that

$$\bar{e}=(\varphi(u))^{-1}\varphi(x)\varphi(t)\varphi(y)=\varphi(x)\varphi(t)\varphi(y)(\varphi(u))^{-1},$$

and by (i) we have $\bar{e}\varphi(x)=\varphi(x)$ and $\varphi(y)\bar{e}=\varphi(y)$. Let $xtyu^{-1}=g\in eSe$. Again by (i), $ex=x$, $u^{-1}e=u^{-1}$, further $\varphi(g)=\bar{e}$ and $(\varphi(g))^{-1}=\bar{e}=\varphi(e)$. Let $u\in e'Se'$, then $\varphi(e')=\bar{e}$ and from $u^{-1}e=u^{-1}$ it follows $e'e=e'$. Now we have $g^{-1}xtyu^{-1}=e$ and therefore $g^{-1}xtye'=eu$ and $e'g^{-1}xtye'=e'eu=e'u=u$. Since T is a normal complex, it follows $e'g^{-1}xt'ye'\in T$ for all $t'\in T$. Applying φ we obtain

$$\varphi(x)\varphi(t')\varphi(y)=\varphi(xt'y)=\bar{e}\varphi(xt'y)\bar{e}=\varphi(e'g^{-1}xt'ye')\in\varphi(T).$$

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The independence of the distributivity conditions in groupoids

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Dedicated to Professor K. Tandori on his 60th birthday

1. Introduction

In the early fifties L. Rédei raised the following problem. The associativity of a binary operation — denoted as multiplication — on a set A means the fulfilment of a set of n^3 equations where $n=\text{card}\, A$, namely, $a(bc)=(ab)c$ for all $(a, b, c)\in A^3$. He asked whether there exists a proper subset $B\subset A^3$ such that if the equation $x(yz)=(xy)z$ holds for all $(x, y, z)\in B$, then the operation is associative on A, i.e., the groupoid $\langle A, \cdot\rangle$ is a semigroup. We say that the associativity conditions are *independent* over the set A, iff there is no such proper subset B of A^3.

This problem was solved by G. Szász [2] in 1953. He proved, that the associativity conditions are independent over any set of at least four elements, but they are not independent over sets of two or three elements.

Analogous notion of independence may also be introduced for other kinds of identities. Thus, in 1954 R. Wiegandt [3] and later R. Wiegandt and J. Wiesenbauer [4] made similar investigations on the distributivity of two binary operations.

Recently we have proved in [1], that the mediality conditions for groupoids — a groupoid is medial (entropic or Abelian according to other terminologies), if it satisfies the identity

$$(xy)(zu) = (xz)(yu)$$

— are independent over a set A if and only if A consists of at least four elements and in sets of at most three elements we have the proper subset mentioned in the original problem.

In the present note we investigate the distributivity conditions for groupoids. A groupoid $\langle A, \cdot\rangle$ is *distributive* if the following two identities hold

(1) $$x(yz) = (xy)(xz),$$

(2) $$(xy)z = (xz)(yz).$$

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If we have (1) (or (2)) only, we say that the groupoid $\langle A, \cdot \rangle$ is *left- (right-) distributive* or simply *semidistributive.*

We prove that the distributivity conditions are independent over a set A if and only if A consists of at least four elements, while the semidistributivity conditions are independent over A exactly when A contains at least three elements.

2. Preliminaries

Let A be any set and let $a, b, c, \ldots$ denote distinct and $x, y, z, \ldots$ arbitrary elements of A. We say that the (ordered) triplet (x, y, z) is *left- (right-) distributive* in the groupoid $\langle A, \cdot \rangle$, if $x(yz)=(xy)(xz)$ $((xy)z=(xz)(yz))$. The triplet (x, y, z) is *distributive* in $\langle A, \cdot \rangle$, if it is both left- and right-distributive. The triplet (x, y, z) is *left- (right-) isolated over the set A,* if there exists a binary operation $\circ$ on A such that all triplets but this one are left- (right-) distributive in $\langle A, \circ \rangle$. The triplet (x, y, z) is *isolated* over A, if there exists a binary operation $*$ on A such that all triplets but this one are distributive in the groupoid $\langle A, * \rangle$. In these cases we say that the triplet (x, y, z) is *left- (right-) isolated* or *isolated* in the *groupoid* $\langle A, \circ \rangle$ or $\langle A, * \rangle$, respectively.

The triplets (x, y, z), $(x', y', z') \in A^3$ are of the same *type*, if there exists a permutation φ on A such that $x'=x\varphi$, $y'=y\varphi$ and $z'=z\varphi$.

Proposition 1. *If the triplet (x, y, z) is left-distributive (left-isolated) in the groupoid $\langle A, \cdot \rangle$, then there exists a binary operation $*$ on A such that (x, y, z) is right-distributive (right-isolated) in $\langle A, * \rangle$.*

Proof. The operation $*$ is defined by $u*v=vu$.

By this proposition in the case of semidistributivity we can restrict ourselves to left-distributivity and left-isolatedness.

Proposition 2. *If the triplets (x, y, z) and (x', y', z') over A are of the same type, moreover, (x, y, z) is distributive (left-distributive) in $\langle A, \cdot \rangle$, then there exists a binary operation $*$ on A such that (x', y', z') is distributive (left-distributive) in $\langle A, * \rangle$.*

Proof. If φ denotes the suitable permutation on A, then put

$$u*v = (u\varphi^{-1} \cdot v\varphi^{-1})\varphi.$$

Corollary. *If the triplets (x, y, z) and (x', y', z') are of the same type, moreover, (x, y, z) is isolated (left-isolated) over A, then (x', y', z') is isolated (left-isolated) over A, too.*

By the above Proposition 2 and its Corollary we have to deal with the following five types of triplets only:

(α) (a, a, a), (β) (a, a, b), (γ) (a, b, a), (δ) (a, b, b), (ε) (a, b, c).

Lemma 1. *If o is a left (right) zero element of the groupoid $\langle A, \cdot\rangle$, then for any $x, y \in A$ the triplet (o, x, y) (the triplet (x, y, o)) is distributive in $\langle A, \cdot\rangle$.*

Lemma 2. *If o is a zero element of $\langle A, \cdot\rangle$, then all triplets over A containing o are distributive.*

Lemma 3. *If e is a left unit element of $\langle A, \cdot\rangle$, then for any $x, y \in A$ the triplet (e, x, y) is left-distributive in $\langle A, \cdot\rangle$.*

Lemma 4. *Let (x, y, z) be an isolated (left-isolated) triplet in the groupoid $\langle A, \cdot\rangle$, and let $A' \supseteqq A$. Then there exists an extension $\langle A', \circ\rangle$ of $\langle A, \cdot\rangle$ such that (x, y, z) is isolated (left-isolated) in $\langle A', \circ\rangle$, too.*

Proof. For any $u, v \in A'$ define

$$u \circ v = \begin{cases} uv & \text{if } u, v \in A, \\ d & \text{otherwise,} \end{cases}$$

where d is an arbitrary fixed element of $A' \setminus A$.

3. Semidistributivity

Theorem 1. *The semidistributivity conditions are independent over a set A if and only if* card $A \geqq 3$.

To prove this theorem our first step will be the following

Lemma 5. *Let $A = \{a, b\}$. If the triplets (a, a, a), (a, b, b), (b, a, a) and (b, b, b) are left-distributive in a groupoid $\langle A, \cdot\rangle$, then this groupoid is left-distributive.*

Proof. There are sixteen binary operations on the set A. Six of them are distributive and additional three are left-distributive. If we examine the remaining seven groupoids, then we get that in each of them one or more triplets mentioned in the lemma are not left-distributive, which proves the lemma.

Proof of Theorem 1. The necessity is implied by Lemma 5. To prove the sufficiency, by Lemma 4, we have to present a suitable binary operation on the set

$A=\{a, b, c\}$ for each of the five types.

Type (α) $\begin{array}{c|ccc} & a & b & c \\ \hline a & c & b & b \\ b & a & b & c \\ c & a & b & c \end{array}$.

Here b and c are left unit elements, therefore, by Lemma 3, it remains to check the triplets (a, x, y). If $x, y \in \{b, c\}$, then $a(xy)=ay=b=bb=(ax)(ay)$, moreover, $a(ax)=ab=b$ and $(aa)(ax)=cb=b$, while $a(xa)=aa=c$ and $(ax)(aa)=bc=c$. Finally, $a(aa)=ac=b$ but $(aa)(aa)=cc=c$, i.e., the triplet (a, a, a) is left-isolated over A.

Type (β) $\begin{array}{c|ccc} & a & b & c \\ \hline a & c & b & c \\ b & c & c & c \\ c & c & c & c \end{array}$.

In this groupoid the only product unequal to c is $ab=b$. Therefore the triplet (a, a, b) is left-isolated.

Type (γ) $\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & c \\ b & b & b & b \\ c & a & b & c \end{array}$.

Now b is a left zero and c is a left unit element, therefore the triplets (b, x, y) and (c, x, y) are left-distributive for any $x, y \in A$ by Lemmas 1 and 3. Furthermore,

$$a(xy)=\begin{cases} a & \text{if} \quad x=y=a \quad \text{or} \quad x=c, \ \ y=a, \\ c & \text{otherwise} \end{cases}$$

and

$$(ax)(ay)=\begin{cases} a & \text{if} \quad x=y=a \quad \text{or} \quad x=b, \ \ y=a \quad \text{or} \quad x=c, \ \ y=a, \\ c & \text{otherwise,} \end{cases}$$

which shows us, that (a, b, a) is left-isolated over A.

Type (δ) $\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & a \\ b & a & b & c \\ c & a & a & a \end{array}$.

In this case a is a right zero element, moreover, b is a left unit element, therefore the triplets (x, y, a) and (b, x, y) are left-distributive for any $x, y \in A$. Since for any $x, y \in A$, $c(xy)=(cx)(cy)=a$, moreover,

$$a(xy)=\begin{cases} c & \text{if} \quad x=y=b, \\ a & \text{otherwise,} \end{cases}$$

and $(ax)(ay)=a$, we have got that the triplet (a, b, b) is left-isolated over A.

Type (ε)

	a	b	c
a	a	a	c
b	a	a	a
c	a	a	a

Here the only product different from a is $ac=c$, thus

$$x(yz) = \begin{cases} c & \text{if } x=y=a \text{ and } z=c, \\ a & \text{otherwise,} \end{cases}$$

and

$$(xy)(xz) = \begin{cases} c & \text{if } x=y=a, \ z=c \text{ or } x=a, \ y=b, \ z=c \\ a & \text{otherwise} \end{cases}$$

which shows the left-isolatedness of the triplet (a, b, c).

4. Distributivity

Theorem 2. *The distributivity conditions are nonindependent over a set A, if* card $A \leqq 3$.

Proof. From Lemma 5 and its dual statement we obtain that the distributivity conditions are nonindependent over a set A, if card $A=2$. To settle the case card $A=3$ we prove the following claim:

Let $A=\{a, b, c\}$ and suppose that all the triplets (a, a, x), (x, a, a), (a, x, x) and (x, x, a) are distributive in the groupoid A, for any $x \in \{b, c\}$. Then the triplet (a, a, a) is also distributive.

It is obvious, that in the case of $aa=a$ the triplet (a, a, a) is distributive. If (a, a, a) is not left-distributive, then we have the following three possibilities:

(i) $\quad a(aa) = a$ and $(aa)(aa) = x$,

(ii) $\quad a(aa) = x$ and $(aa)(aa) = a$,

(iii) $\quad a(aa) = x$ and $(aa)(aa) = y$,

where $x, y \in \{b, c\}$ and $x \neq y$.

We shall show that from each of the above cases a contradiction can be derived.

Case (i) If $aa=x$, then $ax=a$ and $xx=x$, which leads to $a(xx)=ax=a$ and $(ax)(ax)=aa=x$, i.e., the triplet (a, x, x) is not left-distributive. If $aa=z$, where $z \neq a, x$; then $az=a$ and $zz=x$. Consider the triplet (a, a, z), which is distributive under our assumption. But $(aa)z=zz=x$ and $(az)(az)=aa=z$, thus we have a contradiction.

Case (ii). If $aa=x$, then our assumption implies $ax=x$ and $xx=a$. These contradict the left-distributivity of the triplet (a, a, x). Namely, $a(ax)=ax=x$ and $(aa)(ax)=xx=a$. If $aa=z$, where $z\neq a, x$, then we have $az=x$ and $zz=a$. From the right-distributivity of the triplet (a, a, z) we get $xx=a$ and from the left-distributivity of (a, z, z) we obtain $xx=z$, whence $a=z$, a contradiction.

Case (iii). Let $aa=x$. Then $ax=x$ and $xx=y$, which imply the nondistributivity of the triplet (a, a, x), indeed, $a(ax)=ax=x$ and $(aa)(ax)=xx=y$. If $aa=y$, then $ay=x$ and $yy=y$. The right-distributivity of the triplet (a, a, y) implies $xx=y$, while from the left-distributivity of (a, y, y) we infer that $xx=x$, i.e., $x=y$, a contradiction.

If we dualize the above procedure, i.e., consider the triplet (u, v, w) and left- (right-) distributivity instead of the triplet (w, v, u) and right- (left-) distributivity in each step respectively, then we get that the triplet (a, a, a) is right-distributive, thus it is distributive.

Theorem 3. *The distributivity conditions are independent over a set A if and only if* card $A\geqq 4$.

Proof. The necessity is the content of Theorem 2. To prove the sufficiency let $A=\{a, b, c\}$ first, and we define suitable binary operations on A which show the isolatedness of the triplets of types (β), (γ), (δ) and (ε) over A.

Type (β)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & c \\ b & c & c & c \\ c & c & c & c \end{array}.$$

This groupoid is left-distributive, namely, for any $x, y, z\in A$

$$x(yz)=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } x=y=a \quad \text{and} \quad z=b, \\ c & \text{otherwise,} \end{cases}$$

$$(xy)(xz)=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } xy=a, \ \ xz=b, \ \ \text{i.e.,} \ \ x=y=a, \ \ z=b, \\ c & \text{otherwise.} \end{cases}$$

On the other hand

$$(xy)z=\begin{cases} a & \text{if } x=y=z=a, \\ b & \text{if } x=y=a, \ \ z=b, \\ c & \text{otherwise,} \end{cases}$$

$$(xz)(yz)=\begin{cases} a & \text{if } x=y=z=a, \\ c & \text{otherwise,} \end{cases}$$

which shows us, that the triplet (a, a, b) is not right-distributive, i.e., it is isolated.

Type (γ)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & c & c \\ b & a & b & c \\ c & c & c & c \end{array}$$

This groupoid is idempotent, therefore all triplets of the forms of (x, x, y) and (x, y, y) are distributive. Indeed,

$$x(xy) = (xx)(xy) \quad \text{and} \quad (xx)y = xy = (xy)(xy),$$

moreover,

$$x(yy) = xy = (xy)(xy) \quad \text{and} \quad (xy)y = (xy)(yy).$$

Since c is a zero element, thus by Lemma 2 all triplets containing c are distributive. It remains to check the triplets (a, b, a) and (b, a, b). After an easy computation we get that the only nondistributive triplet is (a, b, a), i.e., it is isolated over A.

Type (δ)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & c & a & c \\ b & c & c & c \\ c & c & c & c \end{array}$$

The only product unequal to c is $ab=a$, for this reason $x(xy)=(xy)(xz)=c$ and $(xz)(yz)=c$ for any $x, y, z, \in A$. But

$$(xy)z = \begin{cases} a & \text{if} \quad x = a \quad \text{and} \quad y = z = b, \\ c & \text{otherwise,} \end{cases}$$

which shows the isolatedness of the triplet (a, b, b).

Type (ε)

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & a & b & a \\ b & a & b & c \\ c & a & b & c \end{array}$$

The elements b and c are left units, so the triplets (b, x, y) and (c, x, y) are left-distributive for any $x, y \in A$. Moreover, a and b are right zero elements, therefore, the triplets (x, y, a) and (x, y, b) are distributive for any $x, y \in A$. Since $a(xc)=a$ and $(ax)(ac)=a$ for arbitrary $x \in A$, the groupoid is left-distributive. For any $x \in A$ we have $(xa)c=ac=a$ and $(xc)(ac)=(xc)a=a$, $(xb)c=bc=c$ and

$$(xc)(bc) = (xc)c = \begin{cases} a & \text{if} \quad x = a, \\ c & \text{otherwise,} \end{cases}$$

and $(xc)c=(xc)(cc)$. These show, that the triplet (a, b, c) is isolated in this groupoid.

We have seen in Theorem 2 that the triplets of type (α) are nonisolated over sets of three elements, but they are isolated over four elements sets. Namely, consider the following operation on the set $A=\{a, b, c, d\}$:

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & b & d & d & d \\ b & c & d & d & d \\ c & d & d & d & d \\ d & d & d & d & d \end{array}$$

There are exactly two products different from d: $aa=b$ and $ba=c$. Since $x(yz)=$ $=(xy)(xz)=d$ for all $x, y, z \in A$, this groupoid is left-distributive, but

$$(xy)z = \begin{cases} c & \text{if} \quad x=y=z=a, \\ d & \text{otherwise,} \end{cases}$$

and $(xz)(yz)=d$ for all $x, y, z \in A$, which lead to the isolatedness of the triplet (a, a, a) over A.

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On a problem of Baldwin and Berman

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Introduction. In their paper [1] BALDWIN and BERMAN proved that there exists an algebra where the lengths of the Mal'cev chains are at most two, but in an appropriate factor algebra there exists no bound for the length at all. (For the definitions see the next part of this paper.) In the same paper they asked the following: Is it true for any algebra in which the length of any Mal'cev chain is at most one, that its factors have the same property. This question can be motivated by the following. If not only the length is one, but also the "direction" is given, then for each a and b in the algebra there is a unary algebraic function f, such that $f(a)=b$ and $f(b)=a$. This calls force, however, congruence permutability, which yields the positive answer. However, we shall give in this short paper a negative answer to this question. Instead of giving the construction only we shall show what is beyond it.

Rephrasing the problem. We have to deal with congruences of an algebra and some factors of it. The following is well known:

Proposition 1. *Let $\mathfrak{A}=(A; F)$ be any algebra, S the set of unary algebraic functions of $\mathfrak{A}$ and G the set of nonconstant unary algebraic functions (in the sense of Grätzer). Then S is a semigroup under composition, and the following hold:*

1. *The congruence lattices of $(A; F)$, $(A; S)$, $(A; G)$ coincide.*
2. *The same holds for $(A/\theta; F)$, $(A/\theta; S)$, $(A/\theta; G)$, where θ is any congruence of $\mathfrak{A}$.*

Thus, from now on, we have to deal with unary algebras only. We shall associate to each unary algebra a colored multigraph which will closely keep track of Mal'cev's lemma.

Definition 2. Let $(A; S)$ be a unary algebra. We associate to it a colored undirected multigraph $M(A; S)$ which will be called the Mal'cev graph of the

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algebra. The vertex set of this graph will be A. For distinct vertices a and b the edges of color (a, b) will be all pairs of the form $(f(a), f(b))$. (The edges may have more colors, at each vertex there is a loop of each color.)

It is clear, that the Mal'cev graph of a factor algebra is equal to the factor graph of the original Mal'cev graph, and the kernels are the same. The classes of the principal congruence $\theta(a, b)$ are the connected components of the (a, b)-colored graph. If one of the (a, b)-colored components has diameter at least n, this means that we need a Mal'cev chain of length n when considering $\theta(a, b)$. If n is the maximal length of the Mal'cev chains then the maximal diameter of the components of the (a, b)-colored graph is n. Otherwise, this coloring has infinite diameter.

In their example Baldwin and Berman construct a Mal'cev graph in which each color has diameter at most two, but a factor of it has infinite diameter in one color.

Now we reformulate their question:

Question. Suppose a Mal'cev graph has diameter one in each color. Does each factor of it possess the same property?

Theorem. *There exists a Mal'cev graph such that it has diameter one in each color and a factor of it has infinite diameter in some color.*

Next, we should like to mention, that a Mal'cev graph has diameter one in a color if and only if in this coloring all components are complete.

If the algebra $(A; G)$ is "connected", then it is very hard to describe the action of G. For this reason we investigate the case when it is not connected.

Proposition 3. *Let $(A; G)$ be a unary algebra. Assume G consists of non-constant operations and the diameter of $M(A; G)$ is equal to 1. If A is the disjoint union of B and C such that both $(B; G)$ and $(C; G)$ are subalgebras of $(A; G)$, then G acts semiregularly on A (i.e. for each $a \in A$ and $f, g \in G$, $f(a)=g(a)$ implies $f=g$).*

Proof. Suppose we have $f(a)=g(a)$ for $a \in A$ and $f, g \in G$. We shall show, if a belongs to either one of B and C then the action of f and g are the same on the other among B and C. Thus, a repetition of the argument will prove the statement. Indeed, suppose $a \in C$ and consider any $b \in B$. Since both $(f(a), f(b))$ and $(g(a), g(b))$ have color (a, b) and $f(a)=g(a)$, $(f(b), g(b))$ must have, also, the color (a, b), because of the condition on the diameter. However, $(C; G)$ is a subalgebra, therefore no element of G sends a to either of $f(b)$ and $g(b)$, for they both belong to B. Hence, we must have $f(b)=g(b)$, as it was stated.

Proposition 4. *Let $(A; S)$ be a unary algebra and let G be the subset of the semigroup S consisting of the non-constant operations and suppose G acts semiregularly on A. Then, $M(A; G)$ as well as $M(A; S)$ have diameter 1 in each color if and only if G is a group the elements of which have order at most three.*

Proof. Suppose the Mal'cev graph has diameter 1. First of all we mention that all invertible elements of S belong, obviously, to G. Now, let $f \in G$ be such that $f^2 \neq 1$. Since f is not the identity, semiregularity gives us that for each $a \in A$ the element $b = f(a)$ differs from both a and $c = f(b)$. Again by semiregularity $f^2 \neq 1$ implies $c \neq a$. By the condition on the diameter there exists a $g \in S$ such that $\{g(a), g(b)\}$ equals $\{a, c\}$. Since $b \notin \{a, c\}$, $g(b) \neq b$, i.e. $g \neq 1$, yielding $g(a) \neq a$. Thus, we have $g(a) = c = f^2(a)$ and $gf(a) = g(b) = a$. Now, semiregularity implies $g = f^2$ and $1 = gf = f^2 f = f^3$, i.e. all elements of G have order at most three and they are invertible. Since the invertible elements of a monoid form a group, G satisfies the condition.

Let us suppose, conversely, that the condition on G is satisfied. We have to prove that if (a, b) and (b, c) have some color then so has (a, c). Let us notice that (u, v) always has color $(f(u), f(v))$ for $f \in G$, since G is a group. Hence, we may suppose that the color in question is (a, b). If a, b and c are not all different the statement is clear, so we may assume they are different. Then the condition on the color implies the existence of an $f \in G$ such that $\{b, c\} = \{f(a), f(b)\}$ and the semiregularity yields $f(b) \neq b$, since $f \neq 1$. Hence, we have $b = f(a)$ and $c = f(b) = f(f(a))$. Thus $f^2 \neq 1$ so, by the condition on G, we must have $f^3 = 1$, i.e. $f^2(a) = c$ and $f^2(b) = f^3(a) = 1(a) = a$ proving that the graph has diameter 1.

Remark. In Proposition 4 the conditions on the color and on the nonconstant operations are clearly not equivalent without semiregularity, and they do not imply, even together, the semiregularity. The latter is obvious in the case when A is a three-element set and S consists of all permutations and the constants.

The construction. By Proposition 4 the condition on the diameter will be satisfied if we start with two different copies B and C of a group G such that the order of its elements does not exceed three. It is easy to see, that the restriction of any congruence of such a structure to either one of B and C consists of the left cosets of some subgroups. (The action of the elements of G is defined by $f(g) = fg$.) We have to find two suitable subgroups such that in the factor algebra the diameter of the corresponding Mal'cev graph will be infinite. For the time being we are not interested in the condition on the order of the elements of G.

Proposition 5. *Let B and C be two copies of the group G on which the operations are defined by $f(g) = fg$ and let A be the disjoint union of B and C; further let S be the union of G and the constant maps on A. Consider two subgroups H and K of G.*

Then the equivalence relation on A the classes of which are the left cosets of H on B and the left cosets of K on C is a congruence relation on $(A; S)$ the factor algebra by which will be denoted by $(A'; S)$. Further, the graph of $M(A'; S)$ colored by (H, K) has the following properties:

(i) *Two different vertices of the graph are connected iff they are of the form fH and fK for suitable $f \in G$.*

(ii) *For any subset M of G the set of those elements which are connected with some element bH $(b \in M)$ consists of the elements cK, $c \in MHK$.*

(iii) *If H and K together generate G then the graph is connected.*

(iv) *If, in addition, no power of HK is equal to G then the graph has infinite diameter.*

Proof. The statement on the equivalence relation just as property (i) are trivial.

As far as property (ii) is concerned, it is, clearly, enough to prove it for singletons. By condition (i) bH is connected exactly with those fK for which f is contained in bH, i.e., which are contained in bHK.

Property (iii) is an obvious consequence of property (ii).

By property (ii) the distance between H and some other element is exactly k iff it is contained in a product $HKHKH\ldots$ of $k+1$ factors but not in a product of $k-1$ factors. This proves, obviously, property (iv).

Remark. Just to answer the original question we should not need infinite diameter in the factor, diameter two would do as well. The above idea could give, trivially, such an example. Namely, let $\{1, 2, 3, 4\}$ be the underlying set of the algebra and let the permutation $(12)(34)$ be the only operation and factor out by the congruence generated by $(1, 2)$.

Now, in virtue of Proposition 5, we have to find, only, a group G generated by its subgroups H and K such that no power of HK equals G and the order of the elements does not exceed three. First, we shall construct a group satisfying all the conditions but the one about the order of the elements and then we shall choose it so that the group will have exponent three (i.e., all non-units have order three). In what follows, we shall give a general construction which will give the group in question as a special case. (The form of the presentation of this example was suggested by Ágnes Szendrei.)

We start with infinite quadratic upper triangular matrices over a field F having finitely many off-diagonal non-zero entries. The rows and the columns are indexed by the integers. The place containing the entry indexed by 0, 0 will be called the middle of the matrix. The row and the column containing the middle will be called the middle row and middle column, respectively. These matrices form, obviously, a semigroup under the multiplication.

Now, we shall consider a subsemigroup. We can partite the matrices into nine blocks according to the signs of the indices:

$$\begin{array}{|c|c|c|}\hline A & x & Z \\ \hline o & m & y^{\mathrm{T}} \\ \hline O & o & C \\ \hline\end{array}\quad \left.\begin{array}{l} \} \to \text{negative} \\ \\ \} \to \text{positive}\end{array}\right\}\ \text{indices}$$

Here, A and C are upper triangular matrices indexed by the negative and positive integers, respectively. x and y are column vectors, indexed by the negative integers, and $^{\mathrm{T}}$ stands for the transpose. The element m is the middle entry of the matrix. We shall consider only those matrices where $m=1$ and both A and C are the units of the corresponding matrix rings. Let (x), $(y)^{\mathrm{T}}$ and (Z) denote those matrices which we get when omitting all non-zero entries of the original matrix except those which belong to x, y^{T} and Z, respectively. Thus, the matrices in question are of the form $I+(x)+(y)^{\mathrm{T}}+(Z)$, where I denotes the identity matrix. Computing the product of matrices of this form, we get:

$$\big(I+(x)+(y)^{\mathrm{T}}+(Z)\big)\big(I+(u)+(v)^{\mathrm{T}}+(W)\big) = \big(I+(x+u)+(y+v)^{\mathrm{T}}+(Z+W+x\otimes v)\big).$$

Hence, these matrices form a semigroup G, and I is the unit element of this semigroup. An easy computation shows that $(I+(x)+(y)^{\mathrm{T}}+(Z))^{-1}=I+(-x)+(-y)^{\mathrm{T}}+$ $+(-Z+x\otimes y)$. Since this element belongs to G, G is a group.

It is easy to see, that the elements of the form $I+(x)$, resp. $I+(y)^{\mathrm{T}}$, form a subgroup H, resp. K. In virtue of the equations $\big(I+(x)\big)\big(I+(y)^{\mathrm{T}}\big)\big(I+(-x)\big)\big(I+(-y)^{\mathrm{T}}\big)=$ $=I+(x\otimes y)$ and $I+(x)+(y)^{\mathrm{T}}+\sum\{x_i\otimes y_i|1\leqq i\leqq n\}=\big(I+(y)^{\mathrm{T}}\big)\big(I+(x_1\otimes y_1)\big)\cdot\ldots$ $\ldots\cdot\big(I+(x_n\otimes y_n)\big)\big(I+(x)\big)$ we get that the subgroup generated by $H\cup K$ contains all elements of the form $I+(x)+(y)^{\mathrm{T}}+(Z)$, where Z is a sum of tensor products. Since any matrix having a single non-zero entry is the tensor product of two suitable vectors and the matrix ring in question has elements with finitely many non-zero off-diagonal entries only, the subgroups H and K generate G.

The elements of $H\cup K$ are, clearly, of the form $I+(x)+(y)^{\mathrm{T}}$. We have $\Pi\{(I+(x_i)+(y_i)^{\mathrm{T}})|1\leqq i\leqq n\}=I+(x)+(y)^{\mathrm{T}}+(Z)$ with $x=\sum x_i$, $y=\sum y_i$ and $Z=\sum\{x_i\otimes y_j|1\leqq i<j\leqq n\}$ yielding that the rank of Z is at most $\binom{n}{2}$. Since G has elements $I+(x)+(y)^{\mathrm{T}}+(Z)$ such that the rank of Z is greater than a given integer, no power of $H\cup K$ is equal to G.

The only thing left is to choose the field F so that the exponent of G be equal to three. This is fulfilled whenever F has characteristic three, for $(I+(x)+(y)^{T}+(Z))^3=$ $=(I+3(x)+3(y)^{T}+3(Z+x\otimes y))$.

Problems. 1. Describe all the Mal'cev graphs with diameter 1; or at least those all factors of which have diameter 1, as well.

2. Does there exist a Mal'cev graph with diameter 1 having a factor with unbounded diameter in some color such that each component in this color has finite diameter?

3. Does there exist a Mal'cev graph with diameter 1 having a factor in which each color has finite diameter but the supremum of the diameters is not finite?

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A reduction in case of compact Hamiltonian actions

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Dedicated to Professor K. Tandori on his 60th birthday

The classical results of Jacobi and Liouville on reduction of phase spaces were put in a general setting by E. Cartan which in the up-to-date formulation of R. ABRAHAM and J. MARSDEN ([1], p. 298) runs as follows: Let Q be a smooth manifold and ϱ a closed 2-form on Q then the *characteristic distribution* E of ϱ is given by the subspaces

$$E_z = \{v \mid \iota_v \varrho_z = 0, \quad v \in T_z Q\}, \quad z \in Q,$$

and the 2-form ϱ is said to be *regular* if E is a subbundle of TQ. If ϱ is regular then E proves to be an involutive distribution and thus generates the *characteristic foliation* $\mathscr{F}$ of ϱ on Q. If the quotient space $P=Q/\mathscr{F}$ admits a smooth manifold structure such that the canonical projection

$$\pi\colon Q \to Q/\mathscr{F} = P$$

is a submersion, then there is a unique sympletic form ω on P such that $\varrho=\pi^*\omega$ holds. In this case the symplectic manifold (P, ω) is called a *reduced phase space* and the above procedure is said to be a *reduction* producing it.

The existence of reductions in case of some Hamiltonian actions was observed by J. MARSDEN and A. WEINSTEIN [7]. In fact, let (P, ω) be a symplectic manifold, G a connected Lie group and

$$\Phi\colon G \times P \to P$$

a Hamiltonian action with a momentum mapping $J\colon P \to \mathfrak{g}^*$ which is equivariant with respect to Φ and to the coadjoint action Ad^* of G on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$. Assume that $\mu \in \mathfrak{g}^*$ is a regular value of the momentum mapping J then

$$Q_\mu = J^{-1}(\mu)$$

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is a smooth submanifold of P. Moreover, assume that the action of the isotropy subgroup G_μ on the manifold Q_μ is both free and proper, then the corresponding orbit space

$$P_\mu = Q_\mu/G_\mu$$

admits a smooth manifold structure such that the canonical projection

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu = P_\mu$$

is a submersion. Consider now the restriction ϱ_μ of the symplectic form ω to Q_μ, then the closed 2-form ϱ_μ proves to be regular, the leaves of its characteristic foliation being the orbits of G_μ on Q_μ. Moreover, there is a unique symplectic form ω_μ on P_μ such that

$$\varrho_\mu = \pi_\mu^* \omega_\mu$$

is valid. Thus the reduction procedure applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ). The above procedure is called the *Marsden—Weinstein reduction* and it has several important applications [8].

A generalization of the Marsden—Weinstein reduction is presented below in case of compact Hamiltonian actions. In fact, let (P, ω) be a symplectic manifold, G a compact connected Lie group and

$$\Phi\colon G \times P \to P$$

a Hamiltonian action with a momentum mapping $J\colon P \to \mathfrak{g}^*$ which is equivariant with respect to Φ and Ad^* and has regular elements of $\mathfrak{g}^*$ in its range. It is shown that in case of a $\mu \in \mathrm{Range}\, J$ the set

$$Q_\mu = J^{-1}(\mu)$$

is a smooth submanifold of P provided that $G(z)$ is a non-singular orbit of Φ for any $z \in Q_\mu$. Assuming that the orbits of the isotropy subgroup G_μ on Q_μ are all of the same type it is shown that the orbit space $P_\mu = Q_\mu/G_\mu$ admits a smooth manifold structure such that the canonical projection

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu = P_\mu$$

is a submersion. Moreover, the restriction ϱ_μ of ω to Q_μ proves to be regular and the leaves of its characteristic distribution are shown to be the orbits of G_μ on Q_μ. Thus a unique symplectic form ω_μ on P_μ with $\varrho_\mu = \pi_\mu^* \omega_\mu$ is obtained. Consequently, the reduction procedure applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ). A simple example in order to show that the generalization is essential is presented as well.

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First, a concise account of those facts is given which yield the prerequisites for the proof of the above mentioned result.

Two orbits of the action of a group are said to be of the same type if they have the same conjugacy class of isotropy subgroups. The orbit types of an action are relatively easy to survey in case of actions generated by compact connected Lie groups. Actually, a fundamental result on compact connected Lie group actions, the Principal Orbit Type Theorem, yields the following classification of the orbits of such an action: 1. There are *principal orbits,* they are all of the same type and of maximal dimension; the union of the principal orbits is an open everywhere dense subset of the manifold on which the group acts. 2. There may be *exceptional orbits;* they are also of maximal dimension but not of the same type as the principal ones. 3. There may be *singular orbits:* they are not of maximal dimension [6].

In case of the adjoint action Ad: $G\times\mathfrak{g}\to\mathfrak{g}$ of a compact connected Lie group, the regular elements of $\mathfrak{g}$ have principal orbits, there are no exceptional orbits, and the singular elements of $\mathfrak{g}$ have singular orbits.

If G is a compact Lie group then its Lie algebra $\mathfrak{g}$ is obtainable as a direct sum

$$\mathfrak{g}=\mathfrak{c}\oplus\mathfrak{u}$$

of a commutative and of a semisimple ideal. Consequently, an arbitrarily fixed interior product on $\mathfrak{c}$ and the negative of the Killing—Cartan form of $\mathfrak{u}$ yield an interior product $\langle\,,\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ which is invariant with respect to the adjoint action

$$\mathrm{Ad}\colon\ G\times\mathfrak{g}\to\mathfrak{g}.$$

The interior product $\langle\,,\rangle_{\mathfrak{g}}$ defines a vector space isomorphism $\mathfrak{g}\cong\mathfrak{g}^*$ which is equivariant with respect to the adjoint action and the coadjoint action of G. Thus the Lie algebra $\mathfrak{g}$ will be identified with its dual $\mathfrak{g}^*$ in what follows on account of the above given isomorphism. Consequently, by a momentum mapping the map

$$J\colon\ P\to\mathfrak{g}$$

will be meant subsequently which is obtained from a usual momentum mapping through the above given identification. Moreover, the equivariance of J is understood with respect to the actions Φ and Ad.

Let G be a compact connected Lie group, P a smooth manifold and

$$\Phi\colon\ G\times P\to P$$

a smooth action. It is a well-known fundamental fact, that there is a Riemannian metric $\langle\,,\rangle_P$ on P which is invariant with respect to the action Φ. Assume that there is a symplectic form ω on P which is left invariant by the action Φ; then a unique almost complex structure $\mathbf{J}\colon TP\to TP$ of P can be obtained such that

$$\langle X,Y\rangle_P=\omega(\mathbf{J}X,Y)$$

holds for any vector fields $X, Y \in \mathcal{T}(P)$ applying a basic construction ([1], pp. 172—174; [8]).

Moreover, it can be shown that J is equivariant with respect to the induced tangent action

$$T\Phi\colon G \times TP \to TP;$$

in other words, $\mathbf{J}_{z'} \circ T_z \Phi_g = T_z \Phi_g \circ \mathbf{J}_z$ holds for $z' = \Phi(g, z)$, $g \in G$, $z \in P$ where

$$\Phi_g\colon P \to P$$

is the transformation defined by $\Phi_g(z) = \Phi(g, z)$, $z \in P$, for $g \in G$ as usual [2], [9]. Consider for $z \in P$ the subspace $R_z^0 \subset T_z P$ defined by

$$R_z^0 = \{v \mid T_z \Phi_g v = v \quad \text{for} \quad g \in G_z^0, \quad v \in T_z P\}$$

where G_z^0 is the identity component of the isotropy subgroup G_z. Then the equivariance of J obviously implies that

$$\mathbf{J}_z R_z^0 = R_z^0$$

holds. Assume now that in addition to the former hypotheses the action Φ is Hamiltonian as well and that

$$J\colon P \to \mathfrak{g}$$

is an equivariant momentum mapping for Φ. Then according to earlier observations

$$\text{Kernel}\, T_z J = \mathbf{J}_z N_z G(z)$$

holds at any point $z \in P$ where $N_z G(z)$ is the normal space to the orbit $G(z)$ at z with respect to the Riemannian metric $\langle\, , \rangle_P$ given above [2], [9]. Consider now a point $z \in P$ such that $G(z)$ is a non-singular orbit; then $N_z G(z) \subset R_z^0$ holds and thus

$$\text{Kernel}\, T_z J = \mathbf{J}_z N_z G(z) \subset \mathbf{J}_z R_z^0 = R_z^0$$

holds in consequence of the preceding observations and assertions.

For some part of the subsequent arguments the fact is essential that the Riemannian metric $\langle\, , \rangle_P$ can be chosen so that it becomes Hermitian with respect to the almost complex structure J. In fact, starting with a Φ invariant Riemannian metric $\langle\, , \rangle_P$ and with J defined by $\langle\, , \rangle_P$ and ω the definition

$$2\langle X, Y\rangle_P^H = \langle \mathbf{J}X, \mathbf{J}Y\rangle_P + \langle X, Y\rangle_P, \quad X, Y \in \mathcal{T}(P)$$

yields a Hermitian metric $\langle\, , \rangle_P^H$ which is invariant with respect to the action Φ. Moreover, the equality

$$2\langle X, Y\rangle_P^H = \langle \mathbf{J}Y, \mathbf{J}X\rangle_P + \langle X, Y\rangle_P = \omega(-Y, \mathbf{J}X) + \omega(\mathbf{J}X, Y) = 2\omega(\mathbf{J}X, Y),$$

$$X, Y \in \mathcal{T}(P)$$

shows that $\langle\, , \rangle_P^H$ and $\langle\, , \rangle_P$ are in the same relation to ω; consequently, ω and $\langle\, , \rangle_P^H$ define the same almost complex structure as ω and $\langle\, , \rangle_P$. Thus there is no loss of generality by assuming that $\langle\, , \rangle_P$ is already Hermitian with respect to J.

The following theorem concerns the originally indicated objective, a generalization of the Marsden—Weinstein reduction in case of compact Hamiltonian actions.

Theorem. *Let (P, ω) be a symplectic manifold, G a compact connected Lie group and*

$$\Phi\colon G\times P \to P$$

a Hamiltonian action with an equivariant momentum mapping $J\colon P \to \mathfrak{g}$ which has regular elements of $\mathfrak{g}$ in its range. If $\mu\in\mathfrak{g}$ is in the range of J and such that $G(z)$ is a non-singular orbit for

$$z\in Q_\mu = J^{-1}(\mu)$$

then Q_μ is a smooth submanifold of P. Furthermore, if the orbits of the isotropy subgroup G_μ on Q_μ are all of the same type then the orbit space $P_\mu = Q_\mu/G_\mu$ admits a smooth manifold structure such that the canonical projection

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion and ϱ_μ, the restriction of ω to Q_μ, is regular, the leaves of its characteristic foliation being orbits of G_μ. Moreover, there is a unique symplectic form ω_μ on P_μ such that

$$\varrho_\mu = \pi_\mu^* \omega_\mu$$

holds. Thus the reduction applies to (Q_μ, ϱ_μ) and yields the reduced phase space (P_μ, ω_μ).

Proof. Let P' be the union of the principal orbits of the action Φ, then the isotropy subgroups are all conjugate in points of P'. The fact, that the set of regular elements of $\mathfrak{g}$ is open, the assumption, that the range of J contains regular elements of $\mathfrak{g}$ and the fact, that P' is everywhere dense in P together imply that there is a $z\in P'$ such that $J(z)$ is a regular element of $\mathfrak{g}$. Thus the preceding observations and the equivariance of J entail that the isotropy subgroups in points of P' are all conjugate to a closed subgroup of an arbitrary maximal torus T of G. Obviously the same holds for the identity components of the isotropy subgroups in points of the exceptional orbits of the action Φ.

Consider an element μ in the range of J such that $G(z)$ is a non-singular orbit for

$$z\in Q_\mu = J^{-1}(\mu).$$

It will be shown that the isotropy subalgebra $\mathfrak{g}_z$ as function of $z\in Q_\mu$ is constant on each connected component of the set Q_μ.

In fact, consider a point $z_0 \in Q_\mu$; then since $G(z)$ is a non-singular orbit, the identity component G_z^0 of the isotropy subgroup G_z is commutative by the preceding observation. Furthermore, by an earlier result already mentioned above

$$\operatorname{Ker} T_z J = \mathbf{J}_z N_z G(z) \subset R_z^0$$

holds where $R_z^0 \subset T_z P$ is the subspace of vectors left invariant by the identity component of the isotropy subgroup. Consider the orthogonal decomposition

$$\mathfrak{g}_\mu = \mathfrak{r}_z \oplus \mathfrak{g}_z,$$

then $\mathfrak{r}_z$ is mapped into $\operatorname{Ker}(T_z J \restriction T_z G(z)) \subset R_z^0 \cap T_z G(z)$ under the canonical isomorphism

$$\mathfrak{m}_z \to T_z G(z) \quad (\mathfrak{m}_z \text{ is the orthogonal complement of } \mathfrak{g}_z \text{ in } \mathfrak{g}).$$

Since the above isomorphism is equivariant with respect to the restricted adjoint action of G_z on $\mathfrak{m}_z$ and the isotropy action of G_z on $T_z G(z)$, the following holds

$$[\mathfrak{r}_z, \mathfrak{g}_z] = \{0\}.$$

The preceding observations obviously yield now that the following is valid as well

$$[\mathfrak{g}_\mu, \mathfrak{g}_z] = [\mathfrak{r}_z \oplus \mathfrak{g}_z, \mathfrak{g}_z] \subset [\mathfrak{r}_z, \mathfrak{g}_z] + [\mathfrak{g}_z, \mathfrak{g}_z] = \{0\}.$$

Since the isotropy subalgebras of the restricted action of G_μ on Q_μ are all conjugate in $\mathfrak{g}_\mu$, and since by the equivariance of J they coincide with the isotropy subalgebras of the action of G, the assertion that $\mathfrak{g}_z$ as function of z is constant on the connected components of Q_μ follows.

Consider now an element $\mu \in \mathfrak{g}$ such that the orbits of the points $z \in Q_\mu = J^{-1}(\mu)$ are all non-singular. Let Q_μ^0 be a connected component of Q_μ, then the flat submanifold

$$\mathfrak{q}_\mu^0 = \mu + \mathfrak{g}_z$$

does not depend on the choice of z in Q_μ^0 according to the preceding observation. Fix a conic neighbourhood C of $\mathfrak{m}_z$ in $\mathfrak{g}$ such that $C \cap \mathfrak{g}_z = \{0\}$. Let W be an open and connected neighbourhood of Q_μ^0 which is disjoint from the other components of Q_μ and such that $\mathfrak{m}_x \subset C$ for $x \in W$. It will be shown now that

$$J(W) \cap \mathfrak{q}_\mu^0 = \{\mu\}$$

is valid. In fact, consider an $x \in W$ such that $J(x) = \xi \in \mathfrak{q}_\mu^0$ holds. Then there is a smooth curve $\varphi\colon [0, 1] \to W$ with $\varphi(0) = z \in Q_\mu^0$, $\varphi(1) = x$. Consider now the curve

$$\psi = J \circ \varphi\colon [0, 1] \to \mathfrak{g}.$$

Let $\mathfrak{m}_{\varphi(\tau)}$ be the orthogonal complement of $\mathfrak{g}_{\varphi(\tau)}$, then by preceding stipulations the following holds:

$$\dot{\psi}(\tau) = T_{\varphi(\tau)} J \dot{\varphi}(\tau) \in \mathfrak{m}_{\varphi(\tau)} \subset C \quad \text{for} \quad \tau \in [0, 1].$$

Consequently, $\xi-\mu=\psi(1)-\psi(0)\in C$ holds. On the other hand $\xi-\mu\in\mathfrak{g}_z$ is valid. Thus

$$\xi = \mu$$

follows by the definition of C. Next, it will be shown that the restricted map $J\restriction W$ is transversal to the submanifold $\mathfrak{q}_\mu^0$. In fact, assume that $J(x)\in\mathfrak{q}_\mu^0$ holds for some $x\in W$. Then $x\in Q_\mu^0$ and $J(x)=\mu$ by the preceding observation. Consequently, former assertions yield that

$$T_\mu\mathfrak{g} = \mathfrak{g} = \mathfrak{m}_x\oplus\mathfrak{g}_x = T_x J(T_x W)\oplus T_\mu\mathfrak{q}_\mu^0$$

is valid which yields the transversality of $J\restriction W$. But the transversality of $J\restriction W$ entails that

$$Q_\mu^0 = (J\restriction W)^{-1}(\mathfrak{q}_\mu^0)$$

is a smooth submanifold by a fundamental theorem on transversal maps ([5], pp. 22—23).

Moreover, the same theorem yields that

$$\dim P-\dim Q_\mu^0 = \operatorname{codim} Q_\mu^0 = \operatorname{codim}\mathfrak{q}_\mu^0 = \dim\mathfrak{g}-\dim\mathfrak{g}_z = \dim G(z).$$

Consequently, all the connected components of Q_μ are of the same dimension. Thus Q_μ is a smooth submanifold of P.

The second assertion of the theorem that if the orbits of the points of Q_μ are all of the same type then the orbit space Q_μ/G_μ admits a smooth manifold structure such that

$$\pi_\mu\colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion is a direct consequence of a basic theorem on orbit spaces of actions with a single orbit type ([6], pp. 6—9).

In order to prove the third assertion of the theorem, that ϱ_μ the restriction of the symplectic form ω to Q_μ is regular, consider the above defined invariant Riemannian metric $\langle\,,\rangle_P$ and the almost complex structure $\mathbf{J}$ determined by $\langle\,,\rangle_P$ and ω on P. According to former observations already mentioned above, the following holds

$$T_z Q = \text{Kernel}\, T_z J = \mathbf{J}_z N_z G(z) \quad \text{for} \quad z\in Q_\mu.$$

Let now ϱ_μ be the restriction of ω to the submanifold Q_μ. Then the characteristic distribution E of ϱ_μ is given by the subspaces

$$E_z = \{v \mid \iota_v\varrho_\mu = 0, \quad v\in T_z Q_\mu\}, \quad z\in Q_\mu.$$

According to former observations the following equalities are valid:

$$(\iota_v\varrho_\mu)(u) = \omega(v, u) = \langle\mathbf{J}_z v, u\rangle_P \quad \text{where} \quad u, v\in T_z Q_\mu.$$

Consequently, the subspace E_z is formed by those vectors $v \in T_z Q_\mu$ which satisfy the following condition:

$$\mathbf{J}_z v \perp T_z Q_\mu = \mathbf{J}_z N_z G(z).$$

Since by a former observation $\langle \, , \rangle_P$ is Hermitian therefore $\mathbf{J}_z \colon T_z P \to T_z P$ is an isometry and consequently the preceding condition is equivalent to the following one:

$$v \perp N_z G(z), \quad v \in T_z Q_\mu.$$

Consequently, the characteristic subspace E_z can be given as follows:

$$E_z = \text{Kernel } T_z J \cap T_z G(z) = T_z G_\mu(z).$$

Therefore E is integrable and its leaves are the orbits of the action of G_μ on the submanifold Q_μ; since these orbits are all of the same type by assumption, Q_μ is a fiber bundle over Q_μ/G_μ by a basic theorem ([6], pp. 6—9). Consequently, the characteristic distribution E of ϱ_μ is regular.

The existence of the symplectic form ω_μ is now a direct consequence of the fact that the natural projection

$$\pi_\mu \colon Q_\mu \to Q_\mu/G_\mu$$

is a submersion.

Remark 1. The question that in case of a Hamiltonian action $\Phi \colon G \times P \to P$ of a compact connected Lie group G with an equivariant momentum mapping $J \colon P \to \mathfrak{g}$ having regular elements of $\mathfrak{g}$ in its range, which are those elements of $\mathfrak{g}$ where the preceding theorem applies, seems to be open. In fact, if $G(z)$ is a principal orbit of Φ, then, as it was observed above, G_z is conjugate to a closed subgroup of a maximal torus T of G. Thus, provided that P is compact, a result of Guillemin and Sternberg [3] applies and yields that for any Weyl chamber $\mathfrak{t}_+ \subset \mathfrak{g}$, the set

$$\mathfrak{t}_+ \cap J(P')$$

is the union of a finite number of open r-dimensional convex polytopes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l \subset \subset \mathfrak{t}_+$ where $r = \operatorname{rank} G - \dim G_z$. Thus $\mu \in \mathfrak{t}_+ \cap J(P')$ corresponds to the assumption of the theorem if it is not on the boundary of any one among the polytopes $\mathfrak{p}_1, \ldots, \mathfrak{p}_l$. Moreover, the theorem applies at any point of $\mathfrak{t}_+ \cap J(P')$ provided that the conjecture of Guillemin and Sternberg that

$$\mathfrak{t}_+ \cap J(P')$$

itself is a single r-dimensional open convex polytope [3] proves to be valid. The question, which are those points of $J(P')$ on the boundary of $\mathfrak{t}_+$ where the theorem applies seems to be open, too.

Remark 2. The fact that the Marsden—Weinstein reduction in case of compact Hamiltonian actions is included in the preceding theorem can be verified as follows. Let $\mu \in \mathfrak{g}$ be a regular value of the momentum mapping J. Then $\mathfrak{g}_z = \{0\}$ for any $z \in J^{-1}(\mu)$ by a result of MARSDEN [8]; consequently $G(z)$ is non-singular orbit in case of $z \in J^{-1}(\mu)$. If μ is a regular value of J then the range of J includes a neighbourhood of μ. But the set of regular elements of $\mathfrak{g}$ is everywhere dense in $\mathfrak{g}$ and P' is everywhere dense in P; consequently, there is a $z' \in P'$ such that $J(z')$ is a regular element of $\mathfrak{g}$.

Remark 3. An example is presented below in order to show that there are cases where the Marsden—Weinstein reduction does not apply, however, the one given by the preceding theorem does so.

Let M be a Riemannian manifold with Riemannian metric $\langle\, , \rangle_M$, G a compact connected Lie group and

$$\alpha\colon G \times M \to M$$

an isometric action. Consider the tangent bundle $P = TM$ with its canonical symplectic form ([1], pp. 182—183); then the induced action $\Phi = T\alpha$ of G on P is symplectic. Moreover, the action Φ is Hamiltonian, since an equivariant momentum mapping $J\colon P \to \mathfrak{g}^* \cong \mathfrak{g}$ is defined for Φ by

$$\langle J(v), X \rangle = \langle v, \overline{X}(z) \rangle_M, \quad v \in T_z M, \ X \in \mathfrak{g},$$

according to Noether's Theorem ([1], pp. 282—285); here of course $\overline{X}$ is the infinitesimal generator of α given by X.

Let now $(X_1, \dots, X_n)$ be an orthonormal base of $\mathfrak{g}$ then obviously

$$J(v) = \sum_{i=1}^{n} \langle v, \overline{X}_i(z) \rangle_M X_i, \quad v \in TM,$$

holds. If in particular G is semisimple then $\langle\, , \rangle_{\mathfrak{g}}$ the interior product of $\mathfrak{g}$ is given by the negative of the Killing—Cartan form of $\mathfrak{g}$ according to its definition.

Consider now in particular an m-dimensional Riemannian symmetric space $M = G/H$ where G is compact and semisimple and the Riemannian metric $\langle\, , \rangle_M$ is defined by the negative of the Killing—Cartan form of $\mathfrak{g}$. Consider the canonical decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$$

and let the orthonormal base $(X_1, \dots, X_m)$ of $\mathfrak{g}$ be compatible with this decomposition. Then

$$(\overline{X}_1(0), \dots, \overline{X}_m(0))$$

is an orthonormal base of $T_0 M$ where $o = H \in G/H$. Consequently, the following holds:

$$J(v) = \sum_{i=1}^{m} \langle v, \bar{X}_i(0)\rangle_M X_i, \quad v\in T_0M.$$

But then $J(T_0M)=\mathfrak{m}$ is obviously valid.

A more particular case is obtained as follows. Let A be a compact semisimple Lie group and consider the left action

$$\lambda\colon (A\times A)\times A \to A$$

of the direct product $A\times A$ on A given by $\lambda((g, h), a)=gah^{-1}$ for $a, g, h\in A$. Then A is canonically a Riemannian symmetric space for the action λ ([4], pp. 223—224). In fact, if

$$G = A\times A$$

then A can be obtained as the canonical homogeneous coset space G/H where H is the diagonal in the direct product. Consequently, the canonical decomposition $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$ is given now by

$$\mathfrak{m} = \{(X, -X)\,|\, X\in\mathfrak{a}\}, \quad \mathfrak{h} = \{(X, X)\,|\, X\in\mathfrak{a}\}.$$

An element (X, Y) of the semisimple Lie algebra $\mathfrak{g}$ is regular if and only if both X and Y are regular elements of $\mathfrak{a}$. Therefore, if $X\in\mathfrak{a}$ is a regular element then

$$(X, -X)\in\mathfrak{m}$$

is a regular element of $\mathfrak{g}$. But then the above observation yields that

$$\mathfrak{m} = J(T_0M) \subset J(TM) = J(P)$$

holds and consequently, $J(P)$ contains regular elements of $\mathfrak{g}$. However, Remark 1 does not apply in this case since $P=TM$ is not compact.

In order to show that the momentum mapping J considered above has no regular values, it is sufficient to see that Φ has no discrete isotropy subgroups; since the existence of a regular value of J implies the existence of trivial isotropy algebras by a result of MARSDEN [8]. Since α is transitive action, every orbit of the action $\Phi=T\alpha$ intersects the tangent space $T_0M\cong\mathfrak{m}$. Moreover, the isotropy subgroup of the action Φ at a point

$$(X, -X)\in\mathfrak{m} \cong T_0M$$

is a subgroup of H. But as a simple calculation shows the following holds:

$$\Phi((g, g), (X, -X)) = (\mathrm{Ad}\,(g)X, -\mathrm{Ad}\,(g)X) \quad \text{where} \quad X\in\mathfrak{a},\ g\in A.$$

Consequently, the isotropy subgroup of Φ cannot be discrete at $(X, -X)$; in fact, the principal isotropy subgroups of Φ at points of $\mathfrak{m}$ are given by a suitable, maximal torus T of A as the subgroup $\{(g, g)\,|\,g\in T\}\subset H$. The existence of values $\mu\in J(P)\cap\mathfrak{g}$ where the theorem applies is a consequence of the above observation.

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Techniques of Finsler geometry in the theory of vector bundles

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A topical problem in geometry is the study of the differential geometry of vector bundles. For this study the classical methods are not convenient enough, because of the very complicated analytical expressions of some important geometrical objects defined on the total space of a vector bundle.

Using the ideas from Finsler geometry [10, 11, 12, 13] we can considerably simplify the theory. For this, the notion of Finsler connection on the total space E of a vector bundle $\xi=(E, \pi, M)$ is fundamental. To define it firstly we define the notion of the nonlinear connection N on E, then we use this concept to obtain the algebra of Finsler tensor fields on E. A Finsler connection on the space E is a linear connection ∇ on E which preserves by parallelism the horizontal distribution N and the vertical distribution E^V of ξ. ∇ has four local coefficients which have very simple transformation laws to a change of canonical coordinates on E. Also, its torsion and its curvature have a small number of nontrivial components which are the Finsler tensor fields.

By applying this theory to the Riemannian structure G on E we get a Finsler canonical connection which has a simple form.

In this way the geometrical theory of ξ can be constructed without difficulty. This method was used in our talk "Vector bundles Finsler geometry" presented at the second National Seminar on Finsler spaces [13], at the University of Braşov, Romania, February 15, 1982.

1. Vector bundles

Let $\xi=(E, \pi, M)$ be a vector bundle of the class C^∞. We suppose that the total space E has $n+m$ dimensions, the base M has n dimensions and the local fibre $E_p=\pi^{-1}(p)$, $p\in M$, is a real vector space of dimension n.

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If (U_α, Φ_α) is a vectorial chart of ξ, determined by the chart $(U_\alpha, \varphi_\alpha)$ of the base manifold M, then Φ_α: $\pi^{-1}(U_\alpha)\to U_\alpha\times\mathbf{R}^m$ is a diffeomorphism with the property $\mathrm{pr}_1\circ\Phi_\alpha=\pi$ and $\Phi_{\alpha,p}=\Phi_{\alpha|E_p}$: $E_p\to\mathbf{R}^m$ is an isomorphism of vector spaces.

The mappings $g_{\beta\alpha}$: $U_\beta\cap U_\alpha\to Gl(m, R)$, given by $g_{\beta\alpha}(p)=\Phi_{\beta,p}\circ\Phi_{\alpha,p}^{-1}$, $p\in U_\beta\cap\cap U_\alpha$, are the transition functions of ξ. To the vector chart (U_α, Φ_α) these corresponds a chart on the total space E, $(\pi^{-1}(U_\alpha), \tilde{\Phi}_\alpha)$, where $\tilde{\Phi}_\alpha$ is the diffeomorphism $\tilde{\Phi}_\alpha=(\varphi_\alpha\times 1_{\mathbf{R}^m})\circ\Phi_\alpha$. Therefore, the coordinates of the point $u\in\pi^{-1}(U_\alpha)\subset E$ in this chart, for $u=\Phi_\alpha^{-1}(p, y)$, $(p, y)\in U_\alpha\times\mathbf{R}^m$, are

$$\tilde{\Phi}_\alpha(u) = (\varphi_\alpha\times 1_{\mathbf{R}^m})\circ\Phi_\alpha\circ\Phi_\alpha^{-1}(p, y) = (x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^m)$$

and they are called the canonical coordinates of the point u determinated by the coordinates (x^i) of the point $p=\pi(u)$. Everywhere, the indices $i, j, k, l, \ldots$; $i', j', k', l', \ldots$; $i'', j'', k'', l'', \ldots$ take the values $1, 2, \ldots, n$ and $a, b, c, d, \ldots$; $a', b', c', d', \ldots$; $a'', b'', c'', d'', \ldots$ take the values $1, 2, \ldots, m$.

The transformations of the canonical coordinates $(x, y)\to(x', y')$ of the points of E, are given by:

$$(x', y') = \tilde{\Phi}_\beta\circ\tilde{\Phi}_\alpha^{-1}(x, y) = \big((\varphi_\beta\circ\varphi_\alpha^{-1})(x), g_{\beta\alpha}(p)y\big).$$

We write these transformations in the form:

$$x^{i'} = x^{i'}(x^1, \ldots, x^n),\quad y^{a'} = M_a^{a'}(x)y^a,\quad \det(M_a^{a'}(x)) \neq 0 \tag{1.1}$$

and the inverse transformations:

$$x^i = x^i(x^{1'}, \ldots, x^{n'}),\quad y^a = \tilde{M}_{a'}^a(x')y^{a'},\quad \det\big(\tilde{M}_{a'}^a(x')\big) \neq 0.$$

The map π: $E\to M$ induces the π^T-morphism of the corresponding tangent bundles π^T: $T(E)\to T(M)$. Then $VE=\mathrm{Ker}\,\pi^T$ is a subbundle of $T(E)$ called the *vertical bundle*. VE defines a distribution

$$E^V\colon u\in E \to E_u^V,$$

where E_u^V is the fibre of VE in the point $u\in E$. E^V is called the vertical distribution of ξ. On the open set $\pi^{-1}(U_\alpha)$, $\dfrac{\partial}{\partial y^a}$, $a=1, \ldots, m$, is a local basis of the vertical distribution E^V. Therefore E^V is integrable.

Definition 1.1. A non-linear connection on the total space E of ξ is a differentiable distribution N: $u\in E\to N_u\subset E_u$, with the property

$$E_u = N_u\oplus E_u^V, \tag{1.2}$$

where E_u is the tangent space in the point u to the manifold E.

It follows:

Proposition 1.1. *If the base M of the vector bundle ξ is paracompact, then, on E, there exist the non-linear connections.*

On $\pi^{-1}(U_\alpha)$ the distribution N has a local basis of the form

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^a(x, y)\frac{\partial}{\partial y^a}; \tag{1.3}$$

$N_i^a(x, y)$ are called the coefficients of the non-linear connection N. Also, N can be locally determined by the Pfaff system $\delta y^a = 0$, where

$$\delta y^a = dy^a + N_i^a(x, y)\, dx^i. \tag{1.4}$$

For every vector field X on E there exists a unique decomposition

$$X = X^H + X^V, \quad X_u^H \in N_u, \quad X_u^V \in E_u^V, \quad \forall u \in E. \tag{1.5}$$

X^H is called the horizontal part and X^V the vertical part of X.

If ω is a 1-form field on E, then we can define the 1-form field ω^H on E by the condition:

$$\omega^H(X) = \omega(X^H), \quad \forall X \in \mathscr{X}(E);$$

ω^H is called the horizontal component of ω. Let $\omega^V = \omega - \omega^H$. Then ω has a unique decomposition

$$\omega = \omega^H + \omega^V; \tag{1.6}$$

ω^V is called the vertical component of ω. We have $\omega^H(X^V) = \omega^V(X^H) = 0$, $\forall X \in \mathscr{X}(E)$.

Let us observe that $\frac{\delta}{\delta x^i}$, $i = 1, \dots, n$, being n horizontal independent fields and $\partial/\partial y^a$, $a = 1, \dots, m$, being m vertical independent fields, $(\delta/\delta x^i, \partial/\partial y^a)$ provides a local basis of the module of the vector fields $\mathscr{X}(E)$ and $(dx^i, \delta y^a)$ is a local basis of the module of 1-form fields on E. These bases are dual:

$$\left\langle \frac{\delta}{\delta x^i}, dx^j \right\rangle = \delta_i^j, \ \left\langle \frac{\delta}{\delta x^i}, \delta y^a \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, dx^j \right\rangle = 0, \ \left\langle \frac{\partial}{\partial y^a}, \delta y^b \right\rangle = \delta_a^b. \tag{1.7}$$

2. Algebra of Finsler tensor fields, Finsler connections

Definition 2.1. A tensor field t on the total space E of the vector bundle ξ is called a Finsler tensor field of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ if it has the property

$$\begin{aligned} &t(\underset{1}{\omega}, \dots, \underset{p}{\omega}, \underset{1}{X}, \dots, \underset{q}{X}, \underset{p+1}{\omega}, \dots, \underset{p+r}{\omega}, \underset{q+1}{X}, \dots, \underset{q+s}{X}) = \\ &= t(\underset{1}{\omega^H}, \dots, \underset{p}{\omega^H}, \underset{1}{X^H}, \dots, \underset{q}{X^H}, \underset{p+1}{\omega^V}, \dots, \underset{p+r}{\omega^V}, \underset{q+1}{X^V}, \dots, \underset{q+s}{X^V}), \\ &\qquad \forall \underset{\alpha}{\omega} \in \mathscr{X}^*(E), \quad \forall \underset{\beta}{X} \in \mathscr{X}(E). \end{aligned} \tag{2.1}$$

Proposition 2.1. *A Finsler tensor field of the type* $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ *on* E *has the following local form:*

$$t = t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y)\frac{\delta}{\delta x^{i_1}}\otimes\dots\otimes\frac{\delta}{\delta x^{i_p}}\otimes dx^{j_1}\otimes\dots\otimes dx^{j_q}\otimes \tag{2.2}$$
$$\otimes\frac{\partial}{\partial y^{a_1}}\otimes\dots\otimes\frac{\partial}{\partial y^{a_r}}\otimes\delta y^{b_1}\otimes\dots\otimes\delta y^{b_s}.$$

The coordinate transformation (1.1) changes the coefficients $t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y)$ according to the law:

$$t^{i'_1\dots i'_p a'_1\dots a'_r}_{j'_1\dots j'_q b'_1\dots b'_s}(x', y') = \frac{\partial x^{i'_1}}{\partial x^{i_1}}\dots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\dots\frac{\partial x^{j_q}}{\partial x^{j'_q}}M^{a'_1}_{a_1}\dots M^{a'_r}_{a_r}\tilde{M}^{b_1}_{b'_1}\dots\tilde{M}^{b_s}_{b'_s}t^{i_1\dots i_p a_1\dots a_r}_{j_1\dots j_q b_1\dots b_s}(x, y).$$

Theorem 2.1. *If* w *is a tensor field on* E *of the type* (p, q) *then it determines* 2^{p+q} *Finsler tensor fields on* E *of the type* $\begin{pmatrix} p-r & r \\ q-s & s \end{pmatrix}$ $(r=0, 1, \dots, p; s=0, 1, \dots, q)$.

Proof. The sum

$$w(\underset{1}{\omega^H}+\underset{1}{\omega^V}, \dots, \underset{p}{\omega^H}+\underset{p}{\omega^V}, \underset{1}{X^H}+\underset{1}{X^V}, \dots, \underset{q}{X^H}+\underset{q}{X^V})$$

has 2^{p+q} terms, each of them being a Finsler tensor field of the type mentioned.

The vector field X^H is a Finsler tensor field of the type $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the vector field X^V is a Finsler tensor field of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and ω^H, ω^V are Finsler tensor fields of the type $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively.

A remarkable local vector field is $y = y^a\dfrac{\partial}{\partial y^a}$. It is called the intrinsic vector field of ξ.

If $\underset{F}{\mathcal{T}}{}^{pr}_{qs}(E)$ is the $\mathcal{F}(E)$-module of the Finsler tensor fields of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then the $\mathcal{F}(E)$-module

$$\underset{F}{\mathcal{T}}(E) = \bigoplus_{p,q,r,s=0,1,\dots}\underset{F}{\mathcal{T}}{}^{pr}_{qs}(E)$$

and the product tensor is a graded algebra, called the algebra of Finsler tensor fields on E.

Definition 2.2. A Finsler connection on E is a linear connection ∇ on E with the property that the horizontal linear spaces N_u, $u\in E$, of the distribution N are parallel with respect to ∇ and similarly, the vertical linear spaces E^V_u, $u\in E$, are parallel with respect to ∇.

In what follows we shall prove the existence of a Finsler connection on E. We observe that a linear connection ∇ on E is a Finsler connection on E if and only if

$$(2.3)\qquad (\nabla_X Y^H)^V = 0, \quad (\nabla_X Y^V)^H = 0, \quad \forall X, Y \in \mathscr{X}(E).$$

Then, we have:

Theorem 2.2. *The following statements are equivalent:*
(a) ∇ *is a Finsler connection on* E,
(b) $\nabla_X Y = (\nabla_X Y^H)^H + (\nabla_X Y^V)^V, \quad \forall X, Y \in \mathscr{X}(E)$,
(c) $\nabla_X \omega = (\nabla_X \omega^H)^H + (\nabla_X \omega^V)^V, \quad \forall \omega \in \mathscr{X}^*(E), \forall X \in \mathscr{X}(E)$.

For a Finsler connection ∇ we put:

$$(2.4)\qquad \nabla^H_X Y = \nabla_{X^H} Y, \quad \nabla^V_X Y = \nabla_{X^V} Y, \quad \forall X, Y \in \mathscr{X}(E).$$

The following theorem is easy to prove:

Theorem 2.3. *For any Finsler connection* ∇ *on* E, ∇^H *and* ∇^V *given by* (2.4) *are the covariant derivatives in the algebra* $\underset{F}{\mathscr{T}}(E)$.

∇^H is called the h-covariant derivative and ∇^V is called the v-covariant derivative of the Finsler connection ∇.

Proposition 2.2. *We have:*

(1) $\nabla^H_X f = X^H f, \quad (\nabla^H_X Y^H)^V = 0, \quad (\nabla^H_X Y^V)^H = 0,$

(2) $\nabla^H_X Y = (\nabla^H_X Y^H)^H + (\nabla^H_X Y^V)^V,$

(3) $\nabla^V_X f = X^V f, \quad (\nabla^V_X Y^H)^V = 0, \quad (\nabla^V_X Y^V)^H = 0,$

(4) $\nabla^V_X Y = (\nabla^V_X Y^H)^H + (\nabla^V_X Y^V)^V.$

We have analogous formulas for $\nabla_X \omega$, *too.*

Theorem 2.4. *If* ∇^H *and* ∇^V *are two covariant derivatives in the algebra of Finsler tensor fields* $\underset{F}{\mathscr{T}}(E)$, *having the properties* (1) *and* (3) *from Proposition* 2.2, *then there exists an unique Finsler connection* ∇ *on* E *for which* ∇^H *is the h-covariant derivative and* ∇^V *is the v-covariant derivative of* ∇.

If t is a Finsler tensor field of the type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then for any Finsler connection ∇ the following formulas hold:

$$\nabla_X t = \nabla^H_X t + \nabla^V_X t,$$

$$(\nabla^H_X t)(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) = X^H t(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) - t(\nabla^H_X \underset{1}{\omega}, \ldots, \underset{q+s}{X}) - \ldots - t(\underset{1}{\omega}, \ldots, \nabla^H_X \underset{q+s}{X}),$$

$$(\nabla^V_X t)(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) = X^V t(\underset{1}{\omega}, \ldots, \underset{q+s}{X}) - t(\nabla^V_X \underset{1}{\omega}, \ldots, \underset{q+s}{X}) - \ldots - t(\underset{1}{\omega}, \ldots, \nabla^V_X \underset{q+s}{X}).$$

In the canonical coordinates (x^i, y^a) there exists a well-determined set of differentiable functions on $\pi^{-1}(U)$, $\left(F^i_{jk}(x,y),\ F^a_{bk}(x,y),\ C^i_{ja}(x,y),\ C^a_{bc}(x,y)\right)$ such that

$$
\text{(2.5)}\qquad
\begin{cases}
\nabla^H_{\delta/\delta x^k}\dfrac{\delta}{\delta x^j} = F^i_{jk}(x,y)\dfrac{\delta}{\delta x^i}, & \nabla^H_{\delta/\delta x^k}\dfrac{\partial}{\partial y^b} = F^a_{bk}(x,y)\dfrac{\partial}{\partial y^a},\\[2ex]
\nabla^V_{\partial/\partial y^c}\dfrac{\delta}{\delta x^j} = C^i_{jc}(x,y)\dfrac{\delta}{\delta x^i}, & \nabla^V_{\partial/\partial y^c}\dfrac{\partial}{\partial y^b} = C^a_{bc}(x,y)\dfrac{\partial}{\partial y^a}.
\end{cases}
$$

Let us consider, for simplicity, the Finsler tensor field

$$K = K^{ia}_{jb}(x,y)\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b.$$

Then, if we put for $X^H = X^k\dfrac{\delta}{\delta x^k}$, $X^V = X^c\dfrac{\partial}{\partial y^c}$,

$$\nabla^H_X K = X^k K^{ia}_{jb|k}\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b,\qquad \nabla^V_X K = X^c K^{ia}_{jb}|_c\frac{\delta}{\delta x^i}\otimes\frac{\partial}{\partial y^a}\otimes dx^j\otimes\delta y^b,$$

we have:

$$
\text{(2.6)}\qquad
\begin{aligned}
K^{ia}_{jb|k} &= \frac{\delta K^{ia}_{jb}}{\delta x^k} + F^i_{hk}K^{ha}_{jb} - F^h_{jk}K^{ia}_{hb} + F^a_{ck}K^{ic}_{jb} - F^c_{bk}K^{ia}_{jc},\\
K^{ia}_{jb}|_c &= \frac{\partial K^{ia}_{jb}}{\partial y^c} + C^i_{hc}K^{ha}_{jb} - C^h_{jc}K^{ia}_{hb} + C^a_{dc}K^{id}_{jb} - C^d_{bc}K^{ia}_{jd},
\end{aligned}
$$

which are the local expressions of the h- and v-covariant derivatives.

Proposition 2.3. *To a transformation of the canonical coordinates* (1.1) *the coefficients* (F^i_{jk}, F^a_{bk}) *from* (2.5) *have the following transformation laws:*

$$F^{i'}_{j'k'}(x'\ y') = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}F^i_{jk}(x,y) + \frac{\partial x^{i'}}{\partial x^h}\frac{\partial^2 x^h}{\partial x^{j'}\partial x^{k'}},$$

$$F^{a'}_{b'j'}(x',y') = M^{a'}_a\tilde{M}^b_{b'}\frac{\partial x^j}{\partial x^{j'}}F^a_{bj}(x,y) + M^{a'}_c\frac{\partial\tilde{M}^c_{b'}}{\partial x^{j'}},$$

and the coefficients (C^i_{jc}, C^a_{bc}) *from* (2.5) *are the Finsler tensor fields of the type* $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 0 & 2\end{pmatrix}$, *respectively.*

Remark. If $N^a_i(x,y)$ are the coefficients of the non-linear connection N, then $\dfrac{\partial N^a_k}{\partial y^b}$ have the same transformation law as $F^a_{bk}(x,y)$.

These considerations allow to prove

Theorem 2.5. *If $\xi=(E, \pi, M)$ is a vector bundle with the base M paracompact, and N is a fixed non-linear connection on the total space E, then there exist Finsler connections ∇ on E, which preserve by parallelism the distributions N and E^V.*

The Finsler connection ∇ with the coefficients $F^a_{bj}=\dfrac{\partial N^a_j}{\partial y^b}$, $C^i_{jb}=0$, $C^a_{bc}=0$ and $F^i_{jk}(x, y)$ arbitrary, is called a Berwald connection. Its simplicity made it very malleable in applications.

3. Torsion and curvature of Finsler connections

We consider again a non-linear connection N on the total space E of a vector bundle $\xi=(E, \pi, M)$ and let ∇ be a Finsler connection on E, which preserves by parallelism the distributions N and E^V.

Proposition 3.1. *The torsion tensor field T of the Finsler connection ∇ is characterized by five Finsler tensor fields:*

$$[T(X^H, Y^H)]^H,\quad [T(X^H, Y^H)]^V,\quad [T(X^H, Y^V)]^H,\quad [T(X^H, Y^V)]^V,\quad [T(X^V, Y^V)]^V. \tag{3.1}$$

If T^i_{jk}, T^a_{jk}, P^i_{jb}, P^a_{jb} and S^a_{bc} are their local components $\left(\text{where } T^i_{jk}\dfrac{\delta}{\delta x^i}= =\left[T\left(\dfrac{\delta}{\delta x^k}, \dfrac{\delta}{\delta x^j}\right)\right]^H, \text{ etc.}\right)$, then we have

$$\begin{aligned} &T^i_{jk} = F^i_{jk}-F^i_{kj}, \quad T^a_{jk} = R^a_{jk} = \frac{\delta N^a_j}{\delta x^k}-\frac{\delta N^a_k}{\delta x^j}, \\ &P^i_{jb} = C^i_{jb}, \quad P^a_{jb} = \frac{\partial N^a_j}{\partial y^b}-F^a_{bj}, \quad S^a_{bc} = C^a_{bc}-C^a_{cb}. \end{aligned} \tag{3.2}$$

It follows that the torsions P^i_{jb}, P^a_{jb}, S^a_{bc} of a Berwald connection vanish.

Proposition 3.2. *The Finsler tensor field $[T(X^H, Y^H)]^V$ vanishes if and only if the distribution N is integrable.*

The curvature tensor field R of a Finsler connection ∇ on E has only six non-trivial Finsler components.

Proposition 3.3. *The curvature tensor field R of a Finsler connection ∇ on the total space E of a vector bundle ξ has the property*

$$[R(X, Y)Z^H]^V = [R(X, Y)Z^V]^H = 0, \quad \forall X, Y, Z\in\mathscr{X}(E).$$

Then, we have

Theorem 3.1. *The curvature tensor field R of a Finsler connection ∇ on the total space E of a vector bundle ξ is characterized by the following six Finsler tensor fields:*

$$(3.3)\quad \begin{cases} R(X^H, Y^H)Z^H = \nabla^H_X \nabla^H_Y Z^H - \nabla^H_Y \nabla^H_X Z^H - \nabla^H_{[X^H, Y^H]} Z^H - \nabla^V_{[X^H, Y^H]} Z^H, \\ R(X^H, Y^H)Z^V = \nabla^H_X \nabla^H_Y Z^V - \nabla^H_Y \nabla^H_X Z^V - \nabla^H_{[X^H, Y^H]} Z^V - \nabla^V_{[X^H, Y^H]} Z^V, \end{cases}$$

$$(3.3)'\quad \begin{cases} R(X^V, Y^H)Z^H = \nabla^V_X \nabla^H_Y Z^H - \nabla^H_Y \nabla^V_X Z^H - \nabla^H_{[X^V, Y^H]} Z^H - \nabla^V_{[X^V, Y^H]} Z^H, \\ R(X^V, Y^H)Z^V = \nabla^V_X \nabla^H_Y Z^V - \nabla^H_Y \nabla^V_X Z^V - \nabla^H_{[X^V, Y^H]} Z^V - \nabla^V_{[X^V, Y^H]} Z^V, \end{cases}$$

$$(3.3)''\quad \begin{cases} R(X^V, Y^V)Z^H = \nabla^V_X \nabla^V_Y Z^H - \nabla^V_Y \nabla^V_X Z^H - \nabla^V_{[X^V, Y^V]} Z^H, \\ R(X^V, Y^V)Z^V = \nabla^V_X \nabla^V_Y Z^V - \nabla^V_Y \nabla^V_X Z^V - \nabla^V_{[X^V, Y^V]} Z^V. \end{cases}$$

Let $R^i_{jkl}, R^a_{bkl}, P^i_{jkc}, P^a_{bkc}, S^i_{jbc}, S^a_{bcd}$ be the local components of the Finsler tensor fields (3.3), (3.3)′, (3.3)″, respectively $\left\{R\left(\frac{\delta}{\delta x^l}, \frac{\delta}{\delta x^k}\right)\frac{\delta}{\delta x^j} = R^i_{jkl}\frac{\delta}{\delta x^i}, \text{ etc.}\right\}$. Then, we have:

Theorem 3.2. *The curvature tensor field R of a Finsler connection ∇ with the coefficients $(F^i_{jk}, F^a_{bk}, C^i_{jc}, C^a_{bc})$ is characterized by the Finsler tensor fields* (3.3), (3.3)′, (3.3)″, *whose components are given by*

$$(3.4)\quad \begin{cases} R^i_{jkl} = \dfrac{\delta F^i_{jk}}{\delta x^l} - \dfrac{\delta F^i_{jl}}{\delta x^k} + F^h_{jk} F^i_{hl} - F^h_{jl} F^i_{hk} + C^i_{ja} R^a_{kl}, \\[2ex] R^a_{bkl} = \dfrac{\delta F^a_{bk}}{\delta x^l} - \dfrac{\delta F^a_{bl}}{\delta x^k} + F^c_{bk} F^a_{cl} - F^c_{bl} F^a_{ck} + C^a_{bc} R^c_{kl}, \end{cases}$$

$$(3.4)'\quad \begin{cases} P^i_{jkc} = \dfrac{\partial F^i_{jk}}{\partial y^c} - C^i_{jc|k} + C^i_{jb} P^b_{kc}, \\[2ex] P^a_{bkc} = \dfrac{\partial F^a_{bk}}{\partial y^c} - C^a_{bc|k} + C^a_{bd} P^d_{kc}, \end{cases}$$

$$(3.4)''\quad \begin{cases} S^i_{jbc} = \dfrac{\partial C^i_{jb}}{\partial y^c} - \dfrac{\partial C^i_{jc}}{\partial y^b} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb}, \\[2ex] S^a_{bcd} = \dfrac{\partial C^a_{bc}}{\partial y^d} - \dfrac{\partial C^a_{bd}}{\partial y^c} + C^e_{bc} C^a_{ed} - C^e_{bd} C^a_{ec}. \end{cases}$$

Observe the simplicity of these expressions, as compared to the components of the curvature tensor field R written in the natural frame $\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right)$, [4].

The Ricci and Bianchi identities can be written down without difficulty.

4. Riemannian structures; Finsler connections compatible with a metrical structure

The efficiency of the techniques of Finsler geometry in the study of vector bundles is visible. In particular, its applications to the theory of several geometric structures on the total space of a vector bundles are useful.

We study the Riemannian structure G on the total space E of a vector bundle $\xi=(E,\pi,M)$. If E_u^V is the vertical space at the point u of E, then the vectors orthogonal to E_u^V, with respect to G, uniquely determine the vector subspace N_u complementary to E_u^V. That is, $E_u=N_u\oplus E_u^V$, $u\in E$, and the map $N\colon u\to N_u$ defines, in a geometrical way, a non-linear connection N on the total space E.

Proposition 4.1. *If G is a Riemannian structure on E, then there exists an unique non-linear connection N on E with the property*

$$G(X,Y)=0,\quad \forall X\in N,\quad \forall Y\in E^v. \tag{4.1}$$

Proposition 4.2. *For a Riemannian structure G on E, there exist an unique symmetric Finsler tensor field G^H of the type $\begin{pmatrix}0&0\\2&0\end{pmatrix}$ non-degenerate on the fibres of the horizontal bundle HE, and an unique symmetric Finsler tensor field G^V of the type $\begin{pmatrix}0&0\\0&2\end{pmatrix}$ non-degenerate on the fibres of the vertical bundle VE, such that*

$$G=G^H+G^V. \tag{4.2}$$

Indeed, from (4.1) we get $G(X,Y)=G(X^H,Y^H)+G(X^V,Y^V)$. Putting $G^H(X,Y)=G(X^H,Y^H)$ and $G^V(X,Y)=G(X^V,Y^V)$ one obtains two Finsler tensor fields with the properties mentioned above.

A Finsler connection ∇ on E, which preserves by parallelism the distributions N and E^V, is called compatible with the Riemannian structure G or is called a metrical Finsler connection if $\nabla_X G=0$, $\forall X\in\mathcal{X}(E)$.

Proposition 4.3. *A Finsler connection ∇ on the total space E is a metrical connection, with respect to the Riemannian structure G, if and only if*

$$\nabla_X^H G^H=0,\quad \nabla_X^H G^V=0,\quad \nabla_X^V G^H=0,\quad \nabla_X^V G^V=0. \tag{4.3}$$

Theorem 4.1. *If G is a Riemannian structure on the total space E of a vector bundle $\xi=(E,\pi,M)$, then the following Finsler connection is compatible with the structure G:*

$$2G^H(\nabla_X^H Y^H,Z^H)=X^H G^H(Y^H,Z^H)+Y^H G^H(Z^H,X^H)-Z^H G^H(X^H,Y^H)-$$
$$-G^H(X^H;[Y^H,Z^H]^H)+G^H(Y^H,[Z^H,X^H]^H)+G^H(Z^H,[X^H,Y^H]^H),$$

$$
(4.4)\quad
\begin{aligned}
&\nabla^H_X Y^V = \mathring{\nabla}^H_X Y^V + B(Y^V, X^H), \quad G^V(B(Y^V, X^H), Z^V) = (1/2)(\mathring{\nabla}^H_X G^V)(Y^V, Z^V),\\
&\nabla^V_X Y^H = \mathring{\nabla}^V_X Y^H + D(Y^H, X^V), \quad G^H(D(Y^H, X^V), Z^H) = (1/2)(\mathring{\nabla}^V_X G^H(Y^H, Z^H),\\
&2G^V(\nabla^V_X Y^V, Z^V) = X^V G^V(Y^V, Z^V) + Y^V G^V(Z^V, X^V) - Z^V G^V(X^V, Y^V) -\\
&\qquad - G^V(X^V, [Y^V, Z^V]^V) + G^V(Y^V, [Z^V, X^V]^V) + G^V(Z^V, [X^V, Y^V]^V),
\end{aligned}
$$

where $\mathring{\nabla}$ *is a fixed Finsler connection, which preserves by parallelism the distributions* N *and* E^V.

Proof. We know that there is a Finsler connection $\mathring{\nabla}$ which preserves by parallelism the distributions N and E^V. In the condition $[T(X^H, Y^H)]^H = 0$, by the classical method, the first equality in (4.4) gives, uniquely, $\nabla^H_X Y^H$ and the second one $\nabla^H_X Y^V$. It is easy to see that ∇^H, determined in this way, is a h-covariant derivative in the algebra $\underset{F}{\mathscr{T}}(E)$ and that we have $\nabla^H_X G^H = 0$, $\nabla^H_X G^V = 0$. Analogously, the third equation in (4.4) gives uniquely $\nabla^V_X Y^H$, and the last one, in condition $[T(X^V, Y^V)]^V = 0$, allows to determine uniquely $\nabla^V_X Y^V$. This ∇^V is a v-covariant derivative in the algebra $\underset{F}{\mathscr{T}}(E)$ and it has the properties $\nabla^V_X G^H = 0$, $\nabla^V_X G^V = 0$. Therefore $\nabla = \nabla^H + \nabla^V$ is a Finsler connection on E compatible with the Riemannian structure G.

In canonical coordinates, let $g_{ij} = G\left(\dfrac{\delta}{\delta x^i}, \dfrac{\delta}{\delta x^j}\right)$, $g_{ab} = G\left(\dfrac{\partial}{\partial y^a}, \dfrac{\partial}{\partial y^b}\right)$. The equations (4.4) give the following coefficients of the metrical Finsler connection ∇:

$$
(4.5)\quad
\begin{aligned}
&F^i_{jk}(x, y) = \frac{1}{2} g^{ih}\left(\frac{\delta g_{hj}}{\delta x^k} + \frac{\delta g_{hk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}\right), \quad F^a_{bk}(x, y) = \mathring{F}^a_{bk}(x, y) + \frac{1}{2} g^{ac} g_{bc\mathring{|}k},\\
&C^i_{jb}(x, y) = \mathring{C}^i_{jb}(x, y) + \frac{1}{2} g^{ih} g_{jh}\mathring{|}_b, \quad C^a_{bc}(x, y) = \frac{1}{2} g^{ad}\left(\frac{\partial g_{db}}{\partial y^c} + \frac{\partial g_{dc}}{\partial y^b} - \frac{\partial g_{bc}}{\partial y^d}\right),
\end{aligned}
$$

where $(\mathring{F}^i_{jk}, \mathring{F}^a_{bk}, \mathring{C}^i_{jb}, \mathring{C}^a_{bc})$ are the coefficients of a fixed Finsler connection $\mathring{\nabla}$, and $\mathring{|}$, $\mathring{|}$ are the h- and v-covariant derivatives with respect to $\mathring{\nabla}$.

Observing that the distribution N is geometrically determined by the structure G we get that $F^i_{jk}(x, y)$ and $C^a_{bc}(x, y)$ are well-determined by G. Then, considering as a fixed Finsler connection $\mathring{\nabla}$ the Finsler connection with coefficients

$$
(4.6)\quad \mathring{F}^i_{jk}(x, y) = F^i_{jk}(x, y), \mathring{F}^a_{bk}(x, y) = \frac{\partial N^a_k}{\partial y^b}, \quad \mathring{C}^i_{jb}(x, y) = 0, \quad \mathring{C}^a_{bc} = C^a_{bc},
$$

we have

Theorem 4.2. *The Finsler connection* (4.5), (4.6) *is metrical and depends only on the Riemannian structure* G.

This connection can be called the canonical metrical Finsler connection. We get without difficulty:

Theorem 4.3. *The Riemann—Christoffel connection of a Riemannian structure G on the total space E of a vector bundle ξ coincides with the canonical metrical Finsler connection of G if and only if*

(1) *the horizontal distribution N is integrable,*

(2) *the metrical tensor field G is constant on the fibres of vertical subbundle VE,*

(3) $[T(X^H, Y^V)]^V = 0$.

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Ingham—Jessen's inequality for deviation means

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1. Introduction

Let $\mathbf{R}$, $\mathbf{R}_+$ and $\mathbf{N}$ denote the set of real numbers, positive real numbers and natural numbers, respectively. Let $I\subset\mathbf{R}$ be an interval and let M and N be discrete symmetric means on I. (See PÁLES [11].) We say that M and N satisfy the Ingham—Jessen's inequality if

$$M\big(N(x_{11}, \ldots, x_{1n}), \ldots, N(x_{m1}, \ldots, x_{mn})\big) \leqq$$

$$\leqq N\big(M(x_{11}, \ldots, x_{m1}), \ldots, M(x_{1n}, \ldots, x_{mn})\big)$$

i.e. if

(1) $$M^i\big(N^j(x_{ij})\big) \leqq N^j\big(M^i(x_{ij})\big)$$

for $x_{ij}\in I$, $i\in\{1, \ldots, m\}$, $j\in\{1, \ldots, n\}$, $n, m\in\mathbf{N}$.

This inequality was considered first by JESSEN [7] and INGHAM in the case when M and N are power means.

Define, for $a\in\mathbf{R}$, $x=(x_1, \ldots, x_n)\in\mathbf{R}_+^n$ $(n\in\mathbf{N})$, the a-th power mean $M_a(x)=$ $=M_a^i(x_i)=M_a(x_1, \ldots, x_n)$ by

$$\Big(\sum_{i=1}^n x_i^a/n\Big)^{1/a}, \quad \text{if} \quad a \neq 0,$$

$$\Big(\prod_{i=1}^n x_i\Big)^{1/n}, \quad \text{if} \quad a = 0.$$

Now the result obtained by JESSEN [7] can be formulated as follows (see HARDY—LITTLEWOOD—PÓLYA [6, Th. 26, p. 31]):

Theorem A. *Let* $a, b\in\mathbf{R}$. *In order that the inequality*

(2) $$M_b^i\big(M_a^j(x_{ij})\big) \leqq M_a^j\big(M_b^i(x_{ij})\big)$$

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be valid for any $x_{ij}\in\mathbf{R}_+$, $i\in\{1, \ldots, m\}$, $j\in\{1, \ldots, n\}$, $n, m\in\mathbf{N}$, *it is necessary and sufficient that*

$$a \leqq b. \tag{3}$$

There are a lot of investigations concerning this result. JESSEN [8] considered a more complicated inequality than (1) for power means. (See [6, Th. 137, p. 101].) KALMAN [9] proved a more general inequality than (2). TOYAMA [14] investigated the ratio of the right and left hand sides of (2). Fixing n and m, he gave the greatest lower and least upper bounds of this ratio.

A natural way of generalizing the inequality (2) is to investigate (1) for more general classes of means than power means. In [13], the author considered inequality (1) for homogeneous quasiarithmetic means with continuous weight function. These are the means defined as follows (see ACZÉL—DARÓCZY [1]): for $a, p\in\mathbf{R}$, $x=(x_1, \ldots, x_n)\in\mathbf{R}_+^n$, $n\in\mathbf{N}$, let

$$M_a(x)_p = M_a^i(x_i)_p = M_a(x_1, \ldots, x_n) = \Big(\sum_{i=1}^{n} x_i^{a+p}\Big/\sum_{i=1}^{n} x_i^p\Big)^{1/a}, \quad \text{if} \quad a \neq 0,$$

$$= \exp\Big(\sum_{i+1}^{n} x_i^p \ln x_i\Big/\sum_{i=1}^{n} x_i^p\Big), \quad \text{if} \quad a = 0.$$

For $t\in\mathbf{R}$, denote $t^+=\max\{t, 0\}$, $t^-=\max\{-t, 0\}$.

Concerning these mean values the author obtained the following result (see [13]):

Theorem B. *Let* $a, b, p, q\in\mathbf{R}$. *In order that the inequality*

$$M_b^i(M_a^j(x_{ij})_p)_q \leqq M_a^j(M_b^i(x_{ij})_q)_p \tag{4}$$

be valid for any $x_{ij}\in\mathbf{R}_+$, $i\in\{1, \ldots, m\}$, $j\in\{1, \ldots, n\}$, $n, m\in\mathbf{N}$, *it is necessary and sufficient that*

$$p-a^- \leqq q-b^- \leqq p+a^+ \leqq q+b^+, \quad (p-a^-)(q-b^-)(p+a^+)(q+b^+) = 0. \tag{5}$$

It is easy to see that if $p=q=0$ then we obtain the power means, furthermore, (4) and (5) turn into (2) and (3), respectively.

JESSEN [8] investigated (1) for quasiarithmetic means, too. However he obtained only necessary conditions. (See [6, Th. 136, p. 100].) The aim of the present note is to discuss (1) under very general circumstances. We consider inequality (1) for deviation means. This class of means has many interesting properties (see e.g. DARÓCZY [3], [4], DARÓCZY—PÁLES [5], PÁLES [10], [11], [12]) and contains the well-known classes of means (e.g. power means, quasiarithmetic means, quasiarithmetic means with weight function (ACZÉL—DARÓCZY [1], BAJRAKTAREVIC [2])). If M and N are deviation means then under certain regularity assumptions we obtain necessary and sufficient conditions in order that (1) be valid. We also consider (1) for homogeneous deviation means. In this case the necessary and sufficient conditions are simpler. In the last section we mention some open problems.

2. Notations, definitions and auxiliary results

Let $I \subset \mathbf{R}$ be an open interval. Now, we introduce the basic subclass of deviation functions.

Definition 1. A function $E: I^2 \to \mathbf{R}$ is called a $*$-deviation if it satisfies the following properties:

(i) E is twice differentiable on I^2.

(ii) $\partial E(x, t)/\partial t = \partial_2 E(x, t) < 0$ for $x, t \in I$.

(iii) $E(t, t) = 0$ for $t \in I$.

The class of $*$-deviations will be denoted by $\mathscr{E}(I)$. For $*$-deviations on I we shall also need the following usefull notation: If $E \in \mathscr{E}(I)$ then define E^* by

$$E^*(x, t) = -E(x, t)/\partial_2 E(t, t), \quad x, t \in I.$$

The following theorem and definition is due to DARÓCZY [3], [4].

Theorem C. *Let $E \in \mathscr{E}(I)$, $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$. Then there exists a unique number t_0 in I such that*

$$\operatorname{sgn} \sum_{i=1}^{n} E(x_i, t) = \operatorname{sgn}(t_0 - t) \tag{6}$$

for $t \in I$ and

$$\min_{1 \leq i \leq n} x_i \leq t_0 \leq \max_{1 \leq i \leq n} x_i. \tag{7}$$

Definition 2. Let $E \in \mathscr{E}(I)$, $n \in \mathbf{N}$, $x = (x_1, \ldots, x_n) \in I^n$, and consider the equation

$$\sum_{i=1}^{n} E(x_i, t) = 0. \tag{8}$$

Then, by Theorem C, there exists a unique solution $t = t_0$ of (8) and this solution is called the E-deviation mean of x and is denoted by $\mathfrak{M}_E(x)$ or $\mathfrak{M}_E^i(x_i)$ or $\mathfrak{M}_E(x_1, \ldots, x_n)$. (7) shows that $\mathfrak{M}_E(x)$ is indeed a mean value of x.

Remark. The proof of Theorem C can be found in [3], [4]. However, it can easily be proved using the facts that the function

$$t \to \sum_{i=1}^{n} E(x_i, t), \quad t \in I,$$

is continuous, strictly monoton decreasing and changes sign on the interval I.

The class of $*$-deviations is contained in the class of deviations introduced by DARÓCZY [3], [4]. Theorem C and Definition 2 can very easily be extended to deviations (see [3]).

Examples. 1. Let $\varphi\colon I\to\mathbf{R}$ be a twice differentiable function with positive first derivative and let $f\colon I\to\mathbf{R}_+$ be a positive, twice differentiable function. Set

$$E_{\varphi,f}(x,t)=f(x)\big(\varphi(x)-\varphi(t)\big),\quad x,t\in I. \tag{9}$$

It is obvious that $E_{\varphi,f}\in\mathscr{E}(I)$. If $n\in\mathrm{N}$, $x=(x_1,\dots,x_n)\in I^n$ then $\mathfrak{M}_{E_{\varphi,f}}(x)$ has the following form:

$$\mathfrak{M}_{E_{\varphi,f}}(x)\doteq M_\varphi(x)_f=\varphi^{-1}\Big(\sum_{i=1}^n f(x_i)\varphi(x_i)\Big/\sum_{i=1}^n f(x_i)\Big)$$

i.e. $\mathfrak{M}_{E_{\varphi,f}}$ is a quasiarithmetic mean with weight function (see Bajraktarević [2]). If $f(x)=1$ then $\mathfrak{M}_{E_{\varphi,f}}$ becomes the quasiarithmetic mean M_φ (see Hardy—Littlewood—Pólya [6]).

2. Let $a,p\in\mathbf{R}$ and set

$$E_{a,p}(x,t)=\begin{cases} x^p(x^a-t^a)/a, & \text{if}\quad a\neq 0,\\ x^p(\ln x-\ln t), & \text{if}\quad a=0.\end{cases} \tag{10}$$

Now, for $x\in I^n$, we obtain that $\mathfrak{M}_{E_{a,p}}(x)=M_a(x)_p$. If $p=0$ then we get the power means.

Now we prove a sequence of lemmas which will be needed later on.

Lemma 1. *Let $E\in\mathscr{E}(I)$. Then, for fixed $n\in\mathrm{N}$,*

$$(x_1,\dots,x_n)\to\mathfrak{M}_E(x_1,\dots,x_n) \tag{11}$$

is a continuously differentiable function on I^n and

$$\partial_i\mathfrak{M}_E(x_1,\dots,x_n)=-\partial_1E(x_i,\mathfrak{M}_E(x))\Big/\Big(\sum_{j=1}^n\partial_2E(x_j,\mathfrak{M}_E(x))\Big) \tag{12}$$

for $x=(x_1,\dots,x_n)\in I^n$. (Here ∂_i denotes the partial differentiation with respect to the i-th variable.)

Proof. Let $x_0=(x_{01},\dots,x_{0n})\in I^n$ be fixed and denote by t_0 the mean value $\mathfrak{M}_E(x_0)$. Let

$$F(x,t)=\sum_{i=1}^n E(x_i,t)$$

for $x=(x_1,\dots,x_n)\in I^n$ and $t\in I$. By our assumption (i) on deviations belonging to $\mathscr{E}(I)$, F is continuously differentiable in a neighborhood of (x_0,t_0). (ii) in Definition 1 implies that

$$\partial_tF(x_0,t_0)=\sum_{i=1}^n\partial_2E(x_o,t_0)<0,$$

and we know that $F(x_0, t_0)=0$. Thus the conditions of the implicit function theorem are satisfied. Consequently, the function (11) determined by the equation

$$F(x, \mathfrak{M}_E(x)) = 0$$

is differentiable at the point x_0 and its derivative has the form

$$D\mathfrak{M}_E(x_0) = -(\partial_t F(x_0, t_0))^{-1}\partial_x F(x_0, t_0)$$

i.e. (12) is satisfied at x_0.

The continuity of the function (19) follows from (i). This completes the proof of the lemma.

Lemma 2. *Let $E\in\mathscr{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty}(n-1)(\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1})-t) = E^*(x, t). \tag{13}$$

We omit the proof of this lemma since it is proved in Daróczy [3], [4].

Lemma 3. *Let $E\in\mathscr{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty}(n-1)\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = \partial_1 E^*(x, t). \tag{14}$$

Proof. Let $x, t\in I$ be arbitrary. For $n\in\mathbf{N}$, let

$$t_n = \mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}). \tag{15}$$

Applying Lemma 1, we have

$$\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(x, t_n)}{\partial_2 E(x, t_n)+(n-1)\partial_2 E(t, t_n)}.$$

Hence

$$(n-1)\partial_1\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(x, t_n)}{\partial_2 E(x, t_n)/(n-1)+\partial_2 E(t, t_n)}. \tag{16}$$

By Lemma 2,

$$\lim_{n\to\infty} t_n = t. \tag{17}$$

Therefore, taking the limit $n\to\infty$ in (16) we obtain (14).

Lemma 4. *Let $E\in\mathscr{E}(I)$. Then, for $x, t\in I$,*

$$\lim_{n\to\infty}(n-1)\Big(\sum_{i=2}^{n}\partial_i\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1})-1\Big) = \partial_2 E^*(x, t). \tag{18}$$

Proof. Let $x, t\in I$ be arbitrary and let t_n be defined by (15). Applying Lemma 1 we have

$$\partial_i\mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) = -\frac{\partial_1 E(t, t_n)}{\partial_2 E(x, t_n)+(n-1)\partial_2 E(t, t_n)}$$

for $2 \leqq i \leqq n$. Hence, after a simple calculation, we obtain

$$(19)\qquad (n-1)\Big(\sum_{i=2}^{n} \partial_i \mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) - 1\Big) = -\frac{\partial_2 E(x, t_n) + (n-1)(\partial_1 E(t, t_n) + \partial_2 E(t, t_n))}{\partial_2 E(x, t_n)/(n-1) + \partial_2 E(t, t_n)}.$$

Since $E(t, t) = 0$, we have

$$(20)\qquad \partial_1 E(t, t) + \partial_2 E(t, t) = 0 \quad \text{for} \quad t \in I.$$

If $t \neq x$ then t_n is strictly between x and t. Applying (20) and Lemma 2, we obtain

$$(21)\qquad \begin{aligned} &\lim_{n\to\infty} (n-1)(\partial_1 E(t, t_n) + \partial_2 E(t, t_n)) = \\ &= \lim_{n\to\infty} (n-1)(t_n - t)\left(\frac{\partial_1 E(t, t_n) - \partial_1 E(t, t)}{t_n - t} - \frac{\partial_2 E(t, t_n) - \partial_2 E(t, t)}{t_n - t}\right) = \\ &= E^*(x, t)(\partial_2\partial_1 E(t, t) + \partial_2\partial_2 E(t, t)). \end{aligned}$$

It is easy to see that (21) remains valid if $x = t = t_n$. Now, applying (17) and (21), we can calculate the limit of the right hand side of (19). We get

$$\lim_{n\to\infty} (n-1)\Big(\sum_{i=2}^{n} \partial_i \mathfrak{M}_E(x, \underbrace{t, \dots, t}_{n-1}) - 1\Big) =$$

$$= -\frac{\partial_2 E(x, t) + E^*(x, t)(\partial_2\partial_1 E(t, t) + \partial_2\partial_2 E(t, t))}{\partial_2 E(t, t)} = \partial_2 E^*(x, t).$$

The proof is complete.

Remark. There is a simple connection between Lemma 2 and Lemmas 3 and 4. Differentiating (13) with respect to x and t we obtain (14) and (18), respectively.

3. The main result

In this section we give necessary and sufficient conditions in order that (1) be satisfied with $M = \mathfrak{M}_F$, $N = \mathfrak{M}_E$ where $F, E \in \mathcal{E}(I)$.

Theorem 1. *Let* $E, F \in \mathcal{E}(I)$. *The inequality*

$$(22)\qquad \mathfrak{M}_F^i(\mathfrak{M}_E^j(x_{ij})) \leqq \mathfrak{M}_E^j(\mathfrak{M}_F^i(x_{ij}))$$

holds for any $x_{ij} \in I$, $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, $n, m \in \mathbf{N}$, *if and only if*

$$(23)\qquad \begin{aligned} &E^*(x, y)\partial_1 F^*(y, u) + E^*(z, u)\partial_2 F^*(y, u) \leqq \\ &\leqq F^*(x, z)\partial_1 E^*(z, u) + F^*(y, u)\partial_2 E^*(z, u) \end{aligned}$$

for each $x, y, z, u \in I$.

Proof. Necessity. Let $x, y, z, u \in I$ be arbitrary and let $n, m \in \mathbf{N}$. Define x_{ij} $(1 \leqq i \leqq m, 1 \leqq j \leqq n)$ as follows:

$$\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \dots & x_{mn} \end{pmatrix} = \begin{pmatrix} x & y & \dots & y \\ z & u & \dots & u \\ \vdots & \vdots & & \vdots \\ z & u & \dots & u \end{pmatrix}. \tag{24}$$

Now introduce the following notations:

$$a(n) = \mathfrak{M}_E(x, \underbrace{y, \dots, y}_{n-1}), \qquad A(n) = \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}),$$

$$b(m) = \mathfrak{M}_F(x, \underbrace{z, \dots, z}_{m-1}), \qquad B(m) = \mathfrak{M}_F(y, \underbrace{u, \dots, u}_{m-1}).$$

Applying inequality (22) for the x_{ij}'s defined by (24) we obtain

$$\mathfrak{M}_F(a(n), \underbrace{A(n), \dots, A(n)}_{m-1}) \leqq \mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) \tag{25}$$

for $n, m \in \mathbf{N}$. If $m \to \infty$ then, using Lemma 2, one can easily see that both sides of (25) tend to $A(n)$. Therefore we calculate the following limits:

$$\lim_{m\to\infty} (m-1)\big(\mathfrak{M}_F(a(n), \underbrace{A(n), \dots, A(n)}_{m-1}) - A(n)\big) \doteq R(n), \tag{26}$$

$$\lim_{m\to\infty} (m-1)\big(\mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) - A(n)\big) \doteq L(n). \tag{27}$$

To calculate (26) apply Lemma 2. Then we obtain

$$R(n) = F^*(a(n), A(n)). \tag{28}$$

It is a bit more complicated to determine $L(n)$. By Lemma 2 we have

$$\lim_{m\to\infty} (m-1)(b(m)-z) = F^*(x, z), \quad \lim_{m\to\infty} (m-1)(B(m)-u) = F^*(y, u).$$

Hence, using Lemma 1 and the differentiability of $\mathfrak{M}_E$, we get

$$\begin{aligned} L(n) &= \lim_{m\to\infty} (m-1)\big(\mathfrak{M}_E(b(m), \underbrace{B(m), \dots, B(m)}_{n-1}) - \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1})\big) = \\ &= \partial_1 \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}) F^*(x, z) + \sum_{i=2}^{n} \partial_i \mathfrak{M}_E(z, \underbrace{u, \dots, u}_{n-1}) F^*(y, u). \end{aligned} \tag{29}$$

Inequality (25) implies that, for $n \in \mathbf{N}$,

$$R(n) \leqq L(n). \tag{30}$$

Using Lemmas 3 and 4 we easily obtain that both sides of (30) tend to $F^*(y, u)$. Therefore we calculate the following limits:

$$\lim_{n\to\infty}(n-1)\big(R(n)-F^*(y,u)\big) \doteq R^*, \quad \lim_{n\to\infty}(n-1)\big(L(n)-F^*(y,u)\big) \doteq L^*.$$

Applying Lemma 2 we have

$$\lim_{n\to\infty}(n-1)\big(a(n)-y\big) = E^*(x,y), \quad \lim_{n\to\infty}(n-1)\big(A(n)-u\big) = E^*(z,u).$$

Hence, using the differentiability of F^*, we get

$$R^* = \lim_{n\to\infty}(n-1)\big(F^*(a(n), A(n))-F^*(y,u)\big) = \tag{31}$$
$$= \partial_1 F^*(y,u)E^*(x,y)+\partial_2 F^*(y,u)E^*(z,u).$$

Applying Lemmas 3 and 4 we have

$$\lim_{n\to\infty}(n-1)\partial_1\mathfrak{M}_E\big(z,\underbrace{u,\dots,u}_{n-1}\big) = \partial_1 E^*(z,u),$$

$$\lim_{n\to\infty}(n-1)\Big(\sum_{i=1}^{n}\partial_i\mathfrak{M}_E\big(z,\underbrace{u,\dots,u}_{n-2}\big)-1\Big) = \partial_2 E^*(z,u).$$

Consequently

$$L^* = \partial_1 E^*(z,u)F^*(x,z) = \partial_2 E^*(z,u)F^*(y,u). \tag{32}$$

Inequality (30) implies $R^* \leqq L^*$. This completes the proof of (23).

Sufficiency. Let $n, m \in \mathbb{N}$ and $x_{ij} \in I$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. Further, let

$$y_i = \mathfrak{M}_E^j(x_{ij}), \quad z_j = \mathfrak{M}_F^i(x_{ij}), \quad u = \mathfrak{M}_E^j(\mathfrak{M}_F^i(x_{ij})) = \mathfrak{M}_E^j(z_j). \tag{33}$$

Apply (23) for $x=x_{ij}$, $y=y_i$, $z=z_j$ and add the inequalities obtained. Then we get

$$\sum_{i=1}^{m}\Big\{\partial_1 F^*(y_i,u)\sum_{j=1}^{n}E^*(x_{ij},y_i)\Big\}+\sum_{i=1}^{m}\partial_2 F^*(y_i,u)\sum_{j=1}^{n}E^*(z_j,u) \leqq \tag{34}$$
$$\leqq \sum_{j=1}^{n}\Big\{\partial_1 E^*(z_j,u)\sum_{i=1}^{m}F^*(x_{ij},z_j)\Big\}+\sum_{j=1}^{n}\partial_2 E^*(z_j,u)\sum_{i=1}^{n}F^*(y_i,u).$$

Using (33) and Definition 2 we have

$$\sum_{j=1}^{n}E^*(x_{ij},y_i) = 0, \quad \sum_{i=1}^{m}F^*(x_{ij},z_j) = 0, \quad \sum_{j=1}^{n}E^*(z_j,u) = 0.$$

Therefore (34) simplifies to the following inequality

$$0 \leqq \sum_{j=1}^{n}\partial_2 E^*(z_j,u)\sum_{i=1}^{m}F^*(y_i,u). \tag{35}$$

As we have seen in the proof of Lemma 4,

$$\partial_2 E^*(x, t) = -\frac{\partial_2 E(x, t)+E^*(x, t)\big(\partial_2\partial_1 E(t, t)+\partial_2\partial_2 E(t, t)\big)}{\partial_2 E(t, t)}.$$

Hence, by property (ii) of $*$-deviations,

$$\sum_{j=1}^{n} \partial_2 E^*(z_j, u) = -\sum_{j=1}^{n} \partial_2 E(z_j, u)/\partial_2 E(u, u) < 0. \tag{36}$$

(35) and (36) imply

$$\sum_{i=1}^{m} F(y_i, u) \leqq 0. \tag{37}$$

It follows from Theorem C and from (37) that $\mathfrak{M}^i_F(y_i) \leqq u$ i.e. (22) holds.

Remark. Applying Theorem 1 for the deviations defined by (9) we can easily obtain necessary and sufficient conditions for (1) if M and N are quasiarithmetic means with weight function.

4. Homogeneous means

Let $E \in \mathscr{E}(\mathbf{R}_+)$. The E-deviation mean $\mathfrak{M}_E$ is said to be homogeneous if

$$\mathfrak{M}_E(tx_1, ..., tx_n) = t\mathfrak{M}_E(x_1, ..., x_n)$$

for $t, x_1, ..., x_n \in \mathbf{R}_+$, $n \in \mathbf{N}$.

Concerning homogeneous deviation means Daróczy [3] obtained the following result:

Theorem D. *Let* $E \in \mathscr{E}(\mathbf{R}_+)$. *Then* $\mathfrak{M}_E$ *is homogeneous if and only if*

$$E^*(x, t) = tE^*(x/t, 1)$$

for $x, t \in \mathbf{R}_+$.

For homogeneous deviation means Theorem 1 simplifies to the following form.

Theorem 2. *Let* $E, F \in \mathscr{E}(\mathbf{R}_+)$ *and assume that* $\mathfrak{M}_E$ *and* $\mathfrak{M}_F$ *are homogeneous means. Then the inequality* (22) *is valid for any* $x_{ij} \in \mathbf{R}_+$, $i \in \{1, ..., m\}$, $j \in \{1, ..., n\}$, $n, m \in \mathbf{N}$, *if and only if*

$$t\partial_1^2 E^*(t, 1)\partial_1 F^*(s, 1) \leqq \partial_1 E^*(t, 1)s\partial_1^2 F^*(s, 1) \tag{38}$$

for $s, t \in \mathbf{R}_+$.

Proof. Since $\mathfrak{M}_E$ and $\mathfrak{M}_F$ are homogeneous means, Theorem D implies

$$E^*(x, t) = tE^*(x/t, 1) \doteq te(x/t), \quad F^*(x, t) = tF^*(x/t, 1) \doteq tf(x/t)$$

for $x, t \in \mathbf{R}_+$. By our assumptions on E and F we have that e and f are twice differentiable functions. Then, applying Theorem 1, we obtain that (22) holds if and only if

$$y/uf'(y/u)\{e(x/y)-e(z/u)\} \leqq z/ue'(z/u)\{f(x/z)-f(y/u)\} \tag{39}$$

for $x, y, z, u \in \mathbf{R}_+$.

Replacing x/u, y/u and z/u by r, s and t, respectively, we get

$$0 \leqq te'(t)\{f(r/t)-f(s)\}-sf'(s)\{e(r/s)-e(t)\} \tag{40}$$

for $r, s, t \in \mathbf{R}_+$. It is easy to check that (40) is equivalent to (39). Therefore the proof of the theorem will be complete if we show that (40) holds if and only if (38) is satisfied. Fixing s and t, we denote by $g(r)$ the right hand side of (40).

If (40) is satisfied then $r=st$ is the place of minimum of g. Hence $g''(st) \geqq 0$. This yields (38).

In the other direction we prove that

$$g'(r)(r-st) \geqq 0 \quad \text{for} \quad r>0. \tag{41}$$

Then $g(st)=0$ and (41) implies $g(r)>0$ for $r>0$.

Applying (38) for $s=r/t$ it can be easily seen that the function

$$t \to e'(t)f'(r/t), \quad t>0,$$

is monotone decreasing. Therefore, in the case $r<st$,

$$e'(t)f'(r/t) \leqq e'(r/s)f'(s),$$

i.e. (41) is satisfied in this case. In the case $r>st$ the proof of (41) is similar. The theorem is proved.

Remark. Applying Theorem 2 for the homogeneous means $M=M_{b,q}$, $N=M_{a,p}$ one can easily prove Theorem B. (For details see [13].)

5. Open problems and final remarks

Consider the following more general inequality than (1):

$$M_1^i(N_1^j(x_{ij})) \leqq N_2^j(M_2^i(x_{ij})) \tag{42}$$

for $x_{ij} \in I$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, $n, m \in \mathrm{N}$. (Here M_1, M_2, N_1, N_2 are discrete symmetric means on I.) The following conditions are necessary (but not sufficient) in order that (42) be satisfied:

$$M_1 \leqq M_2, \quad N_1 \leqq N_2, \tag{43}$$

$$N_1 \leqq M_2. \tag{44}$$

If we take $n=1$ and $m=1$ in (42) then we obtain (43). To prove that (44) is also a necessary condition we substitute into (42) the following matrix:

$$(x_{ij}) = \begin{pmatrix} x_1 & x_2 & x_3 \dots & x_n \\ x_2 & x_3 & x_4 \dots & x_1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & x_1 & x_2 \dots & x_{n-1} \end{pmatrix}$$

(whenever $x_1, \dots, x_n \in I$); then we obtain (44).

The system of inequalities (43), (44) is also a sufficient condition if M_1, M_2, N_1, N_2 are power means. (See JESSEN [7], HARDY—LITTLEWOOD—PÓLYA [6, Th. 137, p. 101].) However, it is not sufficient in other classes of means. Finally, we formulate a condition which is sufficient in the class of deviation means.

Let $E_1, E_2, F_1, F_2 \in \mathscr{E}(I)$. If there exist functions $A_1, A_2, B_1, B_2 \colon I^2 \to \mathbf{R}$ such that, for $x_1, \dots, x_n \in I$, $n \in \mathrm{N}$,

$$\sum_{j=1}^{n} B_2(x_j, \mathfrak{M}^j_{E_2}(x_j)) \leqq 0$$

and

$$E_1(x, y)A_1(y, u) + E_2(z, u)A_2(y, u) \leqq F_2(x, z)B_1(z, u) + F_1(y, u)B_2(z, u)$$

for $x, y, z, u \in I$ then (42) is satisfied for $M_1 = \mathfrak{M}_{F_1}$, $M_2 = \mathfrak{M}_{F_2}$, $N_1 = \mathfrak{M}_{E_1}$, $N_2 = \mathfrak{M}_{E_2}$. The proof of this proposition is similar to the proof of the sufficiency part of Theorem 1. We remark that this sufficient condition is also necessary if $E_1 = E_2 = E$, $F_1 = F_2 = F$. (That was the case investigated in Theorem 1.)

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On a class of multiplicative functions

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In a recent paper [1] we proved the following result: Let f be a multiplicative arithmetical function satisfying

$$\sum_{n \leqq x} |f(n)|^2 = O(x). \tag{1}$$

Then, for every irrational α, $(1/x) \sum_{n \leqq x} f(n) e^{2\pi i \alpha n}$ tends to zero as x tends to infinity.

We stated that this result still holds if the exponent 2 in (1) is replaced by any $\lambda > 1$ and that we planned to give the proof elsewhere. It turns out that this follows from the work of INDLEKOFER [3], for the hypothesis that

$$\sum_{n \leqq x} |f(n)|^\lambda = O(x) \quad \text{for some} \quad \lambda > 1 \tag{2}$$

implies that f belongs to his class $\mathcal{L}^*$. However we think it is still interesting to give our proof, which is quite different and enables us to obtain a uniform result, namely the following one.

Theorem. *Let K and λ be fixed real numbers satisfying $K \geqq 1$ and $\lambda > 1$, and let α be a fixed irrational number. Given any $\varepsilon > 0$, there exists a positive $X = X(K, \lambda, \alpha, \varepsilon)$ such that, for any multiplicative arithmetical function satisfying*

$$\sum_{n \leqq x} |f(n)|^\lambda \leqq Kx \quad \textit{for all positive } x, \tag{3}$$

we have

$$\Big|(1/x) \sum_{n \leqq x} f(n) e^{2\pi i \alpha n}\Big| \leqq \varepsilon \quad \textit{for} \quad x \geqq X.$$

For the proof it is obviously sufficient to consider the case when $\lambda \leqq 2$ for, by Hölder's inequality, (3) implies

$$\sum_{n \leqq x} |f(n)|^{\lambda'} \leqq K^{\lambda'/\lambda} x \quad \text{for any positive} \quad \lambda' < \lambda.$$

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Throughout this paper we write $e(t)$ for $e^{2\pi it}$; the letter p is used for a prime number, while the letters m and n are used to denote positive integers. $p^r\|n$ means "$p^r|n$ but $p^{r+1}\nmid n$". An empty sum is assumed to be zero.

1. To prove the theorem we need three lemmas.

1.1. Lemma 1. *Let K, M and ε be real numbers satisfying $M>1$ and $0<\varepsilon<M-1$. There exists $X(K, M, \varepsilon)>e$ such that, if g is any real-valued multiplicative function satisfying*

$$g(n)\geqq 0 \quad \textit{for every } n \textit{ and } \quad \sum_{n\leqq x} g(n)\leqq Kx \quad \textit{for every} \quad x\geqq 1,$$

then for $x\geqq X(K, M, \varepsilon)$

$$\sum_{\substack{p\leqq x\\ g(p)<M}} 1/p \geqq \bigl(1-(1+\varepsilon)/M\bigr)\log\log x.$$

Proof. As in the proof of Lemma 1 of [1] we have for $\sigma>1$ and $x>e$

$$\frac{1}{\log\log x}\sum_{\substack{p\leqq x\\ g(p)<M}}\frac{1}{p}\geqq\frac{1}{\log\log x}\sum_{p\leqq x}\frac{1}{p}-\frac{x^{\sigma-1}}{M\log\log x}\left(A+\log\frac{K\sigma}{\sigma-1}\right),$$

where $A=\sum\left(M/p-\log\left(1+M/p\right)\right)$.

Let η be a positive number such that $e^{\eta}<1+\varepsilon$. If $\sigma=1+\eta/\log x$, then, as x tends to infinity, the right-hand side of the above inequality tends to $1-e^{\eta}/M$. So there exists $X>e$ such that it is $\geqq 1-(1+\varepsilon)/M$ for $x\geqq X$.

1.2. Lemma 2. *Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of complex numbers with the following properties:*

(i) *$|a_n|$ is a multiplicative function of n;*

(ii) *for some $\lambda\in]1, 2]$, $\sum_{n\leqq x}|a_n|^{\lambda}\leqq Kx$ for all positive x.*

Let $\mu=\lambda/(\lambda-1)$ (so that $1/\lambda+1/\mu=1$). If M is any positive number, then we have for every positive x

$$\sum_{\substack{p\leqq x\\ |a_p|\leqq M}}(1/p)\Bigl|(p/x)\sum_{\substack{n\leqq x\\ p\|n}}a_n-(1/x)\sum_{n\leqq x}a_n\Bigr|^{\mu}\leqq C_{\lambda,M}K^{1/(\lambda-1)},$$

where $C_{\lambda,M}$ is a constant which depends only upon λ and M.

Proof. It is known (Elliott [2]) that there exists an absolute contant C such that, if $x_1, x_2, \ldots, x_n, \ldots$ is any sequence of complex numbers, then for every positive x

$$\sum_{\substack{p,r\\ p^r\leqq x}}(1/p^r)\Bigl|(p^r/x)\sum_{\substack{n\leqq x\\ p^r\|n}}x_n-(1/x)\sum_{n\leqq x}x_n\Bigr|^2\leqq C(1/x)\sum_{n\leqq x}|x_n|^2,$$

and therefore

$$\sum_{p\leq x}(1/p)|(p/x)\sum_{\substack{n\leq x\\ p\|n}}x_n-(1/x)\sum_{n\leq x}x_n|^2\leq C(1/x)\sum_{n\leq x}|x_n|^2.$$

We now consider a fixed positive x and we denote by $\mathscr{P}_x$ the set of those primes p which are $\leq x$ and for which $|a_p|\leq M$. The quantity to be estimated is

$$S=\sum_{p\in\mathscr{P}_x}(1/p)X_p^{\mu},\quad \text{where}\quad X_p=\Big|(p/x)\sum_{\substack{n\leq x\\ p\|n}}a_n-(1/x)\sum_{n\leq x}a_n\Big|.$$

We may suppose $S>0$ (and so $\mathscr{P}_x\neq\emptyset$) for the inequality to be proved is trivial if $S=0$. We may write $a_n=\alpha_n|a_n|$, where $|\alpha_n|=1$, and

$$(p/x)\sum_{\substack{n\leq x\\ p\|n}}a_n-(1/x)\sum_{n\leq x}a_n=\omega_pX_p,\quad \text{where}\quad |\omega_p|=1.$$

Define the function Φ on $\mathbf{C}$ by

$$\Phi(z)=(KS)^{-z}\sum_{p\in\mathscr{P}_x}(1/p)\bar{\omega}_pX_p^{\mu z}\Big((p/x)\sum_{\substack{n\leq x\\ p\|n}}\alpha_n|a_n|^{\lambda z}-(1/x)\sum_{n\leq x}\alpha_n|a_n|^{\lambda z}\Big).$$

(Here $0^u=0$ for any complex u). Φ is an entire function and it is bounded in every strip $A\leq\operatorname{Re}z\leq B$. We see that $\Phi(1/\lambda)=K^{-1/\lambda}S^{1/\mu}$ and that $|\Phi(z)|\leq 1+M^\lambda$ when $\operatorname{Re}z=1$, for

$$\sum_{\substack{n\leq x\\ p\|n}}|a_n|^\lambda=\sum_{\substack{mp\leq x\\ p\nmid m}}|a_{mp}|^\lambda=|a_p|^\lambda\sum_{\substack{m\leq x/p\\ p\nmid m}}|a_m|^\lambda\leq|a_p|^\lambda\sum_{m\leq x/p}|a_m|^\lambda.$$

Using Cauchy—Schwarz's inequality and Elliott's inequality, with $x_n=\alpha_n|a_n|^{\lambda z}$, we see that $|\Phi(z)|\leq C^{1/2}$ when $\operatorname{Re}z=1/2$. It follows that, when $1/2\leq\operatorname{Re}z\leq 1$,

$$|\Phi(z)|\leq C^{1-\operatorname{Re}z}(1+M^\lambda)^{2(\operatorname{Re}z-1/2)}.$$

Taking $z=1/\lambda$ we get $K^{-1/\lambda}S^{1/\mu}\leq C^{1/\mu}(1+M^\lambda)^{2/\lambda-1}$, which yields

$$S\leq C(1+M^\lambda)^{(2-\lambda)/(\lambda-1)}K^{1/(\lambda-1)}.$$

1.3. Lemma 3. *Given an arithmetical function f and a real number α, set*

$$C_f(x,\alpha)=(1/x)\sum_{n\leq x}f(n)e(n\alpha)\quad(x>0).$$

Now let $l_1,l_2,\dots,l_r$ be fixed positive numbers satisfying $l_j\geq l_1$ for $j>1$, and let $\alpha_1,\alpha_2,\dots,\alpha_r$ be fixed real numbers satisfying $\alpha_j\not\equiv\alpha_k$ (mod 1) for $j\neq k$. Let

$$T=\sum_{j\neq k}1/|\sin\pi(\alpha_j-\alpha_k)|.$$

Then, for any arithmetical function f satisfying

(4) $$\sum_{n\leq x}|f(n)|^\lambda\leq Kx\quad \textit{for every positive } x\quad(K>0),$$

where $1<\lambda\leqq 2$, *we have for* $x\geqq T/\varepsilon$

$$\sum_{j=1}^{r}(1/l_j)|C_f(x/l_j,\alpha_j)|^\mu \leqq (1+\varepsilon)K^{1/(\lambda-1)}/l_1,$$

where $\mu=\lambda/(\lambda-1)$ *(so that* $1/\lambda+1/\mu=1$*).*

Proof. It is proved in [1] (Lemma 3) that there exists $X(\varepsilon)>0$ such that, for any arithmetical function f satisfying

$$\sum_{n\leqq x}|f(n)|^2\leqq Kx \quad \text{for every positive } x,$$

we have

$$\sum_{j=1}^{r}(1/l_j)|C_f(x/l_j,\alpha_j)|^2\leqq(1+\varepsilon)K/l_1 \quad \text{for} \quad x\geqq X(\varepsilon).$$

Although this is not stated explicitly, it is clear in the proof that $X(\varepsilon)$ may be taken equal to T/ε.

Now let f be any arithmetical function satisfying (4). Consider a fixed $x\geqq T/\varepsilon$ and set

$$|C_f(x/l_j,\alpha_j)|=Y_j,\quad C_f(x/l_j,\alpha_j)=Y_ju_j,\quad \text{where} \quad |u_j|=1,$$

$$f(n)=|f(n)|v_n,\quad \text{where} \quad |v_n|=1,$$

and

$$S=\sum_{j=1}^{r}Y_j^\mu/l_j.$$

We have to prove that $S\leqq(1+\varepsilon)K^{1/(\lambda-1)}/l_1$. Since this is trivially true if $S=0$, we may suppose $S>0$. Define the function Ψ on $\mathbf{C}$ by

$$\Psi(z)=l_1^{1-z}K^{-z}S^{-z}\sum_{j=1}^{r}\bar{u}_jY_j^{\mu z}\big((1/x)\sum_{n\leqq x/l_j}|f(n)|^{\lambda z}v_ne(\alpha_jn)\big).$$

Ψ is an entire function and it is bounded in every strip $A\leqq\operatorname{Re}z\leqq B$. We see that $\Psi(1/\lambda)=l_1^{1/\mu}K^{-1/\lambda}S^{1/\mu}$ and that $|\Psi(z)|\leqq 1$ for $\operatorname{Re}z=1$. If $z=1/2+iy$, where y is real, then by the Cauchy—Schwarz inequality

$$|\Psi(z)|\leqq l_1^{1/2}K^{-1/2}S^{-1/2}\big(\sum_{j=1}^{r}Y_j^\mu/l_j\big)^{1/2}\big(\sum_{j=1}^{r}l_j|(1/x)\sum_{n\leqq x/l_j}|f(n)|^{\lambda(1/2+iy)}v_ne(\alpha_jn)|^2\big)^{1/2},$$

that is

$$|\Psi(z)|\leqq l_1^{1/2}K^{-1/2}\big(\sum_{j=1}^{r}(1/l_j)|C_{f_y}(x/l_j,\alpha_j)|^2\big)^{1/2}, \quad \text{where} \quad f_y(n)=|f(n)|^{\lambda(1/2+iy)}v_n.$$

Since $\sum_{n\leqq x}|f_y(n)|^2=\sum_{n\leqq x}|f(n)|^\lambda\leqq Kx$, it follows from the above quoted result that

$$\sum_{j=1}^{r}(1/l_j)|C_{f_y}(x/l_j,\alpha_j)|^2\leqq(1+\varepsilon)K/l_1.$$

We thus see that $|\Psi(z)| \leqq (1+\varepsilon)^{1/2}$ for Re $z=1/2$. It follows that $|\Psi(z)| \leqq (1+\varepsilon)^{1-\mathrm{Re}\, z}$ for $1/2 \leqq \mathrm{Re}\, z \leqq 1$. In particular $|\Psi(1/\lambda)| \leqq (1+\varepsilon)^{1/\mu}$, or $l_1^{1/\mu} K^{-1/\lambda} S^{1/\mu} \leqq (1+\varepsilon)^{1/\mu}$, which yields the desired result.

2. Proof of the theorem. Let K and λ be fixed real numbers satisfying $K \geqq 1$ and $1<\lambda \leqq 2$, and let α be a fixed irrational number. Let $\mu=\lambda/(\lambda-1)$.

2.1. We first choose $M>1$ and η satisfying $0<\eta<M^\lambda-1$. By Lemma 1 we can choose $X_1>e$ such that, for any multiplicative function f satisfying (3),

$$\sum_{\substack{p \leqq x \\ |f(p)|<M}} 1/p \geqq \left(1-(1+\eta)/M^\lambda\right) \log\log x \quad \text{for} \quad x \geqq X_1. \tag{5}$$

2.2. Now we consider a fixed multiplicative function f satisfying (3) and we denote by $\mathscr{P}$ the set of those primes p for which $|f(p)| \leqq M$. By (5) $\mathscr{P}$ is infinite and its smallest element p_0 is $\leqq X_1$. We remark that, for each prime p,

$$\sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) = \sum_{\substack{mp \leqq x \\ p \nmid m}} f(mp) e(\alpha mp) = f(p) \sum_{\substack{m \leqq x/p \\ p \nmid m}} f(m) e(p\alpha m) =$$

$$= f(p) \Big(\sum_{m \leqq x/p} f(m) e(p\alpha m) - \sum_{\substack{m \leqq x/p \\ p | m}} f(m) e(p\alpha m) \Big).$$

The first sum in the brackets is, in the notation of Lemma 3, $(x/p) C_f(x/p, p\alpha)$. On the other hand we have

$$\Big| \sum_{\substack{m \leqq x/p \\ p|m}} f(m) e(p\alpha m) \Big| \leqq \Big(\sum_{\substack{m \leqq x/p \\ p|m}} |f(m)|^\lambda \Big)^{1/\lambda} \Big(\sum_{\substack{m \leqq x/p \\ p|m}} 1 \Big)^{1/\mu} \leqq$$

$$\leqq (Kx/p)^{1/\lambda} (x/p^2)^{1/\mu} = K^{1/\lambda} x / p^{1+1/\mu}.$$

We thus see that

$$\Big| \sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) \Big| \leqq |f(p)| (x/p) \left(|C_f(x/p, p\alpha)| + K^{1/\lambda}/p^{1/\mu} \right).$$

In particular, if $p \in \mathscr{P}$, then

$$\Big| p/x \sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) \Big| \leqq M \left(|C_f(x/p, p\alpha)| + K^{1/\lambda}/p^{1/\mu} \right).$$

It follows that for every $p \in \mathscr{P}$

$$\Big| (1/x) \sum_{n \leqq x} f(n) e(\alpha n) \Big| \leqq$$

$$\leqq \Big| (p/x) \sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) \Big| + \Big| (p/x) \sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) - (1/x) \sum_{n \leqq x} f(n) e(\alpha n) \Big| \leqq$$

$$\leqq M |C_f(x/p, p\alpha)| + M K^{1/\lambda}/p^{1/\mu} + \Big| (p/x) \sum_{\substack{n \leqq x \\ p \| n}} f(n) e(\alpha n) - (1/x) \sum_{n \leqq x} f(n) e(\alpha n) \Big|$$

and therefore

$$\Big|(1/x)\sum_{n\le x} f(n)e(\alpha n)\Big|^\mu \le 3^{\mu-1}M^\mu |C_f(x/p, p\alpha)|^\mu + 3^{\mu-1}M^\mu K^{1/(\lambda-1)}/p +$$
$$+3^{\mu-1}\Big|(p/x)\sum_{\substack{n\le x\\ p\| n}} f(n)e(\alpha n) - (1/x)\sum_{n\le x} f(n)e(\alpha n)\Big|^\mu.$$

Using Lemma 2 with $a_n = f(n)e(\alpha n)$, we see that, if y is any number $\ge X_1$ (and therefore $\ge p_0$), then we have for $x \ge y$

$$\Big(\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p\Big)\Big|(1/x)\sum_{n\le x} f(n)e(\alpha n)\Big|^\mu \le$$
$$\le 3^{\mu-1}M^\mu \sum_{\substack{p\le y\\ p\in\mathcal{P}}} (1/p)|C_f(x/p, p\alpha)|^\mu + 3^{\mu-1}M^\mu K^{1/(\lambda-1)}\sum 1/p^2 + 3^{\mu-1}C_{\lambda,M}K^{1/(\lambda-1)}.$$

We now remark that, since α is irrational, if p' and p'' are distinct primes, then $p'\alpha \not\equiv p''\alpha \pmod 1$. Define a function T^* for $y \ge 3$ by

(6)
$$T^*(y) = \sum_{\substack{p', p''\le y\\ p'\ne p''}} 1/|\sin \pi\alpha(p''-p')|.$$

It follows from Lemma 3 that, if $x \ge T^*(y)$, then

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}} (1/p)|C_f(x/p, p\alpha)|^\mu \le 2K^{1/(\lambda-1)}/p_0 \le K^{1/(\lambda-1)}.$$

Thus, if $y \ge X_1$ and $x \ge \operatorname{Max}(y, T^*(y))$, then

$$\Big(\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p\Big)\Big|(1/x)\sum_{n\le x} f(n)e(\alpha n)\Big|^\mu \le H,$$

where

(7)
$$H = 3^{\mu-1}K^{1/(\lambda-1)}\big(M^\mu + M^\mu \sum 1/p^2 + C_{\lambda,M}\big).$$

Since

$$\sum_{\substack{p\le y\\ p\in\mathcal{P}}} 1/p \ge \sum_{\substack{p\le y\\ |f(p)|<M}} 1/p \ge \big(1-(1+\eta)/M^\lambda\big)\log\log y,$$

this yields

(8)
$$\Big|(1/x)\sum_{n\le x} f(n)e(\alpha n)\Big|^\mu \le H^{1/\mu}\big(1-(1+\eta)/M^\lambda\big)^{-1/\mu}(\log\log y)^{-1/\mu}.$$

2.3. So far, we have proved the following result: If $y \ge X_1$ and $x \ge \operatorname{Max}(y, T^*(y))$, where $T^*(y)$ is defined by (6), then for any multiplicative function f satisfying (3) we have (8), where H is defined by (7).

Given $\varepsilon > 0$, we can choose $y_0 \ge X_1$ such that the right-hand side of (8) is $\le \varepsilon^\mu$ for $y = y_0$. If $x \ge \operatorname{Max}(y_0, T^*(y_0))$, then for any multiplicative function f satisfying (3)

$$\Big|(1/x)\sum_{n\le x} f(n)e(\alpha n)\Big| \le \varepsilon.$$

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On multiplicative functions that are q-additive

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1. Introduction. We shall say that a complex-valued function $f(n)$ defined on the set of natural numbers is multiplicative if $f(ab)=f(a)f(b)$ holds for every coprime pairs a and b. Let $\mathcal{M}$ be the class of multiplicative functions.

Let $q\geqq 2$ be a fixed integer. Every positive integer can be represented in the form

$$n=a_0+a_1q+\ldots+a_sq^s,\quad a_s\neq 0,\quad a_i\in\{0,1,\ldots,q-1\} \tag{1.1}$$

uniquely. We shall say that a complex-valued function $g(n)$ defined on the set of nonnegative integers is q-additive if $g(0)=0$ and

$$g(n)=g(a_0)+g(a_1q)+\ldots+g(a_sq^s) \tag{1.2}$$

for n having the representation (1.1). Let $\mathcal{A}_q$ be the class of q-additive functions. This notion has been introduced by A. O. GELFOND [1].

It is obvious that $g(n)$ is a q-additive function if and only if

$$g(Aq^k+r)=g(Aq^k)+g(r) \tag{1.3}$$

whenever $0\leqq r<q^k$, $A\geqq 0$. It is obvious that $f(n)=n$ is a multiplicative and q-additive function. The zero function has the same properties.

Our main purpose is to determine all functions in $\mathcal{A}_q\cap\mathcal{M}$.

Theorem 1. *Let $f\in\mathcal{A}_q\cap\mathcal{M}, f(q)\neq 0$. Then $f(n)=n$.*

We shall give all the multiplicative functions $f(n)$ with period q, i.e. those for which

$$f(n+q)=f(n) \tag{1.4}$$

holds for every $n>0$.

Let $q=Q_1^{\omega_1}\ldots Q_r^{\omega_r}$, where $Q_1,\ldots,Q_r$ are distinct primes.

Main lemma. *Let $f\in\mathcal{M}$ satisfy* (1.4), $f(1)=1$.

(1) *Then $f(n)=\chi(n)$ for every n coprime to q, where χ is a multiplicative character* mod q.

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Let $\chi=\chi_1\cdot\ldots\cdot\chi_r$, *where* χ_i *is a multiplicative character* mod $Q_i^{\omega_i}$ $(i=1, 2, \ldots, r)$. *The components* χ_i *are determined by* χ. *Let*

$$\psi_i = \prod_{\substack{j=1 \\ j\neq i}}^{r} \chi_j \quad (i = 1, \ldots, r), \tag{1.5}$$

Let $Q_i^{\varepsilon_i}$ *be the smallest period of* χ_i. *Then* $0\leqq\varepsilon_i\leqq\omega_i$.

(2) *We have*

$$f(Q_i^{\omega_i+h}) = \begin{cases} f(Q_i^{\omega_i})\psi_i(Q_i^h) & \text{if } \chi_i \text{ is principal character,} \\ 0 & \text{if } \chi_i \text{ is not the principal character} \end{cases} \tag{1.6}$$

$(h=1, 2, \ldots)$.

(3) *If* $f(Q_i^{\lambda_i})\neq 0$ *for some* $\lambda_i\in[1, \omega_i-1]$, *then* $\lambda_i\leqq\omega_i-\varepsilon_i$.

Conversely, if $f\in\mathcal{M}$ *satisfies* (1), (2), (3) *then* (1.4) *holds.*

Remark. The assertion stated here may be known. It has an auxiliary character for us. A. Sárközy [2] considered multiplicative arithmetic functions satisfying a linear recursion.

Theorem 2. *Let* $f\in\mathcal{M}\cap\mathcal{A}_q$, $f(1)=1$, $f(q)=0$. *Then* f *is a periodic function with period* q, $f(Q_i^{\omega_i})=0$ *for at least one* i. *The assertions* (1), (2), (3) *in the Main lemma are satisfied.*

Conversely, if these conditions hold and $f\in\mathcal{M}$, *then* $f\in\mathcal{A}_q$.

2. Proof of the Main lemma. Let us assume that $f\in\mathcal{M}, f(1)=1$ and (1.4) holds. It is well known that $f(n)=\chi(n)$ for $(n, q)=1$, χ is a character mod q.

Let i be fixed, $h>0$, $R=q\cdot Q_i^{-\omega_i}$, $n=Q_i^{\omega_i+h}\cdot r$, $(r, q)=1$. Then $n+q=Q_i^{\omega_i}(Q_i^h\cdot r+R)$. Since $(Q_i^h\cdot r+R, q)=1$, we have

$$f(n+q)=f(Q_i^{\omega_i})f(Q_i^h r+R) = f(Q_i^{\omega_i})\chi(Q_i^h r+R) =$$
$$=f(Q_i^{\omega_i})\chi_i(Q_i^h r+R)\psi_i(Q_i^h r+R) = f(Q_i^{\omega_i})\chi_i(Q_i^h r+R)\psi_i(Q_i^h r).$$

Here we observed that ψ_i is a character mod R. Similarly,

$$f(n) = f(Q_i^{\omega_i+h})\chi_i(r)\psi_i(r).$$

Since $\psi_i(r)\neq 0$, therefore from (1.4) we get

$$f(Q_i^{\omega_i+h})\chi_i(r) = f(Q_i^{\omega_i})\psi_i(Q_i^h)\chi_i(Q_i^h r+R). \tag{2.1}$$

This gives immediately that $f(Q_i^{\omega_i})=0$ if and only if $f(Q_i^{\omega_i+h})=0$. Let us assume that $f(Q_i^{\omega_i})\neq 0$. Then $f(Q_i^{\omega_i+h})\neq 0$ for $h=1, 2, \ldots$. By choosing $h=\omega_i$ and observing that $\chi_i(Q_i^{\omega_i}\cdot r+R)=\chi_i(R)$, from (2.1) we get that $\chi_i(r)$ is constant on the set $(r, q)=1$. Since χ_i is a character mod $Q_i^{\omega_i}$, its values depend on r (mod $Q_i^{\omega_i}$), consequently $\chi_i(r)$ is constant for $(r, Q_i)=1$, and so χ_i is the principal character. This proves (2).

Let now $n=Q_i^{\lambda_i}\cdot x$, $(x,q)=1$, $1\leqq\lambda_i<\omega_i$. Then $n+q=Q_i^{\lambda_i}[x+Q_i^{\omega_i-\lambda_i}\cdot R]$. From (1.4) we get

$$f(n)=f(Q_i^{\lambda_i})\chi_i(x)\psi_i(x)=f(n+q)=f(Q_i^{\lambda_i})\chi_i(x+Q_i^{\omega_i-\lambda_i}R)\psi_i(x).$$

Let us assume that $f(Q_i^{\lambda_i})\neq 0$. Then

$$\chi_i(x)=\chi_i(x+Q_i^{\omega_i-\lambda_i}R)(x,q)=1. \tag{2.2}$$

Let $x_0=Ry$, $(y,Q_i)=1$, $x=x_0+t\cdot Q_i^{\varepsilon_i}$, $(t,R)=1$. Hence it follows that $(x,q)=1$, consequently from (2.2) we get

$$\chi_i(x_0)=\chi_i(x)=\chi_i(x+Q_i^{\omega_i-\lambda_i}R)=\chi_i(x_0+Q_i^{\omega_i-\lambda_i}R),$$

and so

$$\chi_i(y)\chi_i(R)=\chi_i(y+Q_i^{\omega_i-\lambda_i})\chi_i(R)\quad\text{for}\quad(y,Q_i)=1.$$

Since $\chi_i(R)\neq 0$, this gives that $Q_i^{\omega_i-\lambda_i}$ is a period of χ_i and so $\omega_i-\lambda_i\geqq\varepsilon_i$. By this (3) is proved.

Now we prove the second assertion. Assume that $f\in\mathcal{M}$ and (1), (2), (3) hold.

We shall assume that $f(Q_i^{\omega_i})=0$ for $i=1,\dots,s$ and $f(Q_i^{\omega_i})\neq 0$ for $i=s+1,\dots,r$, allowing that one of these classes is empty. Then the characters $\chi_{s+1},\dots,\chi_r$ are principal characters with the moduli $Q_j^{\omega_j}$ $(j=s+1,\dots,r)$, respectively.

Let

$$\alpha(n)=\chi_1(n)\cdot\ldots\cdot\chi_s(n),\quad \beta(n)=\chi_{s+1}(n)\cdot\ldots\cdot\chi_r(n),$$

$$q_1=\prod_{i=1}^{s}Q_i^{\omega_i},\quad q_2=\prod_{i=s+1}^{r}Q_i^{\omega_i},\quad q_1^*=\prod_{i=1}^{s}Q_i^{\varepsilon_i}.$$

Then $\beta(n)$ is the principal character with the modulus q_2, $\alpha(n)$ is a character with the modulus q_1, that is periodic with the period q_1^*. Furthermore we may observe that $\psi_i(Q_i^h)=\alpha(Q_i^h)$ for $i=s+1,\dots,r$, $h\geqq 0$.

To prove (1.4) we take $n=m\eta$, $n+q=a\zeta$, where $(\eta,q)=1$, $(\zeta,q)=1$ and m and a are composed from the prime factors of q. Let

$$m=\Big(\prod_{i=1}^{s}Q_i^{\beta_i}\Big)\Big(\prod_{i=s+1}^{k}Q_i^{\beta_i}\Big)\Big(\prod_{i=k+1}^{v}Q_i^{\beta_i}\Big)\Big(\prod_{i=v+1}^{r}Q_i^{\beta_i}\Big)=\Pi_1\Pi_2\Pi_3\Pi_4$$

and

$$a=\Big(\prod_{i=1}^{s}Q_i^{\gamma_i}\Big)\Big(\prod_{i=s+1}^{k}Q_i^{\gamma_i}\Big)\Big(\prod_{i=k+1}^{v}Q_i^{\gamma_i}\Big)\Big(\prod_{i=v+1}^{r}Q_i^{\gamma_i}\Big)=R_1R_2R_3R_4,$$

where in Π_2 $\beta_i>\omega_i$, in Π_3 $\beta_i=\omega_i$, and in Π_4 $\beta_i<\omega_i$. Hence it follows that in R_2 $\gamma_i=\omega_i$, in R_3 $\gamma_i\geqq\omega_i$, in R_4 $\gamma_i=\beta_i$. Consequently $R_4=\Pi_4$. Let U and V be defined by the relations

$$\Pi_2=R_2\prod Q_i^{\beta_i-\omega_i}=R_2U,\quad R_3=\Pi_3\prod Q_i^{\gamma_i-\omega_i}=\Pi_3V.$$

If $\beta_i \geqq \omega_i$ for at least one $i \in [1, s]$, then $f(n)=0$, $\gamma_i \geqq \omega_i$ and so $f(n+q)=0$, i.e. (1.4) is true. If $\omega_i - \varepsilon_i < \beta_i < \omega_i$, then $\beta_i = \gamma_i$, $f(Q_i^{\omega_i})=0$, consequently $f(n)= =f(n+q)=0$. So we may assume that $\beta_i \leqq \omega_i - \varepsilon_i$ for $i=1, \dots, s$. Hence it follows that $\beta_i = \gamma_i$. $\Pi_1 = R_1$. Let us consider now the relation

$$q = a\zeta - m\eta = R_1 R_2 R_3 R_4 \zeta - \Pi_1 \Pi_2 \Pi_3 \Pi_4 \eta = \Pi_1 \Pi_4 R_2 \Pi_3 \{V\zeta - U\eta\}.$$

Since $(\Pi_4 R_2 \Pi_3, q_1)=1$, $\Pi_1 | q_1/q_1^*$, we get that $V\xi \equiv U\eta \pmod{q_1^*}$. Furthermore

$$f(m\eta) = f(\Pi_1)f(\Pi_4)f(R_2 U)f(\Pi_3)f(\eta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3)\alpha(U)\alpha(\eta),$$

$$f(a\zeta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3 V)\alpha(\zeta) = f(\Pi_1)f(\Pi_4)f(R_2)f(\Pi_3)\alpha(V)\alpha(\zeta).$$

By observing that $\alpha(U)\alpha(\eta)=\alpha(V)\alpha(\zeta)$, we get (1.4).

Thus the proof of the Main lemma is complete.

3. Proof of Theorem 1.

Lemma 1. *If $f \in \mathcal{A}_q \cap \mathcal{M}$, $f(1)=1$, then*

$$f(nq^\alpha) = f(n)f(q^\alpha) \tag{3.1}$$

holds for every nonnegative n and α.

Proof. (3.1) is obviously true if $n=0$ or $\alpha=0$. Let $\alpha>0$ and $n>0$. Let us assume that $n=q^\beta$, or $n<q^\alpha$ and $n|q^s$ for a suitable large s. Then $(n, q^\alpha+1)=1$, and hence

$$f(nq^\alpha)+f(n) = f(nq^\alpha+n) = f(n)f(q^\alpha+1) = f(n)f(q^\alpha)+f(n),$$

i.e. (3.1) holds. Let $n<q^\alpha$, $n=n_1n_2$, where $(n_1, q)=1$ and all prime divisors of n_2 divide q. Then

$$f(nq^\alpha) = f(n_1)f(n_2q^\alpha) = f(n_1)f(n_2)f(q^\alpha) = f(n)f(q^\alpha).$$

Let now $n=a_0+a_1q+\dots+a_sq^s$ be an arbitrary positive integer. By using the q-additive property and the results proved earlier we get

$$f(nq^\alpha) = f(a_0q^\alpha+\dots+a_sq^{\alpha+s}) = f(a_0q^\alpha)+\dots+f(a_sq^{\alpha+s}) =$$
$$= f(q^\alpha)[f(a_0)+\dots+f(a_sq^s)] = f(q^\alpha)f(n).$$

The proof of Lemma 1 is finished.

Now we prove Theorem 1. From (3.1) it is obvious that $f(q^\beta)=(f(q))^\beta$ $(\beta=1, 2, \dots)$. Assume that $f(q)\neq 0$. We shall prove that

$$f(2n+1) = f(n+1)+f(n), \quad f(2n) = 2f(n), \tag{3.2}$$

which immediately yields the desired result $f(n)=n$.

Let n be fixed, α be large. Since $(q^\alpha+n+1, q^\alpha+n)=1$, we have

$$f\big((q^\alpha+n+1)(q^\alpha+n)\big) = f(q^\alpha+n+1)f(q^\alpha+n) =$$
$$= \big(f(q^\alpha)+f(n+1)\big)\big(f(q^\alpha)+f(n)\big) = f(q^\alpha)^2+f(q^\alpha)\big(f(n)+f(n+1)\big)+f(n)f(n+1).$$

Furthermore

$$f\big((q^\alpha+n+1)(q^{2\alpha}+n)\big) = f(q^\alpha)+f\big((2n+1)q^\alpha\big)+f\big(n(n+1)\big).$$

Hence we get immediately that $f(2n+1)=f(n+1)+f(n)$. To prove the second relation in (3.2) we consider

$$f(2nq+1) = f(nq+1)+f(nq),$$

whence it follows that

$$f(2n)f(q)+f(1) = 2f(n)f(q)+f(1),$$

and from $f(q)\neq 0$ we get that $f(2n)=2f(n)$.

Theorem 1 is proved.

4. Proof of Theorem 2. Let $f\in\mathcal{A}_q\cap\mathcal{M}$, $f(1)=1$, $f(q)=0$. Since $f(q)=0$, from Lemma 1 we get that $f(nq)=0$ for every n, consequently f is a periodic function with period q. The necessity of the conditions is obvious from the Main lemma. But they are also sufficient, since a periodic multiplicative function f with $f(q)=0$ is q-additive, and so the sufficiency is an immediate consequence of the Main lemma as well.

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On partial asymptotic stability and instability. III (Energy-like Ljapunov functions)

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Dedicated to Professor László Leindler on his 50th birthday

1. Introduction

The main tools in the proofs of the theorems in [1]—[2] were certain invariance principles. Their applications were made possible by the special structure of the basic differential equation: in [1] and [2] the equations were autonomous and asymptotically autonomous, respectively.

In this paper we study the case, when there are no direct restrictions on the right-hand side of the equation. Theorems of such type have been established first by A. M. Ljapunov [3]. Besides some other conditions, he required the Ljapunov function V to be decrescent, i.e. $V(x, t)\to 0$ uniformly in $t\in R_+$ as $x\to 0$. V. V. Rumjancev [4] generalized these theorems to partial asymptotic stability. Since the uncontrolled part z of the coordinates may be unbounded along motions, the condition that V be decrescent has come into these generalizations even in a stronger form: the condition "$V(y, z, t)\to 0$ uniformly in $(z, t)\in R^n\times R_+$ as $y\to 0$" are required in them. Sometimes in practice it is very difficult to construct such a Ljapunov function. For example, the total mechanical energy of a mechanical system is decrescent with respect to the velocities only in that case when no potential forces act on the system. Namely, let us consider again the motion of a material point in a constant field of gravity along a surface under the action of frictional forces [1], [2]. It is a very reasonable conjecture that if the surface is a cup looking upward then the equilibrium is asymptotically stable with respect to the velocities. However, using the generalizations established in [4] one can prove this property only in that case when the surface is a horizontal plane.

In mechanics the total mechanical energy, i.e. the sum of the kinetic and potential energy is often used as a Ljapunov function for stability investigations. These

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applications inspired us to give a sufficient condition for partial asymptotic stability using a non-decrescent Ljapunov function which is the sum of two auxiliary functions. The result will be applied to the study of the conditions of the asymptotic stability with respect to the velocities of equilibrium states in mechanical systems under the action of dissipative and potential forces depending also on the time.

2. The main theorems

Consider the differential equation

$$\dot{x} = X(x, t) \quad (t \in R_+, x \in R^k). \tag{2.1}$$

Let $x=(y, z)$ be a partition of the vector $x \in R^k$ ($y \in R^m$, $z \in R^n$, $1 \leqq m \leqq k$, $n=k-m$) and suppose that the right-hand side of (2.1) satisfies the same conditions as in [1] (see Section 2), i.e. the function X is defined on the set $\Gamma_y(H)$:

$$\Gamma_y(H) := G_y(H) \times R_+, \quad G_y(H) := \{(y, z) \in R^m \times R^n \colon |y| < H\} \; (0 < H \leqq \infty),$$

it is continuous in x and measurable in t, and satisfies the Carathéodory condition locally. The solutions of (2.1) are z-continuable, and $x=0$ is a solution of the equation, i.e. $X(0, t) \equiv 0$ for all $t \in R_+$.

Let $\mathcal{K}$ be the class of continuous strictly increasing functions a: $R_+ \to R_+$ such that $a(0)=0$.

For formulating our main result a new concept is needed. A continuous function φ: $R_+ \to R_+$ is said to be *integrally positive* (see [5], [6]) if $\int_I \varphi(t)\, dt = \infty$ whenever

$$I = \bigcup_{i=1}^{\infty} [\alpha_i, \beta_i], \quad \text{and} \quad \alpha_i < \beta_i < \alpha_{i+1}, \quad \beta_i - \alpha_i \geqq \delta > 0$$

hold for all $i=1, 2, \ldots$ with some positive constant δ.

Denote by $[\alpha]_+$ and $[\alpha]_-$ the positive and the negative parts of the real number α, respectively, i.e. $[\alpha]_+ := \max\{0, \alpha\}$, $[\alpha]_- := \max\{0, -\alpha\}$.

Theorem 2.1. *Suppose that there exist two functions V_1, V_2: $\Gamma_y(H) \to R$ which are continuous, locally Lipschitzian and satisfy the following conditions on the set $\Gamma_y(H)$:*

(i) $V(x, t) := V_1(x, t) + V_2(x, t) \geqq 0$;

(ii) $V_1(x, t) \geqq 0$;

(iii) *the derivative of V with respect to* (2.1) *admits an estimate*

$$\dot{V}(x, t) \leqq -\varphi(t) c(V_1(x, t))$$

with some $c \in \mathcal{K}$ and some integrally positive function φ: $R_+ \to R_+$;

(iv) *for every* $\alpha, \alpha_1 > 0$ *and for every continuous function* $\xi: R_+ \to R^k$ *the inequalities* $V(\xi(t), t) \leqq \alpha$, $V_1(\xi(t), t) \geqq \alpha_1$ $(t \in R_+)$ *imply that the function*

$$\int_0^t [\dot{V}_2(\xi(s), s)]_{+(-)}\, ds$$

is uniformly continuous on R_+, *where the symbol* $[\cdot]_{+(-)}$ *means that either the positive part* $[\cdot]_+$ *or the negative part* $[\cdot]_-$ *is considered for all* $s \in R_+$.

Then for every solution $x(t)$ *of* (2.1) *defined for all t large enough* $\lim_{t\to\infty} V_1(x(t), t) = 0$, *and* $V_2(x(t), t)$ *has a finite limit as* $t \to \infty$.

Proof. Define the functions $v_1(t) := V_1(x(t), t)$, $v_2(t) := V_2(x(t), t)$, $v(t) := v_1(t) + v_2(t)$. Obviously, $v(t)$ is nonincreasing and bounded from below, so $\lim_{t\to\infty} v(t) =: v_0$ exists and is finite. It is sufficient to show that $\lim_{t\to\infty} v_1(t) = 0$.

Suppose the contrary. Then, in consequence of (i)—(iii) we have

$$\liminf_{t\to\infty} v_1(t) < \limsup_{t\to\infty} v_1(t) =: v_1^* \leqq \infty,$$

$$v_{2*} := \liminf_{t\to\infty} v_2(t) = v_0 - v_1^* < v_0 = \limsup_{t\to\infty} v_2(t).$$

Now we show the existence of a sequence of disjoint intervals on which the variations of the function $v_2(t)$ are bounded from below by a positive constant. Indeed, let $\varepsilon := v_1^*/4 > 0$ if $v_1^* < \infty$, and let $\varepsilon > 0$ be arbitrary if $v_1^* = \infty$. There exists a $T \in R_+$ such that $v_0 \leqq v(t) < v_0 + \varepsilon$ for all $t \geqq T$. For the sake of definiteness let us suppose that "plus" sign stands in condition (iv) of the theorem. Obviously, an appropriate sequence $T < t_1' < t_1'' < \ldots < t_i' < t_i'' < \ldots$ has the properties

$$v_1(t_i') = 3\varepsilon, \quad v_1(t_i'') = \varepsilon, \quad \varepsilon \leqq v_1(t) \leqq 3\varepsilon \quad \text{for} \quad t \in [t_i', t_i''] \quad (i = 1, 2, \ldots).$$

Since $v_2(t) = v(t) - v_1(t)$, we obtain

$$v_2(t_i') \leqq v_0 - 2\varepsilon, \quad v_2(t_i'') \geqq v_0 - \varepsilon \quad (i = 1, 2, \ldots).$$

Consequently,

$$0 < \varepsilon \leqq v_2(t_i'') - v_2(t_i') \leqq \int_{t_i'}^{t_i''} [\dot{V}_2(x(t), t)]_+\, dt \quad (i = 1, 2, \ldots).$$

Hence, because of condition (iv), it follows that $t_i'' - t_i' \geqq \delta > 0$ $(i = 1, 2, \ldots)$ with some constant δ. By condition (iii) this implies $v(t) \to -\infty$, which is a contradiction.

The theorem is proved.

If the function V_1 in the theorem is even positive definite with respect to y, then for every solution $x(t) = (y(t), z(t))$ defined for all t large enough $y(t) \to 0$ as $t \to \infty$. If, in addition, $V_1(0, t) \equiv V_2(0, t) \equiv 0$ and $V_2(x, t) \geqq 0$, then $V = V_1 + V_2$ is a positive y-definite Ljapunov-function to (2.1), so the zero solution is even y-stable [4], which leads to the following

Corollary 2.1. *Suppose that there exist two Ljapunov functions*

$$V_1, V_2\colon \Gamma_y(H') \to R \quad (0 < H' < H)$$

satisfying the following conditions on the set $\Gamma_y(H')$:

(i) $V_2(x, t) \geqq 0$;

(ii) *there is a function* $a_1 \in \mathscr{K}$ *such that*

$$a_1(|y|) \leqq V_1(y, z, t).$$

Suppose, in addition, that conditions (iii)—(iv) *in Theorem* 2.1 *are also satisfied.*

Then the zero solution of (2.1) *is asymptotically y-stable and for every solution* $x(t)$ *with sufficiently small initial values the function* $V_2(x(t), t)$ *has a finite limit as* $t \to \infty$.

Let $x = (y', z')$ be another partition of the vector $x \in R^k$: $y' \in R^{m'}$, $z' \in R^{n'}$, $m \leqq m' \leqq \leqq k$, $n' = k - m'$. If V_1 is decrescent with respect to y', then condition (iii) becomes simpler:

Corollary 2.2. *Suppose that there exist two Ljapunov functions*

$$V_1, V_2\colon \Gamma_y(H') \to R \quad (0 < H' < H)$$

satisfying the following conditions on the set $\Gamma_y(H')$:

(i) $V_2(x, t) \geqq 0$;

(ii) *there are functions* $a_1, b_1 \in \mathscr{K}$ *such that*

$$a_1(|y|) \leqq V_1(y, z, t) \leqq b_1(|y'|);$$

(iii) *an inequality*

$$\dot{V}(x, t) \leqq -\varphi(t) c(|y'|)$$

holds with some $c \in \mathscr{K}$ *and some integrally positive* $\varphi\colon R_+ \to R_+$;

(iv) *for every* $\alpha, \beta > 0$ *the function*

$$\int_0^t \sup \{[\dot{V}_2(y', z', s)]_{+(-)}\colon (y', z') \in M_{\alpha,\beta}(s)\}\, ds$$

is unformly continuous on R_+, *where*

$$M_{\alpha,\beta}(s) := \{(y', z') \in R^{m'} \times R^{n'}\colon V(y', z', s) \leqq \alpha, \quad |y'| \geqq \beta\}.$$

Then the statement of Corollary 2.1 *holds.*

Remark 2.1. In condition (iii) of Theorem 2.1 (and Corollaries 2.1—2.2) φ is integrally positive which, roughly speaking, means that it cannot be small in average in any period as $t\to\infty$. It can be formulated also in the following way: for every $\delta>0$

$$\liminf_{t\to\infty}\int_t^{t+\delta}\varphi(s)\,ds>0.$$

Therefore, for example, every nonnegative continuous periodic function not vanishing on any interval is integrally positive. But, obviously, a function tending to zero as $t\to\infty$ cannot be integrally positive even if its integral equals infinity. However, by experiences asymptotic stability may appear also in this case.

Let us relax the condition of integral positivity. We say that a continuous function $\varphi\colon R_+\to R_+$ is *weakly integrally positive* [6] if $\int_I\varphi=\infty$ whenever

$$I=\bigcup_{i=1}^{\infty}(\alpha_i,\beta_i),\quad \alpha_i<\beta_i<\alpha_{i+1},\quad \beta_i-\alpha_i\geqq\delta>0,\quad \alpha_{i+1}-\beta_i\leqq\gamma\quad (i=1,2,\ldots)$$

hold with some positive constants δ, γ.

It is easy to see that any nonincreasing function whose integral on R_+ equals infinity is weakly integrally positive.

We say that equation (2.1) has *property* P with respect to V_1, V_2 if for every $\varepsilon,\alpha>0$ there exist $\eta>0$, $T\in R_+$ such that for every solution $x(t)$ of (2.1) the point $(x(t),t)$ cannot be contained in the set

$$M(\varepsilon,\alpha,\eta):=\{(x,t)\colon\ \alpha\leqq V_1(x,t)+V_2(x,t)<\alpha+\varepsilon,\ V_1(x,t)\geqq\eta\}$$

during any period longer than T.

Analysing the proof of Theorem 2.1 one can show that property P makes it possible to choose the sequence $\{(t_i',t_i'')\}$ in the proof so that the inequality $t_{i+1}'-t_i''\leqq\gamma$ holds for all $i=1,2,\ldots$. Consequently, possessing property P we can assume the function φ in condition (iii) of Theorem 2.1 to be weakly integrally positive instead of integrally positive.

Property P is often guaranteed by means of another auxiliary function [5], [6], [14]. For example, if for some ε, α, η there exists a function $W\colon M(\varepsilon,\alpha,\eta)\to R$ such that for every continuous function $\zeta\colon R_+\to R_+$ with $(\zeta(t),t)\in M(\varepsilon,\alpha,\eta)$ $(t\in R_+)$ the function $W(\zeta(t),t)$ is bounded from above and the condition

$$\lim_{\sigma\to\infty}\int_{t_0}^{t_0+\sigma}\dot W(\zeta(t),t)\,dt=\infty$$

holds uniformly with respect to $t_0\in R_+$, then (2.1) has property P.

3. Applications

I. Consider the generalized Liénard equation

$$\ddot{x}+a(t)g(x,\dot{x})\dot{x}+b(t)f(x)=0, \tag{3.1}$$

where the functions $a\colon R_+\to R_+$, $g\colon R^2\to R_+$, $f\colon R\to R$ are continuous, $b\colon R_+\to R_+$ is continuously differentiable, and $b(t)>0$, $F(x):=\int_0^x f(u)\,du\geqq 0$ for all $t\in R_+$, $x\in R$. This equation, which describes the oscillation of a material point round the origin $x=0$, has been investigated by many authors [7]—[12]. In [13] we obtained sufficient conditions for the asymptotic stability if $0\leqq a(t)\leqq A_0$, $0<b_0\leqq b(t)\leqq B_0$ $(t\in R_+)$, and for the asymptotic x-stability provided that $\lim_{t\to\infty} b(t)=\infty$. Using our results proved in Section 2 of the present paper we can sharpen these theorems and get sufficient conditions for the asymptotic $\dot{x}$-stability, too.

First we define the auxiliary functions

$$V_1(\dot{x}):=\dot{x}^2/2,\quad V_2(x,t):=b(t)F(x).$$

The derivatives of $V:=V_1+V_2$ and V_2 read as follows:

$$\dot{V}(x,\dot{x},t)=-a(t)g(x,\dot{x})\dot{x}^2+\dot{b}(t)F(x)=-2a(t)g(x,\dot{x})V_1(\dot{x})+(\dot{b}(t)/b(t))V_2(x,\dot{x},t),$$

$$\dot{V}_2(x,\dot{x},t)=(\dot{b}(t)/b(t))V_2(x,\dot{x},t)+b(t)\dot{x}f(x).$$

Applying Corollary 2.1 we obtain the following

Corollary 3.1. *Suppose that*

(i) $\dot{b}(t)\leqq 0$ *and* $\dot{b}(t)/b(t)$ *is bounded from below on* R_+;

(ii) $a(t)$ *is integrally positive on* R_+;

(iii) *for every* $0<c_2<C_2$ *there is a* $g_0>0$ *such that*

$$g(u,v)\geqq g_0\quad (u\in R,\ c_2\leqq |v|\leqq C_2);$$

(iv) $f(x)$ *is bounded on* R.

Then the zero solution of (3.1) *is asymptotically* $\dot{x}$*-stable and for every solution* $x(t)$ *the function* $b(t)F(x(t))$ *has a finite limit.*

Some conditions in this corollary become simpler if the solutions are guaranteed to be bounded.

Corollary 3.2. *Suppose that*

(i) $-\beta_0\leqq\dot{b}(t)\leqq 0$, $b(t)\geqq b_0>0$ $(t\in R_+)$;

(ii) $a(t)$ *is integrally positive on* R_+;

(iii) $g(u,v)>0$ *if* $v\neq 0$;

(iv) $\lim_{|x|\to\infty} F(x)=\infty$.

Then the zero solution of (3.1) *is asymptotically* $\dot{x}$*-stable and for every solution* $x(t)$ *the function* $F(x(t))$ *has a finite limit.*

Proof. By conditions (i), (iv) we have $\dot{V}(u, v, t) \leqq 0$, $\lim_{|u|+|v|\to\infty} V(u, v, t) = \infty$. Consequently, x and $\dot{x}$ are bounded on R_+ along every solution, so the conditions of Corollary 3.1 are satisfied.

Surprisingly, the boundedness, even the stability with respect to x, can be guaranteed by a modification of condition (ii) provided $g(u, v) > 0$ on R^2

Corollary 3.3. *Suppose that*

(i) $\dot{b}(t) \leqq 0$ *and* $\dot{b}(t)/b(t)$ *is bounded from below on* R_+;

(ii) $a(t) > 0$ *for* $t \in R_+$, *and* $\int_0^\infty dt/a(t) < \infty$;

(iii) $g(u, v) > 0$ *for* $(u, v) \in R^2$.

Then the zero solution of (3.1) *is stable, asymptotically stable with respect to* $\dot{x}$, *and every solution* $x(t)$ *with sufficiently small initial values* $|x(t_0)|$, $|\dot{x}(t_0)|$ *has a finite limit as* $t \to \infty$.

Proof. All the conditions of Corollary 3.4 in [12] are obviously met by $q := x$, $\dot{q} := \dot{x}$, $A(q) := 1$, $\Pi(t, q) := b(t) F(q)$, $Q(t, q, \dot{q}) := -a(t) g(q, \dot{q}) \dot{q}$, $\alpha = 1$. Consequently, the zero solution of (3.1) is stable and every solution has a finite limit as $t \to \infty$. By Schwarz's inequality, condition (ii) implies the function $a(t)$ to be integrally positive, so all the condition of Corollary 3.1 are satisfied.

The case of nondecreasing function $b(t)$ will be treated for a mechanical system of arbitrary degree of freedom.

II. Consider a holomorphic mechanical system of r degrees of freedom with time-independent constraints under the action of potential and dissipative forces depending on the time, too. Let the motions be described by the Lagrangian equation

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = -g^2 \frac{\partial P^*}{\partial q} + Q \quad (q, \dot{q} \in R^r). \tag{3.2}$$

Here we use the same notations as in [1]—[2]: $T = T(q, \dot{q}) := (1/2) \dot{q}^{\mathrm{T}} A(q) \dot{q}$ is the kinetic energy; $P(q, t) := g^2(t) P^*(q)$ denotes the potential energy in which $g: R_+ \to (0, \infty)$ and $P^*: R^r \to R_+$ are continuously differentiable functions. By $Q = Q(q, \dot{q}, t)$ we denote the resultant of frictional and gyroscopic forces. This means that $Q^{\mathrm{T}}(q, \dot{q}, t) \dot{q} \leqq 0$ for all values of the variables. Assume that $q = \dot{q} = 0$ is an equilibrium state to system (3.2) and $P^*(0) = 0$.

In [1], [2] we investigated the stationary case (i.e. P and Q were independent of t). In the instationary case it gives rise to difficulties that the total mechanical energy is not constant along the motions and the invariance principle cannot be applied any more [5], [6]. L. SALVADORI [14] has given a sufficient condition for the asymptotic stability of the equilibrium $q=\dot{q}=0$ of (3.2) with respect to the coordinates q. Now we are interested in the asymptotic behaviour of the velocities provided that the dissipation is complete in a certain sense. Particularly, we seek for conditions of the asymptotic stability of the equilibrium $q=\dot{q}=0$ with respect to the velocities.

If g is nonincreasing the results proved in paragraph I of the present section for the case of one degree of freedom can be generalized to (3.2). In the sequel g is not supposed to be monotone.

By the transformation $\dot{q}=g(t)y$, introduced by L. SALVADORI [14], system (3.2) can be rewritten into the form

$$\begin{gathered}\dot{q} = g(t)y \\ \frac{d}{dt}\frac{\partial T^*}{\partial y}-g\frac{\partial T^*}{\partial q} = -\frac{\dot{g}}{g}\frac{\partial T^*}{\partial y}-g\frac{\partial P^*}{\partial q}+\frac{Q^*}{g},\end{gathered} \tag{3.3}$$

where

$$T^*(q, y) := (1/2)y^{\mathrm{T}}A(q)y, \quad Q^*(q, y, t) = Q(q, g(t)y, t).$$

Denote by $\lambda(q)$ and $\Lambda(q)$ the smallest and the largest eigenvalue of the positive symmetric matrix $A(q)$, respectively. For $M>0$ let the set $E_M\subset R^r$ be defined by

$$E_M := \{q\in R^r\colon\ P^*(q) \leqq M\}.$$

The derivative of the function $H(q,y):=T^*+P^*$ with respect to (3.3) is

$$\dot{H}(q, y, t) = -2(\dot{g}/g)T^*+(1/g)yQ^*.$$

Making the choice $V_1:=T^*, V_2:=P^*$, from Theorem 2.1 we obtain a lemma, which may be of some independent interest.

Lemma 3.1. *Suppose that for every $M>0$ there exist a function $\varphi\colon R_+\to R_+$ and a constant L such that $\varphi+2\dot{g}/g$ is integrally positive and the following conditions are satisfied:*

(i) $Q^{\mathrm{T}}(q,\dot{q},t)\dot{q}\leqq-\varphi(t)\Lambda(q)|\dot{q}|^2$ *for all* $q\in E_M$, $\dot{q}\in R^r$, $t\in R_+$;

(ii) $|\mathrm{grad}\, P^*(q)|\leqq L[\lambda(q)]^{1/2}$ $(q\in E_M)$;

(iii) *the function* $\int_0^t g(s)\,ds$ *is uniformly continuous on* R_+.

Then for every motion $q=q(t)$ of (3.2) *we have*

$$\dot{q}(t) = o\big(g(t)/[\lambda(q(t))]^{1/2}, \quad P^*(q(t)) \to \text{const.} \quad (t\to\infty). \tag{3.4}$$

If it is "a priori" known that $q(t)$ is bounded along every motion (e.g. $P^*(q)\to\infty$ as $|q|\to\infty$), then $\Lambda(q)$ can be replaced by 1 in (i), condition (ii) is not needed and one can state $\dot{q}(t)=o(g(t))$ $(t\to\infty)$.

Let us now consider the case of the viscous friction, i.e. if $Q(q, \dot{q}, t) = -B(q, t)\dot{q}$ where B is a symmetric positive semi-definite matrix; the smallest eigenvalue of it we denote by $\beta(q, t)$.

Theorem 3.1. *Suppose that for every $M>0$ the following conditions are satisfied:*

(i) *there acts viscous friction on the system such that "the dissipation of the energy is integrally complete", i.e. the function*

$$\inf\{\beta(q, t)/A(q, t)\colon\ q\in E_M\} + 2\dot{g}(t)/g(t)$$

is integrally positive;

(ii) $\inf\{\lambda(q)\colon\ q\in E_M\} > 0$;

(iii) *the functions g and* grad $P^*(q)$ *are bounded on R_+ and E_M, respectively.*

Then the equilibrium $q=\dot{q}=0$ of (3.2) *is asymptotically stable with respect to the velocities.*

Finally, we examine the case of "weakly integrally complete dissipation" starting from Remark 2.1. In order to guarantee property P, let us consider the auxiliary function $W(q, y) = y^{\mathrm{T}} A(q)\,\mathrm{grad}\, P^*(q)$. If g is nondecreasing, A and grad P^* are continuously differentiable, then the derivative $\dot{W}_{(3.3)}$ can be estimated as follows:

$$\begin{aligned}(3.5)\qquad \dot{W}_{(3.3)}(q, y, t) &\leqq -g(t)[\mathrm{grad}\, P^*(q)]^2 + \\ &+ g(t)\left\{d(|y|)\left[\frac{\dot{g}(t)}{g^2(t)} F_1(q) + F_2(q)\right] + \frac{Q(q, g(t)y, t)}{g^2(t)} F_3(q)\right\},\end{aligned}$$

where $d\in\mathscr{K}$ and $F_i\colon R^r\to R_+$ are appropriate continuous functions.

Theorem 3.2. *Suppose that in some neighbourhood $N\subset R^r$ of the origin the following conditions are satisfied:*

(i) *$q=0$ is the only equilibrium position of* (3.2) *in N;*

(ii) *there acts viscous friction on the system with "weakly integrally complete dissipation", i.e. the function*

$$\varphi(t) := \inf\{\beta(q, t)\colon\ q\in N\}$$

is weakly integrally positive on R_+;

(iii) *the function*

$$\frac{1}{t}\int_0^t \sup\{\|B(q, s)\|\colon\ q\in N\}\, ds$$

is bounded on R_+;

(iv) *the function g is nondecreasing and bounded on R_+.*

Then the equilibrium state $q=\dot{q}=0$ *of* (3.2) *is stable, asymptotically stable with respect to the velocities, and for every motion* $q(t)$ *with sufficiently small initial values* $P^*(q(t))\to\text{const.}$ *as* $t\to\infty$.

Proof. By (i) and condition $P^*(q)\geqq 0$ function P is positive definite. Consequently, the solution $q=y=0$ of (3.3) is stable and $q(t)\in N$ for all $t\geqq t_0$ provided that $|q(t_0)|$, $|\dot{q}(t_0)|$ are sufficiently small. In accordance with Remark 2.1 we have only to prove the existence of property P with respect to T^* and P^*.

Let $\varepsilon>0$, α, η $(0<\eta<\alpha)$ be given, and define

$$S(\alpha,\eta) := \{(q,y)\colon\ q\in N,\ T^*(q,y)\leqq\eta,\ P^*(q)\geqq\alpha-\eta\}.$$

Condition (i) implies that

$$m := \inf\{[\operatorname{grad} P^*(q)]^2\colon\ P^*(q)\geqq\alpha-\eta>0,\ q\in N\}>0.$$

Since

$$\int_0^t (\dot{g}(s)/g^2(s))\,ds = 1/g(0)-1/g(t)\leqq\text{const.}\quad (t\geqq 0),$$

by condition (iii) and inequality (3.5) we have the estimate

$$\dot{W}_{(3.3)}(q,y,t)\leqq -g(t)\{m-[c_1+c_2(\dot{g}(t)/g^2(t))]\,d(|y|)-c_3\psi(t)|y|\}$$

on the set $S(\alpha,\eta)\times R_+$, where c_1, c_2, c_3 are positive constants, $\psi\colon R_+\to R_+$ is a continuous function such that $\int_0^t \psi(s)\,ds/t$ is bounded on R_+. Consequently, if η is sufficiently small, then for arbitrary continuous functions $u,v\colon R_+\to S(\alpha,\eta)$ we have

$$\lim_{\sigma\to\infty}\int_{t_0}^{t_0+\sigma}\dot{W}_{(3.3)}(u(t),v(t),t)\,dt=-\infty$$

uniformly with respect to $t_0\in R_+$, which implies property P.

The theorem is proved.

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Об одной алгебре функций

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Посвящается 60-летию со дня рождения К. Тандори — выдающегося специалиста по теории ортогональных рядов

Через Ω будем обозначать множество четных функций $\omega(t)$, которые являются модулями непрерывности при $t\in[0, \infty)$. Хорошо известно, что для любых чисел $\lambda\geqq 0$ и $t\geqq 0$ справедливо неравенство

$$\omega(\lambda t)\leqq\bar\lambda\omega(t),\quad \text{где}\quad \bar\lambda=\begin{cases}\lambda, & \text{если}\quad \lambda\geqq 0\quad \text{и целое}\\ 1+[\lambda] & \text{при остальных}\quad \lambda>0.\end{cases}\tag{1}$$

Здесь $[a]$ обозначает целую часть числа a.

Если $\omega\in\Omega$ и существует постоянная B, для которой интеграл

$$\int\limits_0^\delta \big(\omega(t)/t\big)\,dt\leqq B\omega(\delta)\quad \text{при всех}\quad \delta\in[0, 1],\tag{2}$$

то будем говорить, что функция $\omega(\delta)$ *удовлетворяет условию Бари* ($\omega\in B$) на отрезке $[0, 1]$ (см. [2]). Если выполнено неравенство (2) при всех $\delta\geqq 0$, то будем говорить об условии Бари на полупрямой $[0, \infty)$.

Через $L^\infty=L^\infty(0, 1)$ обозначается пространство существенно ограниченных измеримых функций на отрезке $[0, 1]$ с нормой

$$\|f\|_\infty=\underset{t\in[0,1]}{\text{сущ. верх. гр.}}\,|f(t)|.$$

Пусть функция $f\in L(0, 1)$ и $a_m(f)=(f, \chi_m)$ — коэффициенты Фурье от f по системе Хаара $\{\chi_m\}_{m=1}^\infty$. Положим

$$A_\omega(f)=\sum_{m=2}^\infty\omega\big(a_m(f)\big),\quad A_\omega=\{f\colon A_\omega(f)<\infty\},\quad A_\omega^{(\infty)}=A_\omega\cap L^\infty,$$

$$\chi_m(t)=\chi_n^{(k)}(t)\quad \text{при}\quad m=2^n+k,\quad 1\leqq k\leqq 2^n\quad \text{и}\quad n=0, 1, \ldots.$$

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Через $\mathrm{Lip}_D\,1$ будем обозначать множество всех функций φ, для которых

$$|\varphi(t_1)-\varphi(t_2)|\leqq D|t_1-t_2| \quad \text{при всех} \quad t_1 \quad \text{и} \quad t_2 \quad \text{из} \quad (-\infty, \infty),$$

где $D=\mathrm{const}$.

Линейное множество функций называют *алгеброй*, если поточечные произведения любых двух функций из этого множества также принадлежит ему.

В этой статье мы дадим доказательство двух теорем, сформулированных нами в работе [4] и относящихся к исследованию коэффициентов Фурье—Хаара от суперпозиции функций (по поводу исторических сведений см. также [3]).

Ниже нам понадобятся следующие вспомогательные утверждения.

Лемма 1 (Н. К. Бари [2], стр. 295). *Пусть* $\omega(\delta)\in\Omega$ *и* $\omega(\delta)\not\equiv 0$. *Тогда* $\omega(\delta)$ *удовлетворяет условию Бари на отрезке* $[0, 1]$ *в том и только том случае, когда найдется постоянная* $A>1$ *такая, что*

$$\varliminf_{\delta\to+0} \omega(A\delta)/\omega(\delta) = \gamma > 1. \tag{3}$$

Замечание 1. Если $\omega\in\Omega$ удовлетворяет условию (3), то для любого положительного числа

$$\alpha < \ln\gamma/\ln A = \alpha_0 \tag{4}$$

справедливо соотношение

$$\omega(\delta) = O(\delta^\alpha) \quad \text{при} \quad 0 \leqq \delta \leqq 1. \tag{5}$$

Через $\ln a$ и $\lg a$ обозначаются логарифмы числа a по основанию соответственно e и 2.

В самом деле, возьмем любое $\gamma_0\in(1, \gamma)$ и найдем такое $\delta_0\in(0, 1)$, что (см. (3))

$$\omega(A\delta)/\omega(\delta) \geqq \gamma_0 \quad \text{при всех} \quad \delta\in(0, \delta_0]. \tag{6}$$

Но тогда, если целое $n\geqq 0$ таково, что $A^n\delta\leqq\delta_0$, то применяя (6) n-раз, получим

$$\omega(\delta) \leqq \omega(A^1\delta)/\gamma_0^1 \leqq \omega(A^2\delta)/\gamma_0^2 \leqq \ldots \leqq \omega(A^n\delta)/\gamma_0^n \tag{7}$$

при $A^n\delta\leqq\delta_0$. Выберем n таким, что

$$A^n\delta \leqq \delta_0 < A^{n+1}\delta, \tag{8}$$

т. е.

$$\ln(\delta_0/\delta)/\ln A - 1 < n \leqq \ln(\delta_0/\delta)/\ln A. \tag{9}$$

Так как $\gamma_0>1$, то из (7)—(9) получаем, что

$$\omega(\delta) \leqq \omega(\delta_0)/\gamma_0^n = (\gamma_0\omega(\delta_0))/\gamma_0^{n+1} \leqq (\gamma_0\omega(\delta_0))/\gamma_0^{\ln(\delta_0/\delta)/\ln A} =$$

$$= (\gamma_0\omega(\delta_0))/(\delta_0/\delta)^{\ln\gamma_0/\ln A} = C(\gamma_0, \delta_0, A)\delta^{\ln\gamma_0/\ln A}$$

при всех $\delta\in[0, \delta_0]$, где $C(\gamma_0, \delta_0, A)$ — постоянная, не зависящая от $\delta\in[0, \delta_0]$. Так как γ_0 мы можем брать сколь угодно близким к γ, а $\omega(\delta)$ является модулем непрерывности, то из неравенства (10) вытекает соотношение (5) при условии (4).

Отметим, что из условия (3) не обязано вытекать соотношение (5) при $\alpha=\alpha_0$. Для этого достаточно рассмотреть, например, функцию $\omega_0(\delta)=\delta^\beta \ln(2/\delta)$ при $\delta\in[0, 1]$ и $\omega_0(\delta)=\ln 2$ при $\delta>1$, где $\beta\in(0, 1]$ любое число. Для этой функции выполнено условие (3) с любым $A>1$ и $\gamma=\gamma(A)=A^\beta$. Стало быть в этом случае (см. (4)) число $\alpha_0=\ln\gamma/\ln A=\beta$ и, очевидно, что соотношение (5) для функции $\omega_0(\delta)$ не может быть выполнено при $\alpha=\alpha_0$.

Лемма 2. *Если $\omega(\delta)\in\Omega$, а $\varphi\in\mathrm{Lip}_D 1$, то*

$$A_\omega(\varphi(f)) \leqq A_\omega(Df)+(2/\ln 2)\sum_{m=3}^{\infty}\int_{\|\chi_m\|}^{1}(\omega(Da_m(f)t)/t)\,dt \tag{11}$$

для любой функции $f\in L(0, 1)$, где $\|\chi_m\|_1=\int_0^1|\chi_m(t)|\,dt$.

Это утверждение было сформулировано нами в статье [4] (см. там Теорему 1).

Лемма 3. *Если функция $f(t)=\chi_n^{(1)}(t)$, то для почти всех $t\in[0, 1]$ имеем равенство*

$$|f(t)| = (1/\sqrt{2^n})\{1+\chi_0^{(1)}(t)+\sqrt{2}\,\chi_1^{(1)}(t)+\ldots+\sqrt{2^{n-1}}\,\chi_{n-1}^{(1)}(t)\}. \tag{12}$$

Равенство (12) получается простым подсчетом коэффициентов Фурье—Хаара функции $|f(t)|$.

Лемма 4. *Пусть a_n $(n\geqq 0)$ таковы, что $a_n\geqq a_{n+1}$ при $n\geqq 0$ и $\lim\limits_{n\to\infty} a_n=0$. Тогда найдутся такие b_n, что*

$$\lim_{n\to\infty} b_n = 0, \quad b_n \geqq a_n \text{ и } \Delta^2 b_n \geqq 0 \quad \text{при} \quad n \geqq 0,$$

где $\Delta^2 b_n=\Delta b_n-\Delta b_{n+1}$; $\Delta b_n=b_n-b_{n+1}$.

Эта лемма известна (см. [1], стр. 653—654).

Лемма 5. *Пусть функция $y=f(t)$ определена на отрезке $[0, 1]$, не убывает и $f(+0)=f(0)=0$. Тогда существует функция $y=\psi(t)$, которая выпукла вверх на отрезке $[0, 1]$, $\psi(0)=\psi(+0)=0$ и $f(t)\leqq\psi(t)$ при $t\in[0, 1]$.*

Доказательство почти очевидно. Именно, рассмотрим множество G точек плоскости (t, y), ограниченное отрезком $0\leqq t\leqq 1$ оси t, кривой $y=f(t)$ и отрезком $[(1, 0), (1, f(1))]$, т. е. возьмем график функции $y=f(t)$. Пусть

D — наименьшая выпуклая оболочка для G. «Верхняя» граница D при $0 \leqq t \leqq 1$ и будет искомой функцией $y = \psi(t)$, т. е.

$$\psi(t) = \sup\{y:\ (t, y) \in D\} \quad \text{при} \quad 0 \leqq t \leqq 1.$$

Справедлива

Теорема 1. *Пусть $\omega(\delta) \in \Omega$. Тогда, чтобы множество $A_\omega^{(\infty)}$ было алгеброй, для которой*

$$A_\omega(f^2) \leqq C_\omega A_\omega(f) \quad \textit{при всех} \quad f \in A_\omega^{(\infty)} \ \textit{с} \ \|f\|_\infty = 1, \tag{13}$$

необходимо и достаточно, чтобы $\omega(\delta)$ удовлетворяла условию Бари на отрезке $[0, 1]$.

Через C_ω и т. п. мы обозначаем постоянные, зависящие лишь от указанных параметров.

Доказательство. Достаточность. Пусть выполнено неравенство (2), а функция $f \in A_\omega^{(\infty)}$. Отметим, что

$$|a_m(f)| \leqq \|f\|_\infty \|\chi_m\|_1 \leqq (\sqrt{2}/\sqrt{m})\|f\|_\infty \quad \text{при} \quad m \geqq 3. \tag{14}$$

Положим

$$\varphi_1(t) = \begin{cases} t^2 & \text{при} \quad |t| \leqq \|f\|_\infty, \\ \|f\|_\infty^2 & \text{при} \quad |t| > \|f\|_\infty. \end{cases}$$

Ясно, что $\varphi_1 \in \mathrm{Lip}_D\, 1$ с постоянной $D = D(\varphi_1) = D_1 = 2\|f\|_\infty$. Поэтому найдется натуральное число m_0, такое, что (см. (14))

$$D|a_m(f)| \leqq 2 \quad \text{при всех} \quad m > m_0. \tag{15}$$

Но тогда при $m > m_0$ имеем (см. (1), (2) и (15))

$$\begin{aligned} &\int_{\|\chi_m\|_1}^{1} (\omega(Da_m(f)t)/t)\,dt \leqq 2 \int_{\|\chi_m\|_1}^{1} (\omega((D/2)\,a_m(f)t)/t)\,dt \leqq \\ &\leqq 2 \int_0^{(D/2)|a_m(f)|} (\omega(u)/u)\,du \leqq 2B\omega((D/2)\,a_m(f)) = 2B\omega(\|f\|_\infty a_m(f)). \end{aligned} \tag{16}$$

Стало быть, применяя Лемму 2 к функциям f и φ_1, получаем

$$\begin{aligned} A_\omega(f^2) = A_\omega(\varphi_1(f)) &\leqq (2 + 4B/\ln 2)\, A_\omega(\|f\|_\infty f) + \\ &+ (2/\ln 2) \sum_{m=3}^{m_0} \int_{\|\chi_m\|_1}^{1} (\omega(Da_m(f)t)/t)\,dt. \end{aligned} \tag{17}$$

Так как $f \in A_\omega^{(\infty)}$, то $A_\omega(\|f\|_\infty f) \leqq (1 + \|f\|_\infty) A_\omega(f) < \infty$. Кроме того, все интегралы, стоящие в правой части неравенства (17), конечны, ибо $\|\chi_m\|_1 > 0$. Из сказанного вытекает, что правая часть неравенства (17) является конечной величиной, если $f\in A_\omega^{(\infty)}$. Но тогда и $f^2\in A_\omega^{(\infty)}$ (см. (17)). Итак, мы доказали, что если $f\in A_\omega^{(\infty)}$, то и $f^2\in A_\omega^{(\infty)}$.

Теперь совсем легко убедиться, что множество $A_\omega^{(\infty)}$ является алгеброй. В самом деле, пусть $f\in A_\omega^{(\infty)}$ и $g\in A_\omega^{(\infty)}$. Тогда $|f(t)g(t)|\leqq f^2(t)+g^2(t)$ и потому $A_\omega(fg)\leqq A_\omega(f^2)+A_\omega(g^2)$ т. е. $fg\in A_\omega^{(\infty)}$.

Наконец, отметим, что если $f\in A_\omega^{(\infty)}$ и $\|f\|_\infty\leqq 1$, то в неравенстве (15) число m_0 можно взять равным 2 (см. еще (14)). Это влечет, что неравенства (16) и (17) верны при $m_0=2$, т. е.

(18) $$A_\omega(f^2)\leqq(2+4B/\ln 2)A_\omega(\|f\|_\infty f)\quad\text{при}\quad \|f\|_\infty\leqq 1.$$

Тем самым неравенство (13) установлено для $C_\omega=(2+4B/\ln 2)\equiv C(B)$. Достаточность условия (2) доказана.

Необходимость. Пусть неравенство (13) выполнено для всех $f\in A_\omega^{(\infty)}$ и $\|f\|_\infty=1$. Рассмотрим функцию $f_0(t)=2^{-n/2}\chi_n^{(1)}(t)$ при целом $n\geqq 2$, для которой, очевидно, выполнены свойства

$$f_0\in A_\omega^{(\infty)},\quad \|f_0\|_\infty=1,\quad f_0^2(t)=|f_0(t)|=|\chi_n^{(1)}(t)|/2^{n/2}.$$

В силу Леммы 3 имеем, что

$$f_0^2(t)=(1/2^n)\{1+\chi_0^{(1)}(t)+\sqrt{2}\,\chi_1^{(1)}(t)+\ldots+2^{(n-1)/2}\chi_{n-1}^{(1)}(t)\}$$

и потому должно выполняться неравенство (см. (13))

(19) $$A_\omega(f_0^2)=A_\omega(|f_0|)=\omega(2^{-n})+\sum_{i=0}^{n-1}\omega(2^{-n+i/2})\leqq C_\omega\omega(2^{-n/2})\quad\text{при}\quad n\geqq 2.$$

Если в (19) положить $n=2m$, а в сумме из (19) взять лишь слагаемые с четными $i=0,2,\ldots,2m-2$, то получим неравенство

(20) $$\sum_{j=m+1}^{2m}\omega(2^{-j})\leqq C_\omega\omega(2^{-m})\quad\text{при всех}\quad m\geqq 1.$$

Предположим, что $\omega(\delta)$ не удовлетворяет условию Бари на отрезке $[0,1]$. (В этом случае $\omega(\delta)\not\equiv 0$.) Тогда по Лемме 1

(21) $$\varliminf_{\delta\to+0}\omega(A\delta)/\omega(\delta)=1\qquad\text{при любом}\qquad A>1.$$

Возьмем целое число $p>8C_\omega$ и положим $A=2^p$. В силу (21) существуют такие числа $\delta_i>0$ и натуральные n_i, для которых $\delta_i\downarrow 0$, $n_i>2p$, $1/2^{n_i}\leqq\delta_i<2/2^{n_i}$ и $\omega(A\delta_i)/\omega(\delta_i)\leqq 2$, при всех $i\geqq 1$. Но тогда тем более (см. (1))

$$2\geqq\omega(2^p/2^{n_i})/\omega(2/2^{n_i})\geqq\omega(2^{p-n_i})/2\omega(2^{-n_i})$$

и потому

(22) $$\omega(2^{-n_i}) \geqq \omega(2^{p-n_i})/4 \quad \text{при всех} \quad i \geqq 1.$$

Положив в (20) число $m=n_i-p$, получим (см. еще (22))

$$C_\omega\omega(2^{-m}) = C_\omega\omega(2^{p-n_i}) \geqq \sum_{j=n_i-p+1}^{2(n_i-p)} \omega(2^{-j}) \geqq$$

$$\geqq \sum_{j=n_i-p+1}^{n_i} \omega(2^{-j}) \geqq p\omega(2^{-n_i}) \geqq (p/4)\omega(2^{p-n_i}) = (p/4)\omega(2^{-m}),$$

т. е.

(23) $$C_\omega\omega(2^{-m}) \geqq (p/4)\omega(2^{-m}).$$

Так как $\omega(\delta)\not\equiv 0$, то из (23) вытекает, что $4C_\omega \geqq p$. Но это противоречит неравенству $p>8C_\omega$. Необходимость доказана. Тем самым Теорема 1 полностью доказана.

Замечание 2. Из доказательства Теоремы 1 видно (см. (18)), что если функция $\omega(\delta)$ удовлетворяет условию Бари на отрезке $[0, 1]$, то выполнено неравенство

(24) $$A_\omega(f^2) \leqq C_\omega A_\omega(\|f\|_\infty f)$$

при всех f с $\|f\|_\infty \leqq 1$ и $C_\omega = C(B)$.

В общем случае неравенство (24) может и не выполняться, если мы будем брать *любые* функции $f \in A_\omega^{(\infty)}$, каким бы C_ω ни выбирать.

В качестве примера достаточно взять $\omega_0(\delta)=\delta$ при $0\leqq\delta\leqq 1$ и $\omega_0(\delta)=1$ при $\delta>1$. Ясно, что это $\omega_0(\delta)$ удовлетворяет условию Бари на отрезке $[0, 1]$ с $B=1$. Но, однако, не существует постоянной C, для которой бы выполнялось неравенство

(25) $$A_{\omega_0}(f^2) \leqq CA_{\omega_0}(\|f\|_\infty f) \quad \text{при всех} \quad f \in A_{\omega_0}^{(\infty)}.$$

В самом деле, если бы неравенство (25) было выполнено, то тогда для любой функции $f(t)=Mf_0(t)$ с постоянной $M>0$ мы бы имели

$$f^2(t) = M^2f_0^2(t) = M^2|f_0(t)| = (M^2/2^n)\{1+\ldots+2^{(n-1)/2}\chi_{n-1}^{(1)}(t)\}$$

и потому (см. (25), (19) и (20))

(26) $$\sum_{j=m+1}^{2m} \omega_0(M^2/2^j) \leqq C\omega_0(M^2/2^m) \quad \text{при} \quad m \geqq 1.$$

При $M\to\infty$ левая часть неравенства (26) стремится к m, а правая часть стремится к C. Но число m может быть любым и потому неравенство (26) (а стало быть и (25)) противоречиво.

Справедлива также

Теорема 1'. *Если* $\omega(\delta)\in\Omega$, *то множество* $A_\omega^{(\infty)}$ *является алгеброй с*

$$A_\omega(f^2)\leqq C_\omega A_\omega(\|f\|_\infty f)\quad\text{при всех}\quad f\in A_\omega^{(\infty)} \tag{27}$$

тогда и только тогда, когда $\omega\in B$ *на полупрямой* $[0,\infty)$.

Доказательство. Если $\omega\in B$ на $[0,\infty)$, то (см. доказательство Теоремы 1) неравенство (16) выполненно при всех $m>m_0=2$ и потому (см. (17)) выполнено (27) с $C_\omega=C(B)$. Обратно, пусть (27) выполнено.

Положим $f(t)=(\sqrt{\delta}/2^{n/4})\chi_n^{(1)}(t)$, где $\delta>0$ произвольное число. Тогда $\|f\|_\infty=\sqrt{\delta}\,2^{n/4}$, а $f^2(t)=\delta|\chi_n^{(1)}(t)|$. Стало быть, по Лемме 3 из (27) следует, что

$$\sum_{j=1}^{n}\omega(\delta/2^{j/2})\leqq C_\omega\omega(\delta)\quad\text{при всех}\quad \delta>0,\quad n=1,2,\dots.$$

Поэтому (см. еще (1))

$$C_\omega\omega(\delta)\geqq\sum_{j=1}^{n}\omega(\delta/2^j)\geqq\sum_{j=1}^{n}(1/\ln 2)\int_{\delta/2^{j+1}}^{\delta/2^j}(\omega(t)/t)\,dt=$$

$$=(1/\ln 2)\int_{\delta/2^{n+1}}^{\delta/2}(\omega(t)/t)\,dt\geqq(1/2\ln 2)\int_{\delta/2^{n+1}}^{\delta/2}(\omega(2t)/t)\,dt=(1/2\ln 2)\int_{\delta/2^n}^{\delta}(\omega(u)/u)\,du.$$

Устремляя n к ∞ мы получим неравенство (2) с $B=2C_\omega\ln 2$ при всех $\delta>0$. Что и требовалось доказать.

Замечание 3. Мы рассматриваем подклассы *ограниченных* функций из A_ω. Все дело в том, что если функция $f\in A_\omega$ и не ограничена, то даже для случая $\omega(\delta)=\delta$ функция f^2 уже может не принадлежать A_ω. Для ограниченных же функций f дело обстото иначе (см. по этому поводу [3]).

Теорема 2. *Пусть* $\lambda=\{\lambda_m\}_{m=2}^{\infty}$ *— неотрицательная и неубывающая последовательность. Тогда, чтобы для всех* $\omega\in\Omega$ *и* $\varphi\in\mathrm{Lip}_D 1$ *выполнялось неравенство*

$$A_\omega(\varphi(f))\leqq C_{\omega,\varphi,\lambda}\sum_{m=2}^{\infty}\omega(a_m(f))\lambda_m\quad\text{при всех}\quad f\in L^\infty, \tag{28}$$

необходимо и достаточно, чтобы $\lambda_m\geqq C\lg m$ *при некотором* $C>0$ *и всех* $m=2,3,\dots$. *При этом для всех* $\omega\in\Omega$ *и* $\varphi\in\mathrm{Lip}_D 1$ *величина*

$$A_\omega(\varphi(f))\leqq\sum_{m=2}^{\infty}\omega(Da_m(f))\lg 2(m-1)\quad\text{при}\quad f\in L^\infty. \tag{29}$$

Доказательство. Достаточность. Так как

$$\|\chi_m\|_1 = 1/\sqrt{2^n} \geqq 1/\sqrt{m-1} \quad (m = 2^n+k),$$

то в силу Леммы 2 имеем

$$A_\omega(\varphi(f)) \leqq A_\omega(Df)+(2/\ln 2)\sum_{m=3}^{\infty}\omega(Da_m(f))\ln(1/\|x_m\|_1) \leqq$$

$$\leqq A_\omega(Df)+(2/\ln 2)\sum_{m=3}^{\infty}\omega(Da_m(f))\ln\sqrt{m-1} =$$

$$= A_\omega(Df)+\sum_{m=3}^{\infty}\omega(Da_m(f))\lg(m-1) \leqq \sum_{m=2}^{\infty}\omega(Da_m(f))\lg 2(m-1),$$

т. е. (29) справедливо и потому (28) выполнено, например, при

$$\lambda_m = \lg 2(m-1) \quad \text{и} \quad C_{\omega,\varphi,\lambda} = \bar{D}.$$

Необходимость. Пусть $\lambda_m=\eta_m \lg m$ и неравенство (28) выполнено при всех $\omega\in\Omega$, $\varphi\in\mathrm{Lip}_D\,1$, хотя

(30) $$\varliminf_{m\to\infty}\eta_m = 0.$$

Тем более (28) будет выполнено для функции $\varphi(t)=|t|$. Тогда для каждой $\omega\in\Omega$ найдется постоянная C_ω, для которой

(31) $$A_\omega(|f|) \leqq C_\omega \sum_{m=2}^{\infty}\omega(a_m(f))\eta_m \lg m \quad \text{при всех} \quad f\in L^\infty.$$

Положим $f(t)=\delta\chi_m(t)$, $m=2^n+k$ с $1\leqq k\leqq 2^n$ и $n=1, 2, \ldots$, где δ — любое положительное число. Тогда из (31) имеем (см. также Лемму 3)

(32) $$\sum_{j=1}^{n}\omega(\delta/2^j) \leqq C_\omega\omega(\delta)\eta_m \lg m \leqq 2C_\omega\omega(\delta)\eta_m n \quad \text{при} \quad m \geqq 3.$$

В силу (30) найдутся $m_i=2^{n_i}+k_i$ с $1\leqq k_i\leqq 2^{n_i}$ и $n_i<n_{i+1}$, что $\eta_{m_{i+1}}< <\eta_{m_i}<1/i^2$ при всех $i\geqq 1$. Построим последовательность $\{\mu_j\}_{j=0}^{\infty}$ такую, что $\mu_j>\mu_{j+1}$ $(j\geqq 0)$ и $\mu_{m_i}=\sqrt{\eta_{m_i}}$ при $i\geqq 1$. По Лемме 4 найдем последовательность $\{\alpha_j\}_{j=0}^{\infty}$ такую, что $\alpha_j\to 0$, $\Delta^2\alpha_j\geqq 0$ при $j\geqq 0$ и $\alpha_j\geqq\mu_j$ при $j\geqq 0$. Известно, что тогда

$$\alpha_j-\alpha_{j+1}\downarrow 0 \quad \text{и} \quad \sum_{j=1}^{\infty}(\alpha_j-\alpha_{j+1})<\infty$$

и потому

(33) $$\alpha_j-\alpha_{j+1} = o(1/j) \quad \text{при} \quad j\to\infty.$$

Положим $\beta_j = |j\alpha_j - (j-1)\alpha_{j-1}|$ при $j \geqq 1$. Ясно, что (см. (33))

$$\beta_j \geqq 0 \quad \text{при} \quad j \geqq 1 \quad \text{и} \quad \beta_j \to 0 \quad \text{при} \quad j \to \infty,$$

ибо

$$\beta_j = |j(\alpha_j - \alpha_{j-1}) + \alpha_{j-1}| = |o(1) + \alpha_{j-1}| = o(1) \quad \text{при} \quad j \to \infty.$$

Найдем $\gamma_j > 0$ $(j \geqq 1)$ такие, что $\gamma_j \geqq \beta_j$, $\gamma_j > \gamma_{j+1}$ и $\gamma_j \to 0$ при $j \to \infty$.

Построим функцию $f(t)$ на отрезке $[0, 1]$ такую, что $f(0)=0, f(1)=2\gamma_1 \equiv \gamma_0$, $f(2^{-j}) = \gamma_j$ при $j = 1, 2, \ldots$; $f(t)$ линейна на $[2^{-j-1}, 2^{-j}]$ $(j \geqq 0)$. По Лемме 5 найдем выпуклую вверх функцию $\omega_0(\delta)$ на $[0, 1]$ такую, что $\omega_0(\delta) \geqq f(\delta)$ при $0 \leqq \delta \leqq 1$ и $\omega(+0)=0$. Положим $\omega_0(\delta) = \omega_0(1)$ при $\delta > 1$. Ясно, что $\omega_0(\delta)$ является модулем непрерывности на $[0, \infty)$. Для этой функции имеем оценку

$$\sum_{j=1}^{n} \omega_0(2^{-j}) \geqq \sum_{j=1}^{n} f(2^{-j}) = \sum_{j=1}^{n} \gamma_j \geqq \sum_{j=1}^{n} \beta_j \geqq n\alpha_n \geqq n\mu_n \quad \text{при} \quad n \geqq 2. \tag{34}$$

Но тогда из (32) и (34) вытекает для $\delta = 1$, $n = n_i$ и $m = m_i$, что

$$2C_{\omega_0}\omega_0(1)\eta_{m_i} n_i \geqq n_i \mu_{n_i} \geqq n_i \mu_{m_i} = n_i \sqrt{\eta_{m_i}},$$

т. е. $2C_{\omega_0}\omega_0(1)\sqrt{\eta_{m_i}} \geqq 1$, или

$$2C_{\omega_0}\omega_0(1)(1/i) \geqq 1 \quad \text{при} \quad i \geqq 1.$$

Последнее неравенство противоречиво и потому (см. (30))

$$\varliminf_{m \to \infty} \eta_m > 0. \tag{35}$$

Но $\lambda_m > 0$ при всех $m \geqq 2$ (если бы $\lambda_{m_0} = 0$ при некотором m_0, то неравенство (28) (см. также (31)) не могло бы быть выполненным уже для функции $f = \chi_{m_0}$) и потому из (35) следует, что найдется постоянная $C > 0$, для которой

$$\lambda_m \geqq C \lg m \quad \text{при всех} \quad m \geqq 2. \tag{36}$$

Необходимость, а вместе с ней и Теорема 2, доказана.

З а м е ч а н и е 4. При доказательстве необходимости условия (36) в Теореме 2 использовалось выполнение неравенства (28) лишь для $\varphi(t) = |t|$ и $f(t) = \chi_m(t)$ с $m \geqq 3$.

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Über die Cesàrosche Summierbarkeit von mehrfachen Orthogonalreihen

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1. In dieser Arbeit werden wir uns mit gewissen Divergenzeigenschaften der Cesàroschen Mittel von mehrfachen Orthogonalreihen beschäftigen.

Um die Sätze zu befassen, schicken wir gewisse Bezeichnungen vor.

Für eine Zahlenfolge $a=\{a_k\}_{k=1}^{\infty}$ und für ein Funktionensystem $\varphi=\{\varphi_k(x)\}_{k=1}^{\infty}$ setzen wir

$$s_m(a, \varphi; x) = \sum_{k=1}^{m} a_k \varphi_k(x), \quad \sigma_m(a, \varphi; x) = \sum_{k=1}^{m} (1-(k-1)/m) a_k \varphi_k(x)$$

$$(m = 1, 2, \ldots).$$

Weiterhin, für eine Zahlenfolge $a=\{a_{kl}\}_{k,l=1}^{\infty}$ und für ein Funktionensystem $\varphi=\{\varphi_{kl}(x)\}_{k,l=1}^{\infty}$ seien

$$s_{mn}(a, \varphi; x) = \sum_{k=1}^{m} \sum_{l=1}^{n} a_{kl} \varphi_{kl}(x),$$

$$\sigma_{mn}(a, \varphi; x) = \sum_{k=1}^{m} \sum_{l=1}^{n} (1-(k-1)/m)(1-(l-1)/n) a_{kl} \varphi_{kl}(x) \quad (m, n = 1, 2, \ldots).$$

Im Folgenden bedeutet $a=\{a_k\}_{k=1}^{\infty} \in l^2$, bzw. $a=\{a_{kl}\}_{k,l=1}^{\infty} \in l^2$, dass

$$\sum_{k=1}^{\infty} a_k^2 < \infty, \quad \text{bzw.} \quad \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{kl}^2 < \infty$$

besteht, weiterhin bedeuten $\log \alpha$ den Logarithmus der Zahl α mit der Basis 2, und E den Einheitsquadrat $(0, 1)\times(0, 1)$.

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Für eine Zahlenfolge $a=\{a_{kl}\}_{k,l=1}^{\infty}$ setzen wir

$$A_{pq}=\sqrt{\sum_{k=2^p+1}^{2^{p+1}}\sum_{l=2^q+1}^{2^{q+1}}a_{kl}^2}\quad (p,q=0,1,\ldots),$$

$$A_{-1,q}=\sqrt{\sum_{l=2^q+1}^{2^{q+1}}a_{1l}^2}\ (q=0,1,\ldots),\quad A_{p,-1}=\sqrt{\sum_{k=2^p+1}^{2^{p+1}}a_{k1}^2}\quad (p=0,1,\ldots),$$

$$A_{-1,-1}=|a_{11}|.$$

Der folgende Satz von S. Kaczmarz und D. E. Menchoff ist bekannt. (S. z. B. [2], S. 125.)

Satz A. *Gilt*

(1) $$\sum_{k=1}^{\infty}a_k^2(\log\log(k+3))^2<\infty,$$

dann existiert $\lim\limits_{m\to\infty}\sigma_m(a,\varphi;x)$ *für jedes in einem Maßraum* $(X,\mathcal{A},\mu)$ *orthonormierte System* $\varphi=\{\varphi_k(x)\}_{k=1}^{\infty}$ *in* X μ*-fast überall.*

Der Verf. hat gezeigt, daß für gewisse Koeffizientenfolgen $a=\{a_k\}_{k=1}^{\infty}$ (1) notwendig dafür ist, daß $\lim\limits_{m\to\infty}\sigma_m(a,\varphi;x)$ bei jedem orthonormierten System $\varphi=\{\varphi_k(x)\}_{k=1}^{\infty}$ fast überall existiert. Es gilt nämlich der folgende Satz. (S. z. B. [2], S. 114.)

Satz B. *Gilt* $k^2a_k^2\geqq(k+1)^2a_{k+1}^2$ $(k=1,2,\ldots)$ *und*

$$\sum_{k=1}^{\infty}a_k^2(\log\log(k+3))^2=\infty,$$

so gibt es ein orthonormiertes System $\Phi=\{\Phi_k(x)\}_{k=1}^{\infty}$ *im Intervall* $(0,1)$ *derart, daß die Folge* $\{\sigma_m(a,\Phi;x)\}_{m=1}^{\infty}$ *in* $(0,1)$ *fast überall divergiert.*

Das Analogon des Satzes A hat F. Móricz [4] für mehrfache Orthogonalreihen bewiesen.

Satz C. *Gilt*

(2) $$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}a_{kl}^2(\log\log(k+3))^2(\log\log(l+3))^2<\infty,$$

dann existiert $\lim\limits_{\min(m,n)\to\infty}\sigma_{mn}(a,\varphi;x)$ *für jedes in einem Maßraum* $(X,\mathcal{A},\mu)$ *orthonormierte System* $\varphi=\{\varphi_{kl}(x)\}_{k,l=1}^{\infty}$ *in* X μ*-fast überall.*

F. Móricz hat das Problem aufgeworfen, ob das Analogon des Satzes B für mehrfache Orthogonalreihen richtig ist. Als Antwort werden wir den folgenden Satz beweisen.

Satz I. *Es sei* $a=\{a_{kl}\}_{k,l=1}^{\infty}$ *eine Zahlenfolge mit den Eigenschaften*

$$A_{pq} \geqq A_{p,q+1},\ A_{p+1,q} \quad (p, q = -1, 0, 1, \ldots), \tag{3}$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^2(\log\log(k+3))^2(\log\log(l+3))^2 = \infty. \tag{4}$$

Dann gibt es ein in E orthonormiertes System $\Phi=\{\Phi_{kl}(x, y)\}_{k,l=1}^{\infty}$ *von Treppenfunktionen derart, daß*

$$\overline{\lim_{\min(m,n)\to\infty}} |\sigma_{mn}(a, \Phi;\ x, y)| = \infty$$

in E fast überall besteht.

A. N. Kolmogoroff hat den folgenden Satz bewiesen. (S. z. B. [2], S. 118.)

Satz D. *Gilt* $a=\{a_k\}_{k,}^{\infty}\in l^2$, *dann ist*

$$\lim_{m\to\infty}\left(s_{2^m}(a, \varphi; x)-\sigma_{2^m}(a, \varphi; x)\right) = 0$$

für jedes in einem Maßraum $(X, \mathscr{A}, \mu)$ *orthonormierte System* $\varphi=\{\varphi_k(x)\}_{k=1}^{\infty}$ *in X* μ*-fast überall.*

L. Csernyák [3] hat bewiesen, daß das Analogon des Satzes D für mehrfache Orthogonalreihen im Allgemeinen nicht richtig ist; weiterhin hat F. Móricz [4] den folgenden Satz bewiesen.

Satz E. *Gilt*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^2(\log\log(\max(k, l)+3))^2 < \infty,$$

dann ist bei jedem in einem Maßraum $(X, \mathscr{A}, \mu)$ *orthonormierten System* $\varphi= =\{\varphi_{kl}(x)\}_{k,l=1}^{\infty}$

$$\lim_{\min(m,n)\to\infty}\left(s_{2^m,2^n}(a, \varphi;\ x)-\sigma_{2^m,2^n}(a, \varphi;\ x)\right) = 0$$

in X μ*-fast überall.*

F. Móricz hat das Problem aufgeworfen, ob diese Behauptung ohne díe Bedingung des Satzes E im Allgemeinen nicht richtig ist. Betreffs dieses Problems beweisen wir den folgenden Satz.

Satz II. *Es sei* $\lambda=\{\lambda_k\}_{k=1}^{\infty}$ *eine monoton wachsende Folge von positiven Zahlen mit*

$$\lambda_k = o(\log\log k),$$

und sei $\mu=\{\mu_l\}_{l=1}^{\infty}$ *eine beliebige monoton wachsende Folge von positiven Zahlen.*

1. *Dann gibt es eine Koeffizientenfolge* $a=\{a_{kl}\}_{k,l=1}^{\infty}$ *und ein orthonormiertes System* $\Phi=\{\Phi_{kl}(x)\}_{k,l=1}^{\infty}$ *im Intervall* $(0, 1)$ *derart, daß*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^2\lambda_k^2\mu_l^2<\infty \tag{5}$$

ist, $\lim\limits_{\min(m,n)\to\infty} s_{2^m,2^n}(a, \Phi; x)$ *in* $(0, 1)$ *überall existiert, und*

$$\overline{\lim_{\min(m,n)\to\infty}}\, |\sigma_{2^m,2^n}(a, \Phi; x)|=\infty$$

in $(0, 1/2)$ *fast überall besteht.*

2. *Weiterhin gibt es eine Koeffizientenfolge* $b=\{b_{kl}\}_{k,l=1}^{\infty}$ *und ein orthonormiertes System* $\Psi=\{\Psi_{kl}(x)\}_{k,l=1}^{\infty}$ *in* $(0, 1)$ *derart, dass*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} b_{kl}^2\lambda_k^2\mu_l^2<\infty \tag{6}$$

ist, $\lim\limits_{\min(m,n)\to\infty} \sigma_{2^m,2^n}(b, \Psi; x)$ *in* $(0, 1)$ *fast überall existiert, und*

$$\overline{\lim_{\min(m,n)\to\infty}}\, |s_{2^m,2^n}(b, \Psi; x)|=\infty$$

in $(0, 1/2)$ *fast überall gilt.*

Wir machen den Leser aufmerksam darauf, daß in diesem Satz μ eine beliebig schnell ins Unendliche strebende Folge sein kann.

2. Vorbereitungen. Im Folgenden bezeichnen $c_1, c_2, \dots$ positive absolute Konstanten. Unter Intervall, bzw. Rechteck verstehen wir ein Intervall (a, b) mit $b-a>0$, bzw. ein Rechteck $(a, b)\times(c, d)$ mit $b-a$, $d-c>0$. Die in einem Intervall I definierte Funktion $f(x)$ nennen wir eine Treppenfunktion, wenn eine Zerlegung von I in endlichviele, paarweise disjunkte Intervalle I_i derart existiert, daß f in jedem I_i konstant ist; weiterhin nennen wir die in einem Rechteck T definierte Funktion $f(x, y)$ eine Treppenfunktion, wenn eine Zerlegung von T in endlichviele, paarweise disjunkte Rechtecke T_i derart existiert, daß f in jedem T_i konstant ist. Eine lineare Menge nennen wir einfach, wenn sie die Vereinigung endlichvieler Intervalle ist; weiterhin nennen wir eine zweidimensionale Menge einfach, wenn sie die Vereinigung endlichvieler Rechtecke ist. Es sei $I=(a, b)$ ein Intervall. Für eine in I definierte Funktion $f(x)$ und für eine Menge $H(\subseteq(0, 1))$ seien

$$f(I;\, x)=\begin{cases} f((x-a)/(b-a)), & x\in I,\\ 0 & \text{sonst,}\end{cases}$$

und $H(I)$ diejenige Menge, die aus H mit der Transformation $\bar{x}=(b-a)x+a$ entsteht. Es sei $T=(a, b)\times(c, d)$. Für eine in T definierte Funktion $f(x, y)$ und für eine Menge $H(\subseteqq E)$ sei

$$f(T; x, y) = \begin{cases} f((x-a)/(b-a),\ (y-c)/(d-c)), & (x, y) \in T, \\ 0, & \text{sonst}, \end{cases}$$

und sei $H(T)$ diejenige Menge, die aus H mit der Transformation $\bar{x}=(b-a)x+a$, $\bar{y}=(d-c)y+c$ entsteht. Im Folgenden bezeichnet $r=\{r_k(x)\}_{k=1}^{\infty}$ das Rademachersche Funktionensystem, wobei also $r_k(x)=\operatorname{sign}\sin 2^k\pi x$ $(k=1, 2, \ldots)$ ist.

Zum Beweis unserer Sätze benötigen wir gewisse Hilfssätze.

Von D. E. Menchoff und H. Rademacher stammt der folgende Hilfssatz. (S. z. B. [2], S. 79.)

Hilfssatz I. *Es gibt eine positive Zahl c_1 mit folgender Eigenschaft. Für eine beliebige Folge $\{a_k\}_{k=1}^{N}$ und für ein beliebiges in einem Maßraum $(X, \mathscr{A}, \mu)$ definiertes orthonormiertes System $\{\chi_k(x)\}_{k=1}^{N}$ gilt*

$$\int_X \max_{1 \leqq m \leqq N} \Big(\sum_{k=1}^{m} a_k \chi_k(x)\Big)^2 d\mu \leqq c_1 \log^2(N+1) \sum_{k=1}^{N} a_k^2.$$

Das Analogon des Hilfssatzes I ist richtig für mehrfache Summen von orthogonalen Funktionen. (S. z. B. [1].)

Hilfssatz II. *Es gibt eine positive Zahl c_2 mit folgender Eigenschaft. Für jede Folge a_{kl} $(k=1, \ldots, M,\ l=1, \ldots, N)$ und für jedes in einem Maßraum $(X, \mathscr{A}, \mu)$ orthonormierte System $\chi_{kl}(x)$ $(k=1, \ldots, M,\ l=1, \ldots, N)$ gilt*

$$\int_X \max_{\substack{1 \leqq m \leqq M \\ 1 \leqq n \leqq N}} \Big(\sum_{k=1}^{m}\sum_{l=1}^{n} a_{kl}\chi_{kl}(x)\Big)^2 d\mu \leqq c_2 \log^2(M+1)\log^2(N+1) \sum_{k=1}^{M}\sum_{l=1}^{N} a_{kl}^2.$$

Von D. E. Menchoff stammt der folgende Hilfssatz. (S. z. B. [7].)

Hilfssatz III. *Es gibt positive Konstanten c_3, c_4, c_5 mit den folgenden Eigenschaften. Ist $p(\geqq 2)$ eine ganze Zahl, dann gibt es ein orthonormiertes System $\{f_l(p; x)\}_{l=1}^{2p^2}$ von Treppenfunktionen in $(0, 1)$ und eine einfache Menge $E(p)$ $(\subseteqq(0, 1))$ derart, daß die folgenden Bedingungen erfüllt sind:*

$$\operatorname{mes} E(p) \geqq c_3,$$

$$|f_l(p; x)| \leqq c_4 \quad (x \in (0, 1);\ l = 1, \ldots, 2p^2),$$

und für jeden Punkt $x \in E(p)$ gibt es einen Index $m(x)$ $(p^2 \leqq m(x) < 2p^2)$ mit

$$f_l(p; x) \geqq 0 \quad (x \in E(p);\ l = 1, \ldots, m(x)),$$

und

$$\sum_{l=1}^{m(x)} f_l(p; x) \geqq c_5 p \log p \quad (x \in E(p)).$$

Aus Hilfssatz III ergibt sich unmittelbar der folgende:

Hilfssatz IV. *Es gibt eine positive ganze Zahl p_0 und positive Konstanten c_6, c_7 mit den folgenden Eigenschaften. Ist $p(\geqq p_0)$ eine ganze Zahl, dann gibt es ein orthonormiertes System $\{f_l(p;x)\}_{l=1}^{2p}$ von Treppenfunktionen in $(0,1)$ und eine einfache Menge $E(p)$ $(\subseteqq(0,1))$ derart, daß die folgenden Bedingungen erfüllt sind:*

$$\operatorname{mes} E(p) \geqq c_6,$$

und für jeden Punkt $x\in E(p)$ gibt es einen Index $m(x)$ $(1<m(x)<2p)$ mit

$$f_l(p;\ x)\geqq 0 \quad (x\in E(p);\ l=1,\dots,m(x))$$

und

$$\sum_{l=1}^{m(x)-1} f_l(p;x)\geqq c_7\sqrt{p}\log p \quad (x\in E(p)).$$

Es sei nämlich $\bar{p}$ eine positive ganze Zahl mit $\bar{p}^2\leqq p<(\bar{p}+1)^2$, und wir wenden den Hilfssatz III im Falle $\bar{p}$. Die so erhaltenen Funktionen, bzw. Menge bezeichnen wir mit $\{f_l(p;x)\}_{l=1}^{2\bar{p}^2}$, bzw. mit $E(p)$. Da diese Funktionen Treppenfunktionen sind, können wir leicht solche Treppenfunktionen $f_l(p;x)$ $(l=2\bar{p}^2+1,\dots,2p)$ angeben, daß das System $\{f_l(p;x)\}_{l=1}^{2p}$ in $(0,1)$ orthonormiert ist. Es ist klar, daß für das System $\{f_l(p;x)\}_{l=1}^{2p}$ und für die Menge $E(p)$ alle Erforderungen des Hilfssatzes IV mit geeigneten absoluten Konstanten c_6, c_7 erfüllt werden, wenn nur p genügend groß ist.

Es sei $R=\{r_k(x)r_l(y)\}_{k,l=1}^{\infty}$.

Hilfssatz V. *Ist $a=\{a_{kl}\}_{k,l=1}^{\infty}\notin l^2$, dann gilt*

$$\lim_{\min(m,n)\to\infty} |\sigma_{mn}(a,R;\ x,y)|=\infty$$

fast überall in E.

Diesen Hilfssatz kann man ähnlich, wie den entsprechenden Satz von A. Zygmund über einfache Rademachersche Reihen beweisen. (S. z. B. [8], S. 212.)

Hilfssatz VI. *Unter den Bedingungen*

$$\sum_{k=2^p+1}^{2^{p+1}} a_k^2 \geqq \sum_{k=2^{p+1}+1}^{2^{p+2}} a_k^2 \quad (p=0,1,\dots),$$

$$\sum_{p=0}^{\infty}\Big(\sum_{k=2^p+1}^{2^{p+1}} a_k^2\Big)\log^2(p+2)=\infty,$$

gibt es ein orthonormiertes System $\Phi=\{\Phi_k(x)\}_{k=1}^{\infty}$ von Treppenfunktionen in $(0,1)$ derart, daß die Folge $\{s_{2^p}(a,\Phi;x)\}_{p=0}^{\infty}$ in $(0,1)$ fast überall divergiert.

Dieser Hilfssatz folgt leicht aus bekannten Sätzen. (S. [5], Satz II und [6], Satz VII.) Aus dem Beweis des Satzes II in [5] ergibt sich, daß man die Funktionen $\Phi_k(x)$ als Treppenfunktionen wählen kann.

Aus dem Hilfssatz VI folgt der folgende:

Hilfssatz VII. *Unter den Bedingungen des Hilfssatzes VI gibt es ein orthonormiertes System* $\varphi=\{\varphi_k(x)\}_{k=1}^{\infty}$ *von Treppenfunktionen in* $(0, 1)$ *derart, dass*

$$\overline{\lim_{p\to\infty}}\, |s_{2^p}(a, \varphi;\ x)| = \infty$$

in $(0, 1)$ *fast überall besteht.*

Beweis. Wir sollen zwei Fälle unterscheiden.

Ist $a\notin l^2$, dann gilt

$$\overline{\lim_{p\to\infty}}\, |s_{2^p}(a, r;\ x)| = \infty$$

in $(0, 1)$ fast überall, nach dem zitierten Satz von A. Zygmund. (S. z. B. [8], S. 212.)

Wir nehmen also $a\in l^2$ an. Dann gibt es eine monoton abnehmende, zu 0 strebende Folge $\lambda=\{\lambda_k\}_{k=1}^{\infty}$ von positiven Zahlen mit

$$\sum_{k=2^p+1}^{2^{p+1}} \lambda_k^2 a_k^2 \geqq \sum_{k=2^{p+1}+1}^{2^{p+2}} \lambda_k^2 a_k^2 \quad (p=0, 1, \dots),$$

$$\sum_{p=0}^{\infty} \Big\{\sum_{k=2^p+1}^{2^{p+1}} \lambda_k^2 a_k^2\Big\} \log^2(p+2) = \infty.$$

Auf Grund des Hilfssatzes VI gibt es dann ein in $(0, 1)$ orthonormiertes System φ derart, daß die Folge $\{s_{2^p}(\lambda a, \varphi; x)\}_{p=0}^{\infty}$ in $(0, 1)$ fast überall divergiert, wobei λa die Folge $\{\lambda_k a_k\}_{k=1}^{\infty}$ bezeichnet. Durch Abelsche Umformung ergibt sich, daß für jedes p

$$s_{2^p}(\lambda a, \varphi;\ x) = \sum_{k=1}^{2^p-1} (\lambda_k-\lambda_{k+1}) s_k(a, \varphi;\ x) + \lambda_{2^p} s_{2^p}(a, \varphi;\ x) \quad \big(x\in(0, 1)\big) \tag{7}$$

ist. Da, wegen $a\in l^2$,

$$\sum_{k=1}^{\infty} (\lambda_k-\lambda_{k+1}) \int_0^1 |s_k(a, \varphi;\ x)|\, dx \leqq \sum_{k=1}^{\infty} (\lambda_k-\lambda_{k+1}) \Big\{\sum_{l=1}^{k} a_l^2\Big\}^{1/2} \leqq$$

$$\leqq \Big\{\sum_{l=1}^{\infty} a_l^2\Big\}^{1/2} \sum_{k=1}^{\infty} (\lambda_k-\lambda_{k+1}) = \lambda_1 \Big\{\sum_{l=1}^{\infty} a_l^2\Big\}^{1/2} < \infty$$

ist, konvergiert die Reihe

$$\sum_{k=1}^{\infty} (\lambda_k-\lambda_{k+1}) s_k(a, \varphi;\ x)$$

in $(0, 1)$ fast überall. Auf Grund von (7) erhalten wir, daß die Folge $\{\lambda_{2^p} s_{2^p}(a, \varphi; x)\}_{p=0}^{\infty}$ in $(0, 1)$ fast überall divergiert, woraus die Behauptung des Hilfssatzes VII folgt.

3. Beweis des Satzes I. Anstatt des Satzes I ist es genug den folgenden Satz zu beweisen.

Satz I'. *Wir nehmen an, daß für die Folge $a = \{s_{kl}\}_{k,l=1}^{\infty}$ die Bedingungen des Satzes I erfüllt sind. Dann gibt es ein in E orthonormiertes System $\psi = \{\psi_{kl}(x, y)\}_{k,l=1}^{\infty}$ von Treppenfunktionen derart, daß*

$$\varlimsup_{\min(m,n)\to\infty} |\sigma_{mn}(a, \psi; x, y)| = \infty$$

in einer Untermenge von E mit positivem Maß erfüllt ist.

Wir wollen zeigen, daß Satz I aus Satz I' folgt. Gilt nämlich der Satz I', dann gibt es eine Folge $(0=)r_0 < \ldots < r_i < \ldots$ von ganzen Zahlen, und eine Folge von einfachen Mengen $H_i(\subseteqq E)$ $(i = 1, 2, \ldots)$ derart, daß mit den Bezeichnungen $M_{kl} = \max_{(x,y)\in E} |\psi_{kl}(x, y)|$ $(k, l = 1, 2, \ldots)$ und

$$T(i, m, n) = \{(k, l)\colon k = 1, \ldots, m,\ l = 1, \ldots, n\} \setminus \{(k, l)\colon k, l = 1, \ldots, r_{i-1}\}$$

$(m, n = r_{i-1}+1, \ldots, r_i;\ i = 1, 2, \ldots)$ für jedes $i\ (= 1, 2, \cdots)$ die Bedingungen

$$\text{mes}\, H_i \geqq c_8, \tag{8}$$

und

$$\max_{r_{i-1}<m,n\leqq r_i} \Big|\sum_{(k,l)\in T(i,m,n)} \big(1-(k-1)/m\big)\big(1-(l-1)/n\big) a_{kl}\psi_{kl}(x, y)\Big| \geqq \tag{9}$$
$$\geqq i + \sum_{k=1}^{r_{i-1}} \sum_{l=1}^{r_{i-1}} |a_{kl}| M_{kl} \quad ((x, y)\in H_i)$$

erfüllt werden.

Durch vollständige Induktion definieren wir eine Folge $E_i(\subseteqq E)$ $(i = 1, 2, \ldots)$ von einfachen und stochastisch unabhängigen Mengen, und ein in E orthonormiertes System $\Phi = \{\Phi_{k,l}(x, y)\}_{k,l=1}^{\infty}$ von Treppenfunktionen, die die folgenden Bedingungen erfüllen:

$$\max_{(x,y)\in E} |\Phi_{kl}(x, y)| = M_{kl} \quad (k, l = 1, 2, \ldots), \tag{10}$$

$$\text{mes}\, E_i \geqq c_8 \quad (i = 1, 2, \ldots), \tag{11}$$

$$\max_{r_{i-1}<m,n\leqq r_i} \Big|\sum_{(k,l)\in T(i,m,n)} \big(1-(k-1)/m\big)\big(1-(l-1)/n\big) a_{kl}\Phi_{kl}(x, y)\Big| \geqq \tag{12}$$
$$\geqq i + \sum_{k=1}^{r_{i-1}} \sum_{l=1}^{r_{i-1}} |a_{kl}| M_{kl} \quad ((x, y)\in E_i).$$

Es sei nämlich $\Phi_{kl}(x, y)=\psi_{kl}(x, y)$ $((x, y)\in E;\ k, l=1, \ldots, r_1)$ und $E_1=H_1$. Es sei i_0 eine positive ganze Zahl. Wir nehmen an, daß die einfachen und stochastisch unabhängigen Mengen $E_i(\subseteqq E)$ $(i=1, \ldots, i_0)$ und das in E orthonormierte System $\{\Phi_{kl}(x, y)\}_{k,l=1}^{r_{i_0}}$ von Treppenfunktionen schon derart definiert sind, daß (10)—(12) für $i=1, \ldots, i_0$ erfüllt sind. Dann gibt es eine Einteilung von E in paarweise disjunkte Rechtecke $T_1, \ldots, T_\sigma$ derart, daß jede Funktion $\Phi_{kl}(x, y)$ $(k, l=1, \ldots, r_{i_0})$ in jedem T_s $(s=1, \ldots, \sigma)$ konstant ist, und jede Menge E_i $(i=1, \ldots, i_0)$ die Vereinigung gewisser T_s ist. Jedes Rechteck T_s teilen wir in zwei disjunkte Rechtecke T_s' und T_s'' mit $\operatorname{mes} T_s'=\operatorname{mes} T_s''$ $(s=1, \ldots, \sigma)$. Dann setzen wir

$$\Phi_{kl}(x, y) = \sum_{s=1}^{\sigma} \psi_{kl}(T_s'; x, y)-\sum_{s=1}^{\sigma} \psi_{kl}(T_s''; x, y) \quad ((k, l)\in T(i_0+1, r_{i_0+1}, r_{i_0+1})),$$

$$E_{i_0+1} = \bigcup_{s=1}^{\sigma} \left(H_{i_0+1}(T_s')\cup H_{i_0+1}(T_s'')\right).$$

Es ist klar, daß die Menge $E_{i_0+1}(\subseteqq E)$ einfach ist, die Mengen $E_1, \ldots, E_{i_0+1}$ stochastisch unabhängig sind, die Funktionen $\Phi_{kl}(x, y)$ $((k, l)\in T(i_0+1, r_{i_0+1}, r_{i_0+1}))$ Treppenfunktionen sind, das System $\{\Phi_{kl}(x, y)\}_{k,l=1}^{r_{i_0+1}}$ in E ortonormiert ist, und wegen (8), (9), (10)—(12) im Falle $i=i_0+1$ bestehen. Die Mengenfolge $\{E_i\}_{i=1}^{\infty}$ und das System $\Phi=\{\Phi_{kl}(x, y)\}_{k,l=1}^{\infty}$ mit den erforderten Eigenschaften erhalten wir also durch Induktion.

Wegen (10) und (12) gilt

$$\max_{r_{i-1}<m, n\leqq r_i} |\sigma_{mn}(a, \Phi;\ x, y)| \geqq i \quad ((x, y)\in E_i) \tag{13}$$

für jedes $i(=1, 2, \ldots)$. Es sei $\bar{E}=\overline{\lim\limits_{i\to\infty}} E_i$. Da die Mengen E_i stochastisch unabhängig sind, aus (11) und durch Anwendung des Borel—Cantellischen Lemmas, erhalten wir $\operatorname{mes}\bar{E}=1$. Im Falle $(x, y)\in\bar{E}$ gilt aber (13) für unendlich viele i. So bekommen wir, daß $\overline{\lim\limits_{\min(m,n)\to\infty}} |\sigma_{mn}(a, \Phi; x, y)|=\infty$ in E fast überall besteht.

Damit haben wir bewiesen, daß Satz I sich aus Satz I′ ergibt.

Wir haben also den Satz I′ zu beweisen. Dazu können wir

$$a = \{a_{kl}\}_{k,l=1}^{\infty}\in l^2 \tag{14}$$

voraussetzen, da im entgegengesetzten Falle Satz I auf Grund des Hilfssatzes V folgt.

Wir bemerken, daß die Bedingung (4) mit

$$\sum_{p=-1}^{\infty} \sum_{q=-1}^{\infty} A_{pq}^2 \log^2(p+3)\log^2(q+3) = \infty$$

äquivalent ist. Daraus folgt, daß einer der folgenden zwei Fälle besteht.

a) Eine der zwei Reihen

$$\sum_{p=-1}^{\infty} A_{p,-1}^2 \log^2(p+3), \quad \sum_{q=-1}^{\infty} A_{-1,q}^2 \log^2(q+3)$$

ist divergent.

b) Es gilt

$$\sum_{p=r}^{\infty}\sum_{q=r}^{\infty} A_{pq}^2 \log^2(p+2)\log^2(q+2) = \infty \quad (r = 0, 1, \ldots). \tag{15}$$

a) Beweis des Satzes I′ im Falle a). Wir nehmen an, daß

$$\sum_{p=-1}^{\infty} A_{p,-1}^2 \log^2(p+3) = \infty \tag{16}$$

ist. Den Fall

$$\sum_{q=-1}^{\infty} A_{-1,q}^2 \log^2(q+3) = \infty$$

kann man ähnlicherweise betrachten. Aus (14) folgt

$$a^{(0)} = \{a_{k1}\}_{k=1}^{\infty} \in l^2. \tag{17}$$

Aus (3), (16) und (17), durch Anwendung von Satz D und Hilfssatz VII bekommen wir ein in (0, 1) orthonormiertes System $f=\{f_k(x)\}_{k=1}^{\infty}$ von Treppenfunktionen, für welches

$$\overline{\lim_{m\to\infty}} |\sigma_m(a^{(0)}, f;\ x)| = \infty$$

in (0, 1) fast überall gilt.

Es seien T_{kl} $(k=1, 2, \ldots;\ l=2, 3, \ldots)$ paarweise disjunkte Rechtecke in $(0, 2)\times\times(0, 2)\setminus E$. Wir setzen

$$\bar{\psi}_{k1}(x, y) = \begin{cases} f_k(x), & (x, y)\in E, \\ 0, & (x, y)\in(0, 2)\times(0, 2)\setminus E \end{cases}$$

$(k=1, 2, \ldots)$ und

$$\bar{\psi}_{kl}(x, y) = \begin{cases} 1/\sqrt{\operatorname{mes} T_{kl}}, & (x, y)\in T_{kl}, \\ 0, & (x, y)\in(0, 2)\times(0, 2)\setminus T_{kl} \end{cases}$$

$(k=1, 2, \ldots;\ l=2, 3, \ldots)$.

Es ist klar, daß die Funktionen $\bar{\psi}_{kl}(x, y)$ $(k, l=1, 2, \ldots)$ Treppenfunktionen sind, ein orthonormiertes System in $(0, 2)\times(0, 2)$ bilden, und in E fast überall

$$\overline{\lim_{m\to\infty}} |\sigma_{2^m, 2^m}(a, \bar{\psi};\ x, y)| = \infty$$

erfüllt ist.

Es sei endlich

$$\psi_{kl}(x, y) = 2\bar{\psi}_{kl}(2x, 2y) \quad ((x, y)\in E;\ k, l = 1, 2, \ldots).$$

Für diese Funktionen sind alle Erforderungen des Satzes I′ erfüllt.

b) Beweis des Satzes I′ im Falle b). Während des Beweises von Satzes I′ können wir voraussetzen, daß die Koeffizienten a_{kl} rationelle Zahlen sind.

Unter den Bedingungen des Satzes I′ kann man nämlich solche rationellen Zahlen $\bar{a}_{kl}$ $(k, l=1, 2, \ldots)$ angeben, daß

$$\bar{A}_{pq} \geqq \bar{A}_{p+1,q}, \bar{A}_{p,q+1} \quad (p, q=-1, 0, 1, \ldots),$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \bar{a}_{kl}(\log\log(k+3))^2(\log\log(l+3))^2=\infty, \quad \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|a_{kl}-\bar{a}_{kl}|<\infty$$

erfüllt werden. Dann gilt

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|a_{kl}-\bar{a}_{kl}|\iint_E|\varphi_{kl}(x,y)|\,dx\,dy \leqq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|a_{kl}-\bar{a}_{kl}|<\infty$$

für jedes in E orthonormierte System $\varphi=\{\varphi_{kl}(x,y)\}_{k,l=1}^{\infty}$; daraus folgt, daß $\lim\limits_{\min(m,n)\to\infty} s_{mn}(a-\bar{a}, \varphi; x, y)$ in E fast überall existiert; hieraus ergibt sich aber, daß $\lim\limits_{\min(m,n)\to\infty} \sigma_{mn}(a-\bar{a}; \varphi; x, y)$ auch fast überall in E existiert. Ist also Satz I′ für rationelle Koeffizienten bewiesen, so gibt es ein in E orthonormiertes System $\Phi=\{\Phi_{kl}(x,y)\}_{k,l=1}^{\infty}$ von Treppenfunktionen derart, daß $\varlimsup\limits_{\min(m,n)\to\infty} |\sigma_{mn}(\bar{a}, \Phi; x, y)|=$ $=\infty$ in einer Untermenge von E mit positivem Maß erfüllt ist. Daraus, nach den Obigen folgt $\lim\limits_{\min(m,n)\to\infty} |\sigma_{mn}(a, \Phi; x, y)|=\infty$ in einer Untermenge von E mit positivem Maß.

Zum Beweis des Satzes I′ werden wir also voraussetzen, daß a_{kl} $(k, l=1, 2, \ldots)$ rationell sind, und (3), (14), (15) erfüllt werden. Aus (15), auf Grund von (3) erhalten wir

$$\sum_{p=r}^{\infty}\sum_{q=r}^{\infty} p^2q^2 2^{p+q} A^2_{2^{p+1}-1,2^{q+1}-1}=\infty \quad (r=0, 1, \ldots).$$

Darum gibt es eine Folge $(0)=r_0<\ldots<r_i<\ldots$ von ganzen Zahlen und eine monoton abnehmende, nach 0 strebende Folge $\{s_i\}_{i=0}^{\infty}$ von positiven Zahlen mit

$$\sum_{i=0}^{\infty} s_i^2 \sum_{p=r_i}^{r_{i+1}-1}\sum_{q=r_i}^{r_{i+1}-1} p^2q^2 2^{p+q} A^2_{2^{p+1}-1,2^{q+1}-1}=\infty, \tag{18}$$

$$s_i^2 \sum_{p=r_i}^{r_{i+1}-1}\sum_{q=r_i}^{r_{i+1}-1} p^2q^2 2^{p+q} A^2_{2^{p+1}-1,2^{q+1}-1}\leqq 1 \quad (i=0, 1, \ldots). \tag{19}$$

Für jedes $i=(0, 1, \ldots)$ werden wir einfache Mengen $H_{pq}^{(i)}(\subseteq E)$ $(p, q=r_i, \cdots, r_{i+1})$ und ein in E orthonormiertes System $f_{kl}^{(i)}(x,y)$ $(2^{r_i}\leqq k, l<2^{r_{i+1}})$ von Treppenfunktionen derart angeben, daß die folgenden Bedingungen erfüllt sind:

$$H_{pq}^{(i)}\cap H_{p'q'}^{(i)}=\emptyset \quad ((p,q)\neq(p',q');\ r_i\leqq p, q, p', q'<r_{i+1}), \tag{20}$$

$$\operatorname{mes} H_{pq}^{(i)} \geqq c_9 p^2 q^2 2^{p+q} A^2_{2^{p+1}-1,2^{q+1}-1} s_i^2 \quad (r_i\leqq p, q<r_{i+1}), \tag{21}$$

$$f_{kl}^{(i)}(x,y)=0 \quad ((x,y)\in H_{pq}^{(i)};\ (k,l)\notin[2^p, 2^{p+1})\times[2^q, 2^{q+1});\ r_i\leqq p; q<r_{i+1}), \tag{22}$$

weiterhin für jedes p, q $(r_i \leqq p, q < r_{i+1})$ und $(x, y) \in H_{pq}^{(i)}$ gibt es Indizes $m(x, y)$, $n(x, y)$ $\left(2^p < m(x, y) < 2^{p+1},\ 2^q < n(x, y) < 2^{q+1}\right)$ derart, daß

$$f_{kl}^{(i)}(x, y) \geqq 0 \quad \left((x, y) \in H_{pq}^{(i)};\ 2^p \leqq k \leqq m(x, y),\ 2^q \leqq l \leqq n(x, y)\right) \tag{23}$$

und

$$\sum_{k=2^p}^{m(x,y)-1} \sum_{l=2^q}^{n(x,y)-1} A_{kl} f_{kl}^{(i)}(x, y) \geqq c_{10}/s_i \quad \left((x, y) \in H_{pq}^{(i)}\right) \tag{24}$$

bestehen.

Es sei $i(=0, 1, \ldots)$ eine ganze Zahl, und seien $T_{pq}^{(i)}$ $(r_i \leqq p, q < r_{i+1})$ paarweise disjunkte Rechtecke in E mit

$$\operatorname{mes} T_{pq}^{(i)} = p^2 q^2 2^{p+q} A^2_{2^{p+1}-1,\, 2^{q+1}-1} \quad (r_i \leqq p,\ q < r_{i+1}).$$

Solche Rechtecke existieren wegen (19). Sind p, q ganze Zahlen $(r_i \leqq p, q < r_{i+1})$, dann wenden wir den Hilfssatz IV im Falle 2^{p-1}, bzw. im Falle 2^{q-1} an. Dann seien

$$\bar{f}_{kl}^{(i)}(x, y) = f_{k-2^p+1}(2^{p-1};\ x) \cdot f_{l-2^q+1}(2^{q-1};\ y)$$
$$\left((x, y) \in E;\quad 2^p \leqq k < 2^{p+1},\quad 2^q \leqq l < 2^{q+1}\right),$$
$$\bar{H}_{pq}^{(i)} = E(2^{p-1}) \times E(2^{q-1}),$$

und wir setzen

$$f_{kl}^{(i)}(x, y) = \left(1/\sqrt{\operatorname{mes} T_{pq}^{(i)}}\right) \bar{f}_{kl}^{(i)}(T_{pq}^{(i)};\ x, y) \quad (2^p \leqq k < 2^{p+1},\ 2^q \leqq l < 2^{q+1}),$$
$$H_{pq}^{(i)} = \bar{H}_{pq}^{(i)}(T_{pq}^{(i)}).$$

Aus dem Hilfssatz IV folgt, daß für diese Mengen und Funktionen (20)—(24) bei jedem $i(=0, 1, \ldots)$ erfüllt sind.

Für jede ganze Zahl $i(=0, 1, \ldots)$ werden wir dann die einfachen Mengen $F_{pq}^{(i)} (\subseteqq E)$ $(r_i \leqq p, q < r_{i+1})$ und ein in E orthonormiertes System der Treppenfunktionen $g_{\alpha\beta}^{(i)}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}})$ derart angeben, daß die folgenden Bedingungen erfüllt sind:

$$F_{pq}^{(i)} \cap F_{p'q'}^{(i)} = \emptyset \quad \left((p, q) \neq (p', q');\ r_i \leqq p, q, p', q' < r_{i+1}\right), \tag{25}$$

$$\operatorname{mes} F_{pq}^{(i)} \geqq c_{11} p^2 q^2 2^{p+q} A^2_{2^{p+1}-1,\, 2^{q+1}-1} s_i^2 \quad (r_i \leqq p,\ q < r_{i+1}), \tag{26}$$

$$g_{\alpha\beta}^{(i)}(x, y) = 0 \quad \left((x, y) \in F_{pq}^{(i)};\ (\alpha, \beta) \notin (2^{2^p}, 2^{2^{p+1}}] \times (2^{2^q}, 2^{2^{q+1}}];\ r_i \leqq p, q < r_{i+1}\right), \tag{27}$$

weiterhin für jede p, q $(r_i \leqq p, q < r_{i+1})$, und $(x, y) \in F_{pq}^{(i)}$ gibt es Indizes $m(x, y)$, $n(x, y)$ $\left(2^p < m(x, y) < 2^{p+1},\ 2^q < n(x, y) < 2^{q+1}\right)$ derart, daß

$$a_{\alpha\beta}\, g_{\alpha\beta}^{(i)}(x, y) \geqq 0 \quad \left((x, y) \in F_{pq}^{(i)};\ 2^{2^p} < \alpha \leqq 2^{m(x,y)+1},\quad 2^{2^q} < \beta \leqq 2^{n(x,y)+1}\right), \tag{28}$$

und

$$\sum_{\alpha=2^{2^p}+1}^{2^{m(x,y)}} \sum_{\beta=2^{2^q}+1}^{2^{n(x,y)}} a_{\alpha\beta}\, g_{\alpha\beta}^{(i)}(x, y) \geqq c_{12}/s_i \quad \left((x, y) \in F_{pq}^{(i)}\right) \tag{29}$$

erfüllt werden.

Es sei i eine nichtnegative ganze Zahl. Da die Koeffizienten $a_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots)$ rationell sind, gibt es eine positive ganze Zahl Q_i derart, daß für jede ganzen Zahlen k, l $(2^{r_i} \leqq k, l < 2^{r_{i+1}})$

$$a_{\alpha\beta}^2 / A_{kl}^2 = P(\alpha, \beta;\ k, l)/Q_i \quad (2^k < \alpha \leqq 2^{k+1},\ 2^l < \beta \leqq 2^{l+1})$$

mit gewissen ganzen Zahlen $P(\alpha, \beta; k, l)$ erfüllt ist.

Es sei T_u $(u = 1, \ldots, Q_i)$ eine Einteilung von E in paarweise disjunkte Rechtecke mit

$$\operatorname{mes} T_u = 1/Q_i \quad (u = 1, \ldots, Q_i).$$

Sind k, l positive ganze Zahlen $(2^{r_i} \leqq k < 2^{r_{i+1}},\ 2^{r_i} \leqq l < 2^{r_{i+1}})$ und α, β ganze Zahlen mit $2^k < \alpha \leqq 2^{k+1}$, $2^l < \beta \leqq 2^{l+1}$, dann sei

$$g_{\alpha\beta}^{(i)}(x, y) = (A_{kl}/a_{\alpha\beta}) \sum_{u=M(\alpha,\beta)+1}^{N(\alpha,\beta)} f_{kl}^{(i)}(T_u;\ x, y),$$

wobei

$$M(\alpha, \beta) = \sum_{a=2^k+1}^{2^{k+1}} \sum_{b=2^l+1}^{\beta-1} P(a, b;\ k, l) + \sum_{a=2^k+1}^{\alpha-1} P(a, \beta;\ k, l),$$

$$N(\alpha, \beta) = \sum_{a=2^k+1}^{2^{k+1}} \sum_{b=2^l+1}^{\beta-1} P(a, b;\ k, l) + \sum_{a=2^k+1}^{\alpha} P(a, \beta;\ k, l),$$

und sei

$$F_{pq}^{(i)} = \bigcup_{u=1}^{Q_i} H_{pq}^{(i)}(T_u) \quad (r_i \leqq p,\ q < r_{i+1}).$$

Auf Grund von (20)—(24) sind (25)—(29) für jedes $i(=0, 1, \ldots)$ erfüllt.

Endlich werden wir eine Folge von einfachen und stochastisch unabhängigen Mengen $G_i(\subseteqq E)$ $(i = 0, 1, \ldots)$ und ein orthonormiertes System von Treppenfunktionen $\Psi_{\alpha\beta}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}};\ i = 0, 1, \ldots)$ in $(0, 2)\times(0, 2)$ mit folgenden Eigenschaften definieren. Für jedes $i(=0, 1, \ldots)$ gelten

$$\operatorname{mes} G_i = \sum_{p=r_i}^{r_{i+1}-1} \sum_{q=r_i}^{r_{i+1}-1} \operatorname{mes} F_{pq}^{(i)}, \tag{30}$$

$$\Psi_{\alpha\beta}(x, y) = 0 \quad ((x, y) \notin E;\ 2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}}), \tag{31}$$

weiterhin für jedes $(x, y) \in G_i$ gibt es ganze Zahlen p, q $(r_i \leqq p, q < r_{i+1})$ und Indizes $m(x, y), n(x, y)$ $(2^p < m(x, y) < 2^{p+1},\ 2^q < n(x, y) < 2^{q+1})$ mit

$$a_{\alpha\beta}\Psi_{\alpha\beta}(x, y) \geqq 0, \quad \text{oder} \quad a_{\alpha\beta}\Psi_{\alpha\beta}(x, y) \leqq 0 \tag{32}$$
$$(2^{2^p} < \alpha \leqq 2^{m(x,y)+1},\quad 2^{2^q} < \beta \leqq 2^{n(x,y)+1}),$$

$$\Bigl|\sum_{\alpha=2^{2^p}+1}^{2^{m(x,y)}} \sum_{\beta=2^{2^q}+1}^{2^{n(x,y)}} a_{\alpha\beta}\Psi_{\alpha\beta}(x, y)\Bigr| \geqq c_{12}/s_i, \tag{33}$$

$$\Psi_{\alpha\beta}(x, y) = 0 \quad ((x, y) \in G_i;\ 2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}};\ (\alpha, \beta) \notin (2^{2^p}, 2^{2^{p+1}}] \times (2^{2^q}, 2^{2^{q+1}}]). \tag{34}$$

Die Mengen G_i und die Funktionen $\bar{\psi}_{\alpha\beta}(x, y)$ definieren wir durch Induktion. Es sei

$$G_0 = \sum_{p=r_0}^{r_1-1} \sum_{q=r_0}^{r_1-1} F_{pq}^{(0)},$$

und

$$\bar{\psi}_{\alpha\beta}(x,y) = \begin{cases} g_{\alpha\beta}^{(0)}(x, y), & (x, y)\in E, \\ 0, & (x, y)\in(0, 2)\times(0, 2)\setminus E \end{cases}$$

$(2^{2^{r_0}} < \alpha, \beta \leqq 2^{2^{r_1}})$.

Es sei i_0 eine nichtnegative ganze Zahl. Wir nehmen an, daß die einfachen und stochastisch unabhängigen Mengen $G_i(\subseteqq E)$ $(i=0, \dots, i_0)$ und die in $(0, 2)\times(0, 2)$ orthonormierten Treppenfunktionen $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}}; i=0, \dots, i_0)$ schon derart definiert sind, daß (30)—(34) im Falle $i=0, \dots, i_0$ erfüllt sind.

Dann gibt es eine Einteilung von E in paarweise disjunkte Rechtecke T_s $(s=1, \dots, \sigma)$ derart, daß jede Funktion $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}}; i=0, \dots, i_0)$ in jedem T_s $(s=1, \dots, \sigma)$ konstant ist, und jede Menge G_i $(i=0, \dots, i_0)$ die Vereinigung von gewissen T_s ist. Wir teilen T_s in zwei disjunkte Rechtecke T_s', T_s'' mit $\operatorname{mes} T_s' = \operatorname{mes} T_s''$ $(s=1, \dots, \sigma)$. Dann setzen wir

$$G_{i_0+1} = \bigcup_{s=1}^{\sigma} \Big(\bigcup_{p,q} \big(F_{pq}^{(i_0+1)}(T_s') \cup F_{pq}^{(i_0+1)}(T_s'')\big)\Big), \quad r_{i_0+1} \leqq p, q < r_{i_0+2}$$

und

$$\bar{\psi}_{\alpha\beta}(x, y) = \begin{cases} \sum_{s=1}^{\sigma} g_{\alpha\beta}^{(i_0+1)}(T_s';\ x, y) - \sum_{s=1}^{\sigma} g_{\alpha\beta}^{(i_0+1)}(T_s'';\ x, y), & (x, y)\in E, \\ 0, \quad (x, y)\in(0, 2)\times(0, 2)\setminus E \end{cases}$$

$$(2^{2^{r_{i_0+1}}} < \alpha, \beta \leqq 2^{2^{r_{i_0+2}}}).$$

Es ist klar, daß die Menge $G_{i_0+1}(\subseteqq E)$ einfach ist, $G_0, \dots, G_{i_0+1}$ stochastisch unabhängig sind, die Funktionen $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_{i_0+1}}} < \alpha, \beta \leqq 2^{2^{r_{i_0+2}}})$ Treppenfunktionen sind, und die Funktionen $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}}; i=0, \dots, i_0+1)$ in $(0, 2)\times(0, 2)$ ein orthonormiertes System bilden. Weiterhin, auf Grund der Definition der Menge G_{i_0+1}, bzw. der Funktionen $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_{i_0+1}}} < \alpha, \beta \leqq 2^{2^{r_{i_0+2}}})$ und der Relationen (25)—(29) sind (30)—(34) auch im Falle $i=i_0+1$ erfüllt. Die Mengenfolge $\{G_i\}_{i=0}^{\infty}$ und die Funktionen $\bar{\psi}_{\alpha\beta}(x, y)$ $(2^{2^{r_i}} < \alpha, \beta \leqq 2^{2^{r_{i+1}}}; i=0, 1, \dots)$ mit den erforderten Eigenschaften erhalten wir durch Induktion.

Wir bemerken, daß auf Grund von (18) eine der Reihen

$$\sum_{i=0}^{\infty} s_{2i}^2 \sum_{p=r_{2i}}^{r_{2i+1}-1} \sum_{q=r_{2i}}^{r_{2i+1}-1} p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2,$$

$$\sum_{i=0}^{\infty} s_{2i+1}^2 \sum_{p=r_{2i+1}}^{r_{2i+2}-1} \sum_{p=r_{2i+1}}^{r_{2i+2}-1} p^2 q^2 2^{p+q} A_{2^{p+1}-1, 2^{q+1}-1}^2$$

divergiert. Wir nehmen an, daß

$$(35)\qquad \sum_{i=0}^{\infty} s_{2i}^2 \sum_{p=r_{2i}}^{r_{2i+1}-1} \sum_{q=r_{2i}}^{r_{2i+1}-1} p^2 q^2 2^{p+q} A_{2^{p+1}-1,\,2^{q+1}-1}^2 = \infty$$

ist. Den anderen Fall können wir nämlich ähnlicherweise betrachten.

Wir definieren ein Funktionensystem $\psi^* = \{\psi_{kl}^*(x, y)\}_{k,l=1}^{\infty}$ folgenderweise.

Es sei $N_2 = \{(\alpha, \beta)\colon \alpha, \beta = 1, 2, \ldots\}$ und es sei N die Menge der geordneten Paare (α, β) von positiven ganzen Zahlen, für die

$$(\alpha, \beta) \notin \bigcup_{i=0}^{\infty} (2^{2^{r_{2i}}}, 2^{2^{r_{2i+1}}}] \times (2^{2^{r_{2i}}}, 2^{2^{r_{2i+1}}}]$$

ist. Es seien weiterhin T_{uv} $((u, v) \in N)$ paarweise disjunkte Rechtecke in $(0, 2) \times \times (0, 2) \setminus E$. Wir setzen

$$\psi_{\alpha\beta}^*(x, y) = \bar{\psi}_{\alpha\beta}(x, y), \quad (x, y) \in (0, 2) \times (0, 2) \quad ((\alpha, \beta) \in N_2 \setminus N),$$

und

$$\psi_{\alpha\beta}^*(x, y) = \begin{cases} 1/\sqrt{\operatorname{mes} T_{uv}}, & (x, y) \in T_{uv}, \\ 0, & (x, y) \in (0, 2) \times (0, 2) \setminus T_{uv} \end{cases}$$

$((u, v) \in N)$.

Es ist klar, daß diese Funktionen Treppenfunktionen sind, und in $(0, 2) \times (0, 2)$ ein orthonormiertes System bilden.

Auf Grund von (26), (30) und (35) gilt

$$\sum_{i=0}^{\infty} \operatorname{mes} G_{2i} = \infty.$$

Es sei $G = \overline{\lim_{i \to \infty}}\, G_{2i}$. Da die Mengen G_{2i} stochastisch unabhängig sind, durch Anwendung des Borel—Cantellischen Lemmas erhalten wir daraus für die Menge $G (\subseteqq E)$

$$(36)\qquad \operatorname{mes} G = 1.$$

Auf Grund der Definition der Mengen G_{2i}, bzw. der Funktionen $\psi_{\alpha\beta}^*(x, y)$ und von (32)—(34) ergibt sich folgendes. Für jedes $i (= 0, 1, \ldots)$ gibt es im Falle $(x, y) \in G_{2i}$ Indizes $m_i(x, y)$, $n_i(x, y)$ $(2^{r_{2i}} < m_i(x, y), n_i(x, y) < 2^{r_{2i+1}})$ derart, daß

$$\left| \sum_{\alpha=2^{2^{r_{2i}}}+1}^{2^{m_i(x,y)+1}} \sum_{\beta=2^{2^{r_{2i}}}+1}^{2^{n_i(x,y)+1}} (1 - (\alpha-1)/2^{m_i(x,y)+1})(1 - (\beta-1)/2^{n_i(x,y)+1}) a_{\alpha\beta} \psi_{\alpha\beta}^*(x, y) \right| \geqq$$

$$\geqq c_{12}/s_{2i} \quad ((x, y) \in G_{2i}).$$

Für beliebiges $(x, y) \in G$, gilt diese Ungleichung für unendlich viele i, und so bekommen wir

(37)

$$\varlimsup_{i\to\infty} \max_{2^{2^{r}2i}<m,n\leqq 2^{2^{r}2i+1}} \left| \sum_{\alpha=2^{2^{r}2i}+1}^{m} \sum_{\beta=2^{2^{r}2i}+1}^{n} (1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\psi_{\alpha\beta}^{*}(x,y)\right| = \infty$$

$$((x,y)\in G).$$

Wir werden noch die Funktionen $\psi_{\alpha\beta}^{*}(x,y)$ ein wenig verändern.

Auf Grund von (14) gilt

$$\sum_{i=0}^{\infty}\int_0^2\int_0^2 \Big(\sum_{\alpha=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} \sum_{\beta=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} a_{\alpha\beta}\psi_{\alpha\beta}^{*}(x,y)\Big)^2 dx\,dy = \sum_{i=0}^{\infty}\Big(\sum_{\alpha=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} \sum_{\beta=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} a_{\alpha\beta}^2\Big) < \infty.$$

So konvergiert die Reihe

$$\sum_{i=0}^{\infty}\Big(\sum_{\alpha=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} \sum_{\beta=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} a_{\alpha\beta}\psi_{\alpha\beta}^{*}(x,y)\Big)^2$$

in $(0,2)\times(0,2)$ fast überall. Daraus, durch Anwendung des wohlbekannten Satzes von A. N. Kolmogoroff und J. Khintchine ergibt sich, daß die Reihe

(38)
$$\sum_{i=0}^{\infty} r_i(t)\Big(\sum_{\alpha=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} \sum_{\beta=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} a_{\alpha\beta}\psi_{\alpha\beta}^{*}(x,y)\Big)$$

bei fast jedem $(x,y)\in(0,2)\times(0,2)$ für fast jedes $t\in(0,1)$ konvergiert. Es sei H die Menge der Punkte (x,y,t) $((x,y)\in(0,2)\times(0,2);\ t\in(0,1))$, für welche die Reihe (38) konvergiert. Wir setzen

$$H_t = \{(x,y)\in(0,2)\times(0,2)\colon (x,y,t)\in H\} \quad (t\in(0,1)),$$
$$H_{xy} = \{t\in(0,1)\colon (x,y,t)\in H\} \quad ((x,y)\in(0,2)\times(0,2)).$$

Die charakteristische Funktion von H bezeichnen wir mit $\chi(x,y,t)$. Nach dem Obigen gilt

$$\int_0^2\int_0^2\Big(\int_0^1 \chi(x,y,t)\,dt\Big)dx\,dy = \int_0^2\int_0^2 (\operatorname{mes} H_{xy})\,dx\,dy = 1.$$

Nach dem Fubinischen Satz erhalten wir

$$\int_0^1 (\operatorname{mes} H_t)\,dt = \int_0^1\Big(\int_0^2\int_0^2 \chi(x,y,t)\,dx\,dy\Big)dt = \int_0^2\int_0^2\Big(\int_0^1 \chi(x,y,t)\,dt\Big)dx\,dy = 1,$$

woraus folgt, daß die Reihe (38) bei fast jedem $t\in(0,1)$ für fast jedes $(x,y)\in(0,2)\times\times(0,2)$ konvergiert. Es sei $t_0\in(0,1)$ ein Wert, für welchen die Reihe

$$\sum_{i=0}^{\infty} r_i(t_0)\Big(\sum_{\alpha=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} \sum_{\beta=2^{2^{r}2i}+1}^{2^{2^{r}2i+1}} a_{\alpha\beta}\psi_{\alpha\beta}^{*}(x,y)\Big)$$

in $(0, 2)\times(0, 2)$ fast überall konvergiert; man kann annehmen, daß t_0 nicht dyadisch ist.

Dann setzen wir in $(0, 2)\times(0, 2)$

$$\tilde{\psi}_{\alpha\beta}(x, y) = \begin{cases} r_i(t_0)\psi^*_{\alpha\beta}(x, y), & 2^{2^{r_{2i}}} < \alpha, \beta \leqq 2^{2^{r_{2i+1}}}; \quad i = 0, 1, \ldots, \\ \psi^*_{\alpha\beta}(x, y), & (\alpha, \beta)\in N. \end{cases}$$

Es ist klar, daß die Funktionen $\tilde{\psi}_{\alpha\beta}(x, y)$ $(\alpha, \beta = 1, 2, \ldots)$ Treppenfunktionen sind und in $(0, 2)\times(0, 2)$ ein orthonormiertes System bilden, und nach (37)

(39)
$$\varlimsup_{i\to\infty} \max_{2^{2^{r_{2i}}} < m, n \leqq 2^{2^{r_{2i+1}}}} \Big| \sum_{\alpha=2^{2^{r_{2i}}}+1}^{m} \sum_{\beta=2^{2^{r_{2i}}}+1}^{n} (1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x, y)\Big| = \infty$$
$$((x, y)\in G)$$

gilt. Weiterhin, auf Grund der Definition der Funktionen $\tilde{\psi}_{\alpha\beta}(x, y)$ konvergiert die Reihe

(40)
$$\sum_{i=0}^{\infty} \Big(\sum_{\alpha=2^{2^{r_{2i}}}+1}^{2^{2^{r_{2i+1}}}} \sum_{\beta=2^{2^{r_{2i}}}+1}^{2^{2^{r_{2i+1}}}} a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x, y)\Big)$$

in $(0, 2)\times(0, 2)$ fast überall.

Für die positiven ganzen Zahlen m, n mit $(m, n)\in N_2\setminus N$ sei $i(m, n)$ diejenige nichtnegative ganze Zahl, für die

$$2^{2^{r_{2i(m,n)}}} < m, n \leqq 2^{2^{r_{2i(m,n)+1}}}$$

gilt. Auf Grund der Definition der Funktionen $\tilde{\psi}_{kl}(x, y)$ ist

(41)
$$\sigma_{mn}(a, \tilde{\psi};\ x, y) = \sum_{\alpha=2^{2^{r_{2i(m,n)}}}+1}^{m} \sum_{\beta=2^{2^{r_{2i(m,n)}}}+1}^{n} (1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x, y) +$$
$$+ \sum_{j=0}^{i(m,n)-1} \Big(\sum_{\alpha=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} \sum_{\beta=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} (1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x, y)\Big) \quad ((x, y)\in E)$$

im Falle $(m, n)\in N_2\setminus N$. Es sei

(42)
$$R(m, n;\ x, y) = \sum_{j=0}^{i(m,n)-1} \sum_{\alpha=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} \sum_{\beta=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} (1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x, y).$$

Wir werden zeigen, daß

(43)
$$\Big\{ \lim_{\substack{\min(m,n)\to\infty \\ (m,n)\in N_2\setminus N}} R(m, n;\ x, y) \Big\}$$

in $(0,2)\times(0,2)$ fast überall existiert. Wir setzen für $j(=0,1,\ldots)$

$$s_{kl}(j;\ x,y)=\sum_{\alpha=2^{2^{r_{2j}}}+1}^{k}\ \sum_{\beta=2^{2^{r_{2j}}}+1}^{l} a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x,y)\quad (2^{2^{r_{2j}}}<k,\ l\leqq 2^{2^{r_{2j+1}}}).$$

Durch Abelsche Umformung bekommen wir

$$\sum_{\alpha=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}}\ \sum_{\beta=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}}(1-(\alpha-1)/m)(1-(\beta-1)/n)a_{\alpha\beta}\tilde{\psi}_{\alpha\beta}(x,y)=\tag{44}$$

$$=\frac{1}{mn}\sum_{k=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}\ \sum_{l=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{kl}(j;\ x,y)+\frac{1}{m}\sum_{k=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{k,2^{2^{r_{2j+1}}}}(j;\ x,y)+$$

$$+\frac{1}{n}\sum_{l=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{2^{2^{r_{2j+1}}},l}(j;\ x,y)+$$

$$+(1-(2^{2^{r_{2j+1}}}-1)/m)(1-(2^{2^{r_{2j+1}}}-1)/n)s_{2^{2^{r_{2j+1}}},2^{2^{r_{2j+1}}}}(j;\ x,y).$$

Wir setzen

$$R^{(1)}(m,n,j;\ x,y)=\frac{1}{mn}\sum_{k=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}\ \sum_{l=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{kl}(j;\ x,y)+$$

$$+\frac{1}{m}\sum_{k=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{k,2^{2^{r_{2j+1}}}}(j;\ x,y)+\frac{1}{n}\sum_{l=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}-1}s_{2^{2^{r_{2j+1}}},l}(j;\ x,y),$$

$$R^{(2)}(m,n,j;\ x,y)=(1-(2^{2^{r_{2j+1}}}-1)/m)(1-(2^{2^{r_{2j+1}}}-1)/n)s_{2^{2^{r_{2j+1}}},2^{2^{r_{2j+1}}}}(j;\ x,y).$$

Daraus und aus (42), (44) ergibt sich

$$R(m,n;\ x,y)=\sum_{j=0}^{i(m,n)-1}R^{(1)}(m,n,j;\ x,y)+\sum_{j=0}^{i(m,n)-1}R^{(2)}(m,n,j;\ x,y)\tag{45}$$

im Falle $(m,n)\in N_2\setminus N$. Es sei

$$\delta_j(x,y)=\max_{2^{2^{r_{2j}}}<k,\,l\leqq 2^{2^{r_{2j+1}}}}|s_{kl}(j;\ x,y)|\quad (j=0,1,\ldots).$$

Auf Grund der Definition von $R^{(1)}(m,n,j;x,y)$ ergibt sich

$$\left|\sum_{j=0}^{i-1}R^{(1)}(m,n,j,x,y)\right|\leqq\sum_{j=0}^{i-1}\left(\frac{(2^{2^{r_{2j+1}}})^2}{mn}+\frac{2^{2^{r_{2j+1}}}}{m}+\frac{2^{2^{r_{2j+1}}}}{n}\right)\delta_j(x,y)\quad (i=1,2,\ldots).$$

Daraus folgt für jedes $i(=1, 2, \ldots)$

$$\tilde{\delta}_i(x, y) = \max_{2^{2^{r_{2i}}} < m, n \leqq 2^{2^{r_{2i+1}}}} \left|\sum_{j=0}^{i-1} R^{(1)}(m, n, j;\ x, y)\right| \leqq$$

(46)
$$\leqq \sum_{j=0}^{i-1} \left[\left(\frac{2^{2^{r_{2j+1}}}}{2^{2^{r_{2i}}}}\right)^2 + 2\frac{2^{2^{r_{2j+1}}}}{2^{2^{r_{2i}}}}\right] \delta_j(x, y) \leqq c_{13} \frac{1}{2^{2^{r_{2i-1}}}} (\delta_0(x, y) + \ldots + \delta_{i-1}(x, y)).$$

Nach dem Hilfssatz II gilt

(47)
$$\int_0^2 \int_0^2 \delta_j^2(x, y)\, dx\, dy \leqq c_2 2^{4r_{2j+1}} \sum_{\alpha=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} \sum_{\beta=2^{2^{r_{2j}}}+1}^{2^{2^{r_{2j+1}}}} a_{\alpha\beta}^2 \quad (j = 0, 1, \ldots).$$

Auf Grund von (14) und (46) erhalten wir daraus

$$\sum_{i=1}^{\infty} \int_0^2 \int_0^2 \tilde{\delta}_i^2(x, y)\, dx\, dy \leqq c_{14} \sum_{i=1}^{\infty} i \frac{2^{4r_{2i-1}}}{2^{2^{r_{2i-1}}}} < \infty.$$

Folglich ist die Reihe

$$\sum_{i=1}^{\infty} \tilde{\delta}_i^2(x, y)$$

in $(0, 2)\times(0, 2)$ fast überall konvergent, und so gilt

(48)
$$\lim_{i\to\infty} \max_{2^{2^{r_{2i}}} < m, n \leqq 2^{2^{r_{2i+1}}}} \left|\sum_{j=0}^{i-1} R^{(1)}(m, n, j;\ x, y)\right| = 0$$

in $(0, 2)\times(0, 2)$ fast überall.

Auf Grund der Definition von $R^{(2)}(m, n, j; x, y)$ gilt für jedes $i(=1, 2, \ldots)$

$$\sum_{j=0}^{i-1} R^{(2)}(m, n, j;\ x, y) = \sum_{j=0}^{i-1} s_{2^{2^{r_{2j+1}}}, 2^{2^{r_{2j+1}}}}(j;\ x, y) +$$

$$+ \sum_{j=0}^{i-1} \left(\frac{(2^{2^{r_{2j+1}}} - 1)^2}{mn} - \frac{2^{2^{r_{2j+1}}} - 1}{m} - \frac{2^{2^{r_{2j+1}}} - 1}{n}\right) s_{2^{2^{r_{2j+1}}}, 2^{2^{r_{2j+1}}}}(j;\ x, y).$$

Wir setzen

$$F(i;\ x, y) = \sum_{j=0}^{i-1} s_{2^{2^{r_{2j+1}}}, 2^{2^{r_{2j+1}}}}(j;\ x, y),$$

$$G(m, n, i;\ x, y) =$$

$$= \sum_{j=0}^{i-1} \left(\frac{(2^{2^{r_{2j+1}}} - 1)^2}{mn} - \frac{2^{2^{r_{2j+1}}} - 1}{m} - \frac{2^{2^{r_{2j+1}}} - 1}{n}\right) s_{2^{2^{r_{2j+1}}}, 2^{2^{r_{2j+1}}}}(j;\ x, y).$$

Dann ist für jedes $i(=1, 2, \ldots)$

(49) $$\sum_{j=0}^{i-1} R^{(2)}(m, n, j;\ x, y) = F(i;\ x, y) + G(m, n, i;\ x, y).$$

Da die Reihe (40) in $(0, 2)\times(0, 2)$ fast überall konvergiert, auf Grund der Definition von $F(i;\ x, y)$ erhalten wir, daß

(50) $$\lim_{i\to\infty} F(i;\ x, y)$$

in $(0, 2)\times(0, 2)$ fast überall existiert. Andererseits, auf Grund der Definition von $G(m, n, i;\ x, y)$ ergibt sich für jedes $i(=1, 2, \ldots)$

$$\tilde{\tilde{\delta}}_i(x, y) = \max_{2^{2^{r_{2i}}}<m, n\leq 2^{2^{r_{2i+1}}}} |G(m, n, i;\ x, y)| \leq$$

$$\leq \sum_{j=0}^{i-1}\left(\left(\frac{2^{2^{r_{2j+1}}}}{2^{2^{r_{2i+1}}}}\right)^2 + 2\frac{2^{2^{r_{2j+1}}}}{2^{2^{r_{2i+1}}}}\right)\delta_j(x, y) \leq \frac{1}{2^{2^{r_{2i}-1}}}(\delta_0(x, y) + \ldots + \delta_{i-1}(x, y)).$$

Daraus und aus (47) folgt

$$\sum_{i=1}^{\infty}\int_0^2\int_0^2 \tilde{\tilde{\delta}}_i^2(x, y)\, dx\, dy \leq c_{15}\sum_{i=1}^{\infty} i\frac{2^{4r_{2i-1}}}{2^{2^{r_{2i}}}} < \infty,$$

auf Grund von (14). Daraus ergibt sich, daß die Folge

$$\sum_{i=1}^{\infty}\tilde{\tilde{\delta}}_i^2(x, y)$$

in $(0, 2)\times(0, 2)$ fast überall konvergiert, und so gilt

(51) $$\lim_{i\to\infty}\max_{2^{2^{r_{2i}}}<m, n\leq 2^{2^{r_{2i+1}}}} |G(m, n, i;\ x, y)| = 0$$

in $(0, 2)\times(0, 2)$ fast überall.

Da nach (45) und (49) im Falle $(m, n)\in N_2\setminus N$

$$R(m, n;\ x, y) = \sum_{j=0}^{i(m,n)-1} R^{(1)}(m, n, j;\ x, y) + F(i(m, n);\ x, y) + G(m, n, i(m, n);\ x, y)$$

gilt, aus (48), (50) und (51) folgt, daß der Limes (43) in $(0, 2)\times(0, 2)$ fast überall existiert. Daraus, und aus (39), (41) ergibt sich, daß

$$\varlimsup_{\min(m,n)\to\infty} |\sigma_{mn}(a, \tilde{\psi};\ x, y)| = \infty$$

im Falle $(x, y)\in G$ besteht. Wegen (36) besteht diese Relation in E fast überall.

Wir setzen endlich

$$\psi_{kl}(x, y) = 2\tilde{\psi}_{kl}(2x, 2y) \quad ((x, y)\in E;\ k, l = 1, 2, \ldots).$$

Dieses System $\psi = \{\psi_{kl}(x, y)\}_{k,l=1}^{\infty}$ befriedigt also alle Erforderungen des Satzes I'.

Damit haben wir Satz I' bewiesen.

4. Beweis des Satzes II. Zum Beweis des Satzes können wir ohne Beschränkung der Allgemenheit $\lambda_0, \mu_0 \geqq 1$ voraussetzen.

1. Beweis des Falles 1. Durch vollständige Induktion können wir eine Folge $(2=)M_1 < \ldots < M_r < \ldots$ von ganzen Zahlen angeben, für die

$$c_5 (\log M_{r+1})/(r+1)\lambda_{2^{2M_{r+1}^2}+1}\mu_{2^{r+2}} \geqq \max\left(r+1, 33c_4 \sum_{\varrho=1}^{r} 2M_\varrho^2\right) \quad (r=1, 2, \ldots) \tag{52}$$

gilt.

Die Koeffizientenfolge a definieren wir folgenderweise. Es sei

$$a_{kl} = \begin{cases} 1/rM_r\lambda_{2^{2M_r^2}+1}\mu_{2^r+1}, & l = 2^r+1, \ k = 2^1+1, \ldots, 2^{2M_r^2}+1, \qquad r = 1, 2, \ldots, \\ -1/rM_r\lambda_{2^{2M_r^2}+1}\mu_{2^{r+1}}, & l = 2^{r+1}-1, \ k = 2^2-1, \ldots, 2^{2^{2M_r^2+1}}-1, \quad r = 1, 2, \ldots, \\ 0, \text{ sonst.} \end{cases}$$

Es ist klar, daß

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}^2\lambda_k^2\mu_l^2 =$$

$$= \sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2} a_{2^s+1,2^r+1}^2 \lambda_{2^s+1}^2 \mu_{2^r+1}^2 + \sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2} a_{2^{s+1}-1,2^{r+1}-1}\lambda_{2^{s+1}-1}^2\mu_{2^{r+1}-1} \leqq$$

$$\leqq 2\sum_{r=1}^{\infty} \lambda_{2^{2M_r^2}+1}^2 \mu_{2^r+1}^2 2M_r^2/r^2M_r^2\lambda_{2^{2M_r^2}+1}^2\mu_{2^r+1}^2 = 4\sum_{r=1}^{\infty} 1/r^2 < \infty$$

gilt, also besteht (5).

Jetzt werden wir das entsprechende Funktionensystem Φ definieren.

Für jede positive ganze Zahl r wenden wir den Hilfssatz III im Falle $p = M_r$ an. Die so erhaltene Menge, bzw. das so erhaltene Funktionensystem bezeichnen wir mit $F(r)$, bzw. mit $\{g_l(r; x)\}_{l=1}^{2M_r^2}$ $(r=1, 2, \ldots)$. Auf Grund des Hilfssatzes III gelten die folgenden Behauptungen für jedes $r = 1, 2, \ldots$.

$F(r)(\subseteqq (0, 1))$ ist einfach, und gilt

$$\text{mes } F(r) \geqq c_3. \tag{53}$$

Die Funktionen $g_l(r; x)$ sind Treppenfunktionen und bilden in $(0, 1)$ ein beschränktes System:

$$|g_l(r; x)| \leqq c_4 \quad (x \in (0, 1);\ l = 1, \ldots, 2M_r^2). \tag{54}$$

Weiterhin, für jedes $x \in F(r)$ gibt es einen Index $m(r; x)$ mit

$$\begin{aligned} &\text{(i)} && M_r^2 \leqq m(r; x) < 2M_r^2, \\ (55)\quad &\text{(ii)} && g_l(r; x) \geqq 0 \quad (x \in F(r);\ r = 1, \ldots, m(r; x)), \\ &\text{(iii)} && \sum_{l=1}^{m(r;x)} g_l(r; x) \geqq c_5 M_r \log M_r \quad (x \in E(r)). \end{aligned}$$

Es sei I_{kl} $(k, l=1, 2, \ldots)$ eine Einteilung des Intervalls $(1, 2)$ in paarweise disjunkte Intervalle.

Wir definieren Treppenfunktionen

$$\varphi_{2^s+1,2^r+1}(x),\ \varphi_{2^{s+1}-1,2^{r+1}-1}(x)\quad (s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots)$$

und Mengen $H_1, H_2, \ldots$ mit den folgenden Eigenschaften:

Die Mengen $H_r(\subseteqq(0, 1))$ sind einfach und stochastisch unabhängig, weiterhin für jedes r gilt

$$\text{(56)}\qquad \operatorname{mes} H_r \geqq c_3.$$

Die Funktionen bilden ein orthonormiertes System in $(0, 2)$, weiterhin für jedes r besteht

$$\text{(57)}\qquad |\varphi_{2^s+1,2^r+1}(x)|,\ |\varphi_{2^{s+1}-1,2^{r+1}-1}(x)| \leqq c_4/\sqrt{2}\quad (x\in(0, 1);\ s=1, \ldots, 2M_r^2)$$

und für jedes $x\in H_r$ gibt es einen Index $m(r; x)$ mit

$$\text{(58)}\qquad \begin{aligned} &\text{(i)} && M_r^2 \leqq m(r;\ x) < 2M_r^2,\\ &\text{(ii)} && \varphi_{2^s+1,2^r+1}(x) \geqq 0\quad (x\in H_r;\ s=1, \ldots, m(r;\ x)),\\ &\text{(iii)} && \sum_{s=1}^{m(r;x)} \varphi_{2^s+1,2^r+1}(x) \geqq (c_5/\sqrt{2})M_r \log M_r\quad (x\in H_r). \end{aligned}$$

Weiterhin, für jedes r gelten

$$\text{(59)}\qquad \varphi_{2^s+1,2^r+1}(x) \equiv \varphi_{2^{s+1}-1,2^{r+1}-1}(x)\quad (x\in(0, 1);\ s=1, \ldots, 2M_r^2),$$

(60)

$$\varphi_{2^s+1,2^r+1}(x) \equiv \varphi_{2^{s+1}-1,2^{r+1}-1}(x) \equiv 0\ \left(x\in(1, 2)\setminus(I_{2^s+1,2^r+1}\cup I_{2^{s+1}-1,2^{r+1}-1})\right)$$
$$(s=1, \ldots, 2M_r^2).$$

Wir setzen nämlich

$$\varphi_{2^s+1,2^1+1}(x) = \begin{cases} g_s(1;\ x)/\sqrt{2}, & x\in(0, 1),\\ 1/2\sqrt{\operatorname{mes} I_{2^s+1,2^1+1}}, & x\in I_{2^s+1,2^1+1},\\ 1/2\sqrt{\operatorname{mes} I_{2^{s+1}-1,2^2-1}}, & x\in I_{2^{s+1}-1,2^2-1},\\ 0, & \text{sonst}, \end{cases}$$

$$\varphi_{2^{s+1}-1,2^2-1}(x) = \begin{cases} g_s(1;\ x)/\sqrt{2}, & x\in(0, 1),\\ -1/2\sqrt{\operatorname{mes} I_{2^s+1,2^1+1}}, & x\in I_{2^s+1,2^1+1},\\ -1/2\sqrt{\operatorname{mes} I_{2^{s+1}-1,2^2-1}}, & x\in I_{2^{s+1},2^2-1},\\ 0, & \text{sonst} \end{cases}$$

$(s=1, \ldots, 2M_1^2)$, und $H_1=F(1)$.

Auf Grund der Eigenschaften von $F(1)$ und $g_s(1; x)$ ist $H_1(\subseteqq(0, 1))$ einfach, die Funktionen $\varphi_{2^s+1,2^1+1}(x)$, $\varphi_{2^{s+1}-1,2^2-1}(x)$ $(s=1, \ldots, 2M_1^2)$ sind Treppenfunktionen, und sie bilden ein orthonormiertes System in $(0, 1)$. Weiterhin, wegen der Definition dieser Menge, bzw. dieser Funktionen, aus (53)—(55) folgt, daß (56)—(60) im Falle $r=1$ erfüllt sind.

Es sei r_0 eine positive ganze Zahl. Wir nehmen an, daß die einfachen und stochastisch unabhängigen Mengen $H_r(\subseteqq(0, 1))$ $(r=1, \ldots, r_0)$, und in $(0, 2)$ orthonormierte Treppenfunktionen

$$\varphi_{2^s+1,2^r+1}(x),\ \varphi_{2^{s+1}-1,2^{r+1}-1}(x) \quad (s=1, \ldots, 2M_r^2;\ r=1, \ldots, r_0)$$

schon derart definiert sind, daß (56)—(60) für $r=1, \ldots, r_0$ erfüllt sind. Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle I_p $(p=1, \ldots, P)$ derart, daß jede Funktion

$$\varphi_{2^s+1,2^r+1}(x),\ \varphi_{2^{s+1}-1,2^{r+1}-1}(x) \quad (s=1, \ldots, 2M_r^2;\ r=1, \ldots, r_0)$$

in jedem I_p $(p=1, \ldots, P)$ konstant ist, und jede Menge H_r $(r=1, \ldots, r_0)$ die Vereinigung gewisser I_p ist. Die zwei Hälften von I_p bezeichnen wir mit I_p' und I_p'' $(p=1, \ldots, P)$.

Dann setzen wir

$$\varphi_{2^s+1,2^{r_0+1}+1}(x) =$$

$$= \begin{cases} (1/\sqrt{2})\left(\sum\limits_{p=1}^{P} g_s(r_0+1;\ I_p';\ x) - \sum\limits_{p=1}^{P} g_s(r_0+1;\ I_p'';\ x)\right), & x\in(0, 1), \\ 1/2\sqrt{\operatorname{mes} I_{2^s+1,2^{r_0+1}+1}}, & x\in I_{2^s+1,2^{r_0+1}}, \\ 1/2\sqrt{\operatorname{mes} I_{2^{s+1}-1,2^{r_0+2}-1}}, & x\in I_{2^{s+1}-1,2^{r_0+2}-1}, \\ 0, & \text{sonst}, \end{cases}$$

$$\varphi_{2^{s+1}-1,2^{r_0+2}-1}(x) =$$

$$= \begin{cases} (1/\sqrt{2})\left(\sum\limits_{p=1}^{P} g_s(r_0+1;\ I_p';\ x) - \sum\limits_{p=1}^{P} g_s(r_0+1;\ I_p'';\ x)\right), & x\in(0, 1), \\ -1/2\sqrt{\operatorname{mes} I_{2^s+1,2^{r_0+1}+1}}, & x\in I_{2^s+1,2^{r_0+1}+1}, \\ -1/2\sqrt{\operatorname{mes} I_{2^{s+1}-1,2^{r_0+2}-1}}, & x\in I_{2^s+1-1,2^{r_0+2}-1}, \\ 0, & \text{sonst} \end{cases}$$

$(s=1, \ldots, 2M_{r_0+1}^2)$, und

$$H_{r_0+1} = \bigcup_{p=1}^{P} \left(F(r_0+1;\ I_p') \cup F(r_0+1;\ I_p'')\right).$$

Es ist klar, daß die Menge $H_{r_0+1}(\subseteqq(0, 1))$ einfach ist, die Funktionen Treppenfunktionen sind, die Mengen $H_1, \ldots, H_{r_0+1}$ stochastisch unabhängig sind, die Funktionen $\varphi_{2^s+1,2^r+1}(x)$, $\varphi_{2^s+1-1,2^{r+1}-1}(x)$ $(s=1, \ldots, 2M_r^2;\ r=1, \ldots, r_0+1)$ in $(0, 1)$ ein orthonormiertes System bilden, und daß, weiterhin, auf Grund der

Definition von H_{r_0+1}, bzw. von den Funktionen $\varphi_{2^s+1, 2^{r_0+1}+1}(x), \varphi_{2^{s+1}-1, 2^{r_0+2}-1}(x)$ $(s=1, \ldots, 2M^2_{r_0+1})$, wegen (53)—(55), die Bedingungen (56)—(60) im Falle $r=r_0+1$ auch erfüllt sind.

Die Mengenfolge $\{H_r\}_{r=1}^{\infty}$ und die Funktionen $\varphi_{2^s+1, 2^r+1}(x)$, $\varphi_{2^{s+1}-1, 2^{r+1}-1}(x)$ $(s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots)$ mit den erforderten Eigenschaften erhalten wir also durch Induktion.

Endlich sei

$$\varphi_{kl}(x)=\begin{cases} 1/\sqrt{\operatorname{mes} I_{kl}}, & x\in I_{kl}, \\ 0, & \text{sonst} \end{cases}$$

im Falle $(k, l)\neq(2^s+1, 2^r+1)$, $(2^{s+1}-1, 2^{r+1}-1)$ $(s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots)$. Es ist klar, daß das System $\varphi=\{\varphi_{kl}(x)\}_{k,l=1}^{\infty}$ orthonormiert in $(0, 2)$ ist.

Wegen der Definition der Folge a und wegen (59) gilt

$$s_{2^m, 2^n}(a, \varphi;\ x)\equiv 0\quad (x\in(0, 1);\ m, n=0, 1, \ldots).$$

Weiterhin, wegen der Definition der Funktionen $\varphi_{kl}(x)$ für jedes $x\in(1, 2)$ gibt es eine Zahl $m(x)$ derart, daß im Falle $m, n\geqq m(x)$

$$s_{2^m, 2^n}(a, \varphi;\ x)=$$

$$=\begin{cases} a_{kl}\varphi_{kl}(x), & x\in I_{kl}, \\ & (k, l)\neq(2^s+1, 2^r+1),\ (2^{s+1}-1, 2^{r+1}-1), \\ & s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots, \\ a_{2^s+1, 2^r+1}\varphi_{2^s+1, 2^r+1}(x)+a_{2^{s+1}-1, 2^{r+1}-1}\varphi_{2^{s+1}-1, 2^{r+1}-1}(x), & \\ & x\in I_{2^s+1, 2^r+1}\cup I_{2^{s+1}-1, 2^{r+1}-1}, \\ & s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots \end{cases}$$

ist. So folgt, daß $\lim\limits_{\min(m,n)\to\infty} s_{2^m, 2^n}(a, \varphi;x)$ in $(0, 2)$ überall existiert.

Es sei nun r_0 eine positive ganze Zahl, und $x\in H_{r_0+1}$. Es sei $m(x)=m(r_0+1;x)$ derjenige Index, für welchen (58) im Falle $r=r_0+1$ gilt. Dann folgt, daß auf Grund von (52), (57)—(59)

$$|\sigma_{2^{m(x)+1}, 2^{r_0+2}}(a, \varphi;\ x)|\geqq \tag{61}$$

$$\geqq\left|\sum_{s=1}^{m(x)}\left(1-\frac{2^s}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+1}}{2^{r_0+2}}\right)a_{2^s+1, 2^{r_0+1}+1}\varphi_{2^s+1, 2^{r_0+1}+1}(x)+\right.$$

$$\left.+\sum_{s=1}^{m(x)}\left(1-\frac{2^{s+1}-2}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+2}-2}{2^{r_0+2}}\right)a_{2^{s+1}-1, 2^{r_0+2}-1}\varphi_{2^{s+1}-1, 2^{r_0+2}-1}(x)\right|-$$

$$-\left|\sum_{r=1}^{r_0}\sum_{s=1}^{m(x)}\left(1-\frac{2^s}{2^{m(x)+1}}\right)\left(1-\frac{2^r}{2^{r_0+2}}\right)a_{2^s+1, 2^r+1}\varphi_{2^s+1, 2^r+1}(x)-\right.$$

$$\left.-\sum_{r=1}^{r_0}\sum_{s=1}^{m(x)}\left(1-\frac{2^{s+1}-2}{2^{m(x)+1}}\right)\left(1-\frac{2^{r+1}-2}{2^{r_0+2}}\right)a_{2^{s+1}-1, 2^{r+1}-1}\varphi_{2^{s+1}-1, 2^{r+1}-1}(x)\right|\geqq$$

$$\geqq \left|\sum_{s=1}^{m(x)} \left(\left(1-\frac{2^s}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+1}}{2^{r_0+2}}\right)-\right.\right.$$

$$\left.\left.-\left(1-\frac{2^{s+1}-2}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+2}-2}{2^{r_0+2}}\right)\right) a_{2^s+1,\,2^{r_0+1}+1}\varphi_{2^s+1,\,2^{r_0+1}+1}(x)\right| - 2\frac{1}{\sqrt{2}}\, c_4 \sum_{r=1}^{r_0} 2M_r^2 \geqq$$

$$\geqq \left|\sum_{s=1}^{m(x)-1} \left(\left(1-\frac{2^s}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+1}}{2^{r_0+2}}\right)-\right.\right.$$

$$\left.-\left(1-\frac{2^{s+1}-2}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+2}-2}{2^{r_0+2}}\right)\right) a_{2^s+1,\,2^{r_0+1}+1}\varphi_{2^s+1,\,2^{r_0+1}+1}(x)+$$

$$\left.+a_{2^{m(x)}+1;\,2^{r_0+1}+1}\,\varphi_{2^{m(x)}+1,\,2^{r_0+1}+1}(x)\right| - \left|\left(\left(1-\frac{2^{m(x)}}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+1}}{2^{r_0+2}}\right)-\right.\right.$$

$$\left.\left.-\left(1-\frac{2^{m(x)+1}-2}{2^{m(x)+1}}\right)\left(1-\frac{2^{r_0+2}-2}{2^{r_0+2}}\right)\right) a_{2^{m(x)}+1,\,2^{r_0+1}+1}\varphi_{2^{m(x)}+1,\,2^{r_0+1}+1}(x)\right| -$$

$$-\left|a_{2^{m(x)}+1,\,2^{r_0+1}+1}\varphi_{2^{m(x)}+1,\,2^{r_0+1}+1}(x)\right| - \sqrt{2}\, c_4 \sum_{r=1}^{r_0} 2M_r^2 \geqq$$

$$\geqq \frac{c_5}{8\sqrt{2}} M_{r_0+1}\log M_{r_0+1}\frac{1}{(r_0+1)M_{r_0+1}\lambda_{2^{2M^2_{r_0+1}+1}}\mu_{2^{r_0+2}}} - \sqrt{2}\,c_4 - \sqrt{2}\,c_4 \sum_{r=1}^{r_0} 2M_r^2 \geqq$$

$$\geqq \frac{c_5}{8\sqrt{2}}\frac{\log M_{r_0+1}}{(r_0+1)\lambda_{2^{2M^2_{r_0+1}+1}}\mu_{2^{r_0+2}}} - 2\sqrt{2}\, c_4 \sum_{r=1}^{r_0} 2M_r^2 \geqq$$

$$\geqq (1/8\sqrt{2})\left(\max\left(r_0+1, 33c_4\sum_{r=1}^{r_0} 2M_r^2\right) - 32c_4\sum_{r=1}^{r_0} 2M_r^2\right) \geqq (1/8\sqrt{2})(r_0+1),$$

wegen $|a_{kl}| \leqq 1$.

Es sei nun $H = \overline{\lim\limits_{r\to\infty}}\, H_r$. Da die Mengen H_r stochastisch unabhängig sind, und (56) für jedes r gilt, ergibt sich durch Anwendung des Borel—Cantellischen Lemmas:

$$\text{mes}\, H = 1.$$

Im Falle $x \in H$ gilt aber (61) für unendlich viele r mit $m(x) = m(r;\, x)$ $(\geqq M_r^2)$. So erhalten wir

$$\overline{\lim_{\min(m,n)\to\infty}}\, |\sigma_{2^m,\,2^n}(a,\, \varphi;\; x)| = \infty$$

fast überall in $(0, 1)$.

Es sei

$$\Phi_{kl}(x) = \sqrt{2}\,\varphi_{kl}(2x) \quad (x \in (0,\, 1);\; k,\, l = 1, 2, \ldots).$$

Für das System $\Phi = \{\Phi_{kl}(x)\}_{k,l=1}^{\infty}$ und für die Folge a sind aber alle Erforderungen des Falles 1 erfüllt.

2. Beweis des Falles 2. Durch vollständige Induktion werden wir zwei Folgen $(2\leqq)M_1<\ldots<M_r<\ldots$, $(1=)N_1<\ldots<N_r<\ldots$ von positiven ganzen Zahlen angeben, für die die Ungleichungen

$$r+1>(\log M_r)/r\lambda_{2^{2M_r^2}}\mu_{2^{N_r+1}}\geqq r, \tag{62}$$

$$(1/2^{N_r})\sum_{\varrho=1}^{r-1}M_\varrho^2\leqq 1/r \tag{63}$$

bei jedem $r=1,2,\ldots$ erfüllt sind.

Wir setzen nämlich $N_1=1$. Dann sei M_1 gleich der kleinsten ganzen Zahl $k(\geqq 2)$, für die

$$2>(\log k)/\lambda_{2^{2k^2}}\mu_{2+1}\geqq 1$$

gilt. Dann sind (62), (63) im Falle $r=1$ erfüllt.

Es sei r_0 eine positive ganze Zahl. Wir nehmen an, daß die ganzen Zahlen $(2\leqq)M_1<\ldots<M_{r_0}$, $(1=)N_1<\ldots<N_{r_0}$ schon derart definiert sind, daß (62), (63) im Falle $r=1,\ldots,r_0$ erfüllt sind. Dann sei N_{r_0+1} gleich der kleinsten ganzen Zahl $l(>N_{r_0})$, für die

$$(1/2^l)\sum_{\varrho=1}^{r_0}M_\varrho^2\leqq 1/(r_0+1)$$

ist. Weiterhin sei M_{r_0+1} gleich der kleinsten ganzen Zahl $k(>M_{r_0})$, für die

$$r_0+2>(\log k)/(r_0+1)\lambda_{2^{2k^2}}\mu_{2^{N_{r_0+1}}+1}\geqq r_0+1$$

ist. Für die ganzen Zahlen $(2=)M_1<\ldots<M_{r_0+1}$, $(1=)N_1<\ldots<N_{r_0+1}$ bestehen also (62), (63) auch im Falle $r=1,\ldots,r_0+1$. Die Zahlenfolgen $\{M_r\}_{r=1}^{\infty}$, $\{N_r\}_{r=1}^{\infty}$ mit den erforderten Eigenschaften erhalten wir durch Induktion.

Wir definieren die Koeffizientenfolge b folgenderweise. Es sei

$$b_{kl}=\begin{cases}1/rM_r\lambda_{2^{2M_r^2}}\mu_{2^{N_r}+1}, & l=2^{N_r};\ k=2^1,\ldots,2^{2M_r^2};\ r=1,2,\ldots,\\ -1/rM_r\lambda_{2^{2M_r^2}}\mu_{2^{N_r}+1}, & l=2^{N_r}+1;\ k=2^1,\ldots,2^{2M_r^2};\ r=1,2,\ldots,\\ 0, & \text{sonst.}\end{cases}$$

Es ist klar, daß

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}b_{kl}^2\lambda_k^2\mu_l^2=\sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2}b_{2^s,2^{N_r}}^2\lambda_{2^s}^2\mu_{2^{N_r}}^2+\sum_{r=1}^{\infty}\sum_{s=1}^{2M_r^2}b_{2^s,2^{N_r}+1}^2\lambda_{2^s}^2\mu_{2^{N_r}+1}^2\leqq$$

$$\leqq 2\sum_{r=1}^{\infty}\lambda_{2^{2M_r^2}}^2\mu_{2^{N_r}+1}^2 2M_r^2(1/r^2M_r^2\lambda_{2^{2M_r^2}}^2\mu_{2^{N_r}+1}^2)=4\sum_{r=1}^{\infty}1/r^2<\infty$$

gilt, also besteht (6).

Wir definieren das in $(0,2)$ orthonormierte System von Treppenfunktionen

$$\psi_{2^s,2^{N_r}}(x),\ \psi_{2^s,2^{N_r}+1}(x)\quad(s=1,\ldots,2M_r^2;\ r=1,2,\ldots)$$

und die Folge von einfachen und stochastisch unabhängigen Mengen $G_r(\subseteqq(0,1))$ $(r=1,2,\ldots)$ mit folgenden Eigenschaften.

Für jedes r gilt

$$\text{(64)} \qquad \operatorname{mes} G_r \geqq c_3,$$

und

$$\text{(65)} \qquad |\psi_{2^s,2^{N_r}}(x)|,\ |\psi_{2^s,2^{N_r+1}}(x)| \leqq c_4/\sqrt{2} \quad (x\in(0,1);\ s=1,\ldots,2M_r^2).$$

Für jedes $x\in G_r$ gibt es einen Index $m(r;x)$ mit

$$\text{(66)} \quad \begin{array}{ll} \text{(i)} & M_r^2 \leqq m(r;x) < 2M_r^2, \\ \text{(ii)} & \psi_{2^s,2^{N_r}}(x) \geqq 0 \quad (x\in G_r;\ s=1,\ldots,m(r;x)), \\ \text{(iii)} & \displaystyle\sum_{s=1}^{m(r;x)} \psi_{2^s,2^{N_r}}(x) \geqq (c_5/\sqrt{2})\, M_r \log M_r \quad (x\in G_r). \end{array}$$

Weiterhin besteht für jedes r

$$\text{(67)} \qquad \psi_{2^s,2^{N_r}}(x) \equiv \psi_{2^s,2^{N_r+1}}(x) \quad (x\in(0,1);\ s=1,\ldots,2M_r^2),$$

$$\text{(68)} \qquad \psi_{2^s,2^{N_r}}(x) \equiv \psi_{2^s,2^{N_r+1}}(x) \equiv 0 \quad (x\in(1,2)\setminus(I_{2^s,2^{N_r}}\cup I_{2^s,2^{N_r+1}}))$$
$$(s=1,\ldots,2M_r^2).$$

Wir setzen

$$\psi_{2^s,2^{N_1}}(x) = \begin{cases} g_s(1;\,x)/\sqrt{2}, & x\in(0,1), \\ 1/2\sqrt{\operatorname{mes} I_{2^s,2^{N_1}}}, & x\in I_{2^s,2^{N_1}}, \\ 1/2\sqrt{\operatorname{mes} I_{2^s,2^{N_1+1}}}, & x\in I_{2^s,2^{N_1+1}}, \\ 0, & \text{sonst}, \end{cases}$$

$$\psi_{2^s,2^{N_1+1}}(x) = \begin{cases} g_s(1;\,x)/\sqrt{2}, & x\in(0,1), \\ -1/2\sqrt{\operatorname{mes} I_{2^s,2^{N_1}}}, & x\in I_{2^s,2^{N_1}}, \\ -1/2\sqrt{\operatorname{mes} I_{2^s,2^{N_1+1}}}, & x\in I_{2^s,2^{N_1+1}}, \\ 0, & \text{sonst} \end{cases}$$

$(s=1,\ldots,2M_1^2)$, und $G_1=F(1)$.

Auf Grund der Eigenschaften von $F(1)$ und $g_s(1;x)$ ist $G_1(\subseteqq(0,1))$ einfach, diese Funktionen sind Treppenfunktionen, und sie bilden ein orthonormiertes System in $(0,2)$. Weiterhin, auf Grund der Definition von G_1, bzw. von diesen Funktionen, wegen (53)—(55) sind (64)—(68) im Falle $r=1$ erfüllt.

Es sei r_0 eine positive ganze Zahl. Wir nehmen an, daß die einfachen und stochastisch unabhängigen Mengen $G_r(\subseteqq(0,1))$ $(r=1,\ldots,r_0)$ und die orthonormierten Treppenfunktionen $\psi_{2^s,2^{N_r}}(x)$, $\psi_{2^s,2^{N_r+1}}(x)$ $(s=1,\ldots,2M_r^2;\ r=1,\ldots,r_0)$ in $(0,2)$ schon derart definiert sind, daß (64)—(68) im Falle $r=1,\ldots,r_0$ erfüllt werden.

Dann gibt es eine Einteilung des Intervalls $(0, 1)$ in paarweise disjunkte Intervalle J_q $(q=1, \ldots, Q)$ derart, daß jede Funktion

$$\psi_{2^s, 2^{N_r}}(x), \psi_{2^s, 2^{N_r}+1}(x) \quad (s=1, \ldots, 2M_r^2; r=1, \ldots, r_0)$$

in jedem Intervall J_q $(q=1, \ldots, Q)$ konstant ist, und jede Menge G_r $(r=1, \ldots, r_0)$ die Vereinigung von gewissen J_q ist. Die zwei Hälfte von J_q bezeichnen wir mit J_q', bzw. mit J_q'' $(q=1, \ldots, Q)$.

Dann setzen wir

$$\psi_{2^s, 2^{N_{r_0+1}}}(x) =$$

$$= \begin{cases} (1/\sqrt{2})\left(\sum_{q=1}^{Q} g_s(r_0+1;\ J_q';\ x) - \sum_{q=1}^{Q} g_s(r_0+1;\ J_q'';\ x)\right), & x \in (0, 1), \\ 1/2\sqrt{\operatorname{mes} I_{2^s, 2^{N_{r_0+1}}}}, & x \in I_{2^s, 2^{N_{r_0+1}}}, \\ 1/2\sqrt{\operatorname{mes} I_{2^s, 2^{N_{r_0+1}}+1}}, & x \in I_{2^s, 2^{N_{r_0+1}}+1}; \\ 0, & \text{sonst}, \end{cases}$$

und

$$\psi_{2^s, 2^{N_{r_0+1}}+1}(x) =$$

$$= \begin{cases} (1/\sqrt{2})\left(\sum_{q=1}^{Q} g_s(r_0+1;\ J_q';\ x) - \sum_{q=1}^{Q} g_s(r_0+1;\ J_q'';\ x)\right), & x \in (0, 1), \\ -1/2\sqrt{\operatorname{mes} I_{2^s, 2^{N_{r_0+1}}}}, & x \in I_{2^s, 2^{N_{r_0+1}}}, \\ -1/2\sqrt{\operatorname{mes} I_{2^s, 2^{N_{r_0+1}}+1}}, & x \in I_{2^s, 2^{N_{r_0+1}}+1}, \\ 0, & \text{sonst} \end{cases}$$

$(s = 1, \ldots, 2M_{r_0+1}^2)$, und

$$G_{r_0+1} = \bigcup_{q=1}^{Q} \left(F(r_0+1;\ J_q') \cup F(r_0+1;\ J_q'')\right).$$

Es ist klar, daß die Menge $G_{r_0+1}(\subseteq (0, 1))$ einfach ist, die Mengen G_r $(r=1, \ldots, r_0+1)$ stochastisch unabhängig sind, diese Funktionen Treppenfunktionen sind, und die Funktionen $\psi_{2^s, 2^{N_r}}(x)$, $\psi_{2^s, 2^{N_r}+1}(x)$ $(s=1, \ldots, 2M_r^2;\ r=1, \ldots, r_0+1)$ in $(0, 2)$ ein orthonormiertes System bilden. Weiterhin, nach der Definition der Menge G_{r_0+1} und der Funktionen $\psi_{2^s, 2^{N_{r_0+1}}}(x)$, $\psi_{2^s, 2^{N_{r_0+1}}+1}(x)$ $(s=1, \ldots, 2M_{r_0+1}^2)$ aus (53)—(55) folgt, daß (64)—(68) auch im Falle $r=r_0+1$ bestehen.

Die Mengenfolge $\{G_r\}_{r=1}^{\infty}$ und die Funktionen

$$\psi_{2^s, 2^{N_r}}(x), \psi_{2^s, 2^{N_r}+1}(x) \quad (s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots)$$

erhalten wir also durch Induktion.

Endlich sei

$$\psi_{kl}(x) = \begin{cases} 1/\sqrt{\operatorname{mes} I_{kl}}, & x \in I_{kl}, \\ 0, & \text{sonst} \end{cases}$$

im Falle $(k, l) \neq (2^s, 2^{N_r}), (2^s, 2^{N_r}+1)$ $(s=1, \ldots, 2M_r^2;\ r=1, 2, \ldots)$. Damit haben wir das ganze orthonormierte System $\psi = \{\psi_{kl}(x)\}_{k,l=1}^{\infty}$ in $(0, 2)$ definiert.

Es sei $G = \varlimsup_{r\to\infty} G_r$. Da die Mengen G_r stochastisch unabhängig sind, und (64) für jedes r besteht, durch Anwendung des Borel—Cantellischen Lemmas bekommen wir:

$$\operatorname{mes} G = 1.$$

Es sei r_0 eine positive ganze Zahl, und $x \in G_{r_0}$. Es sei weiterhin $m(x) = m(r_0; x)$ derjenige Index, für welchen (66) im Falle $r = r_0$ besteht. Auf Grund der Definition der Folge b, aus (62), (66) und (67) ergibt sich:

$$|s_{2^{m(x)}, 2^{N_{r_0}}}(b;\ \psi;\ x)| = \Big|\sum_{s=1}^{m(x)} b_{2^s, 2^{N_{r_0}}} \psi_{2^s, 2^{N_{r_0}}}(x)\Big| =$$

(69)
$$= (1/r_0 M_{r_0} \lambda_{2^{2M_{r_0}^2}} \mu_{2^{N_{r_0}+1}}) \Big|\sum_{s=1}^{m(x)} \psi_{2^s, 2^{N_{r_0}}}(x)\Big| \geqq$$

$$\geqq (c_5/\sqrt{2})\big((\log M_{r_0})/r_0 \lambda_{2^{2M_{r_0}^2}} \mu_{2^{N_{r_0}+1}}\big) \geqq (c_5/\sqrt{2}) r_0 \quad (x \in G_{r_0}).$$

Im Falle $x \in G$ gilt aber (69) für unendlich viele r_0, und so ist

$$\varlimsup_{\min(m,n)\to\infty} |s_{2^m, 2^n}(b,\ \psi;\ x)| = \infty$$

fast überall in $(0, 1)$, wegen (66) (i).

Auf Grund von (68) gibt es für jedes $x \in (1, 2)$ eine Zahl $m(x)$ derart, daß im Falle $m, n \geqq m(x)$

$$\sigma_{2^m, 2^n}(b,\ \psi;\ x) =$$

$$= \begin{cases} \left(1 - \dfrac{k-1}{2^m}\right)\left(1 - \dfrac{l-1}{2^n}\right) a_{kl} \psi_{kl}(x), & x \in I_{kl},\ (k, l) \neq (2^s, 2^{N_r}),\ (2^s, 2^{N_r}+1); \\ & s = 1, \ldots, 2M_r^2;\ r = 1, 2, \ldots, \\ \left(1 - \dfrac{2^s-1}{2^m}\right)\left(1 - \dfrac{2^{N_r}-1}{2^m}\right) a_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x) + & \\ \quad + \left(1 - \dfrac{2^s-1}{2^m}\right)\left(1 - \dfrac{2^{N_r}}{2^n}\right) a_{2^s, 2^{N_r+1}} \psi_{2^s, 2^{N_r+1}}(x), & \\ & x \in I_{2^s, 2^{N_r}} \cup I_{2^s, 2^{N_r+1}}; \\ & s = 1, \ldots, 2M_r^2;\ r = 1, 2, \ldots \end{cases}$$

ist. So folgt, daß $\lim_{\min(m,n)\to\infty} \sigma_{2^m, 2^n}(b, \psi; x)$ in $(1, 2)$ überall existiert.

Es seien m, n positive ganze Zahlen, und sei $r(n)$ diejenige ganze Zahl, für die

$$N_{r(n)} \geqq n < N_{r(n)+1}$$

besteht; offensichtlich gilt

$$\lim_{n\to\infty} r(n) = \infty.$$

Für $x\in(0, 1)$, auf Grund der Definition der Folge b und von (67) gilt

$$\sigma_{2^m, 2^n}(b, \psi;\ x) =$$
$$= \sum_{s=1}^{\min(2M_{r(n)}^2, m)} (1-(2^s-1)/2^m)(1-(2^{N_{r(n)}}-1)/2^{N_{r(n)}})\, b_{2^s, 2^{N_{r(n)}}} \psi_{2^s, 2^{N_{r(n)}}}(x) +$$
$$+(1/2^{N_{r(n)}}) \sum_{r=1}^{r(n)-1} \sum_{s=1}^{\min(2M_r^2, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x)$$

im Falle $n=N_{r(n)}$, bzw.

$$\sigma_{2^m, 2^n}(b, \psi;\ x) = (1/2^n) \sum_{s=1}^{\min(2M_{r(n)}^2, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_{r(n)}}} \psi_{2^s, 2^{N_{r(n)}}}(x) +$$
$$+(1/2^n) \sum_{r=1}^{r(n)-1} \sum_{s=1}^{\min(2M_r^2, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x)$$

im Falle $n>N_{r(n)}$. Also besteht immer

$$\sigma_{2^m, 2^n}(b, \psi;\ x) = (1/2^n) \sum_{s=1}^{\min(2M_{r(n)}^2, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_{r(n)}}} \psi_{2^s, 2^{N_{r(n)}}}(x) +$$
(70)
$$+(1/2^n) \sum_{r=1}^{r(n)-1} \sum_{s=1}^{\min(2M_r, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x)$$

in $(0, 1)$.

Aus $|b_{kl}|\leqq 1$ und aus (63), (65) ergibt sich

$$(1/2^n)\Big|\sum_{r=1}^{r(n)-1} \sum_{s=1}^{\min(2M_r^2, m)} (1-(2^s-1)/2^m)\, b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x)\Big| \leqq$$

$$\leqq (1/2^{N_{r(n)}}) \sum_{r=1}^{r(n)-1} \sum_{s=1}^{2M_r^2} |b_{2^s, 2^{N_r}}|\,|\psi_{2^s, 2^{N_r}}(x)| \leqq (c_4/\sqrt{2}\, 2^{N_{r(n)}}) \sum_{r=1}^{r(n)-1} M_r^2 \leqq (c_4/\sqrt{2})(1/r(n)),$$

woraus folgt, daß überall in $(0, 1)$

(71) $$\lim_{n\to\infty} \frac{1}{2^n} \sum_{r=1}^{r(n)-1} \sum_{s=1}^{\min(2M_r^2, m)} ((1-(2^s-1)/2^m)\, b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x) = 0$$

gleichmäßig in m besteht.

Für eine positive ganze Zahl r setzen wir

$$\delta_r(x) = \max_{1\leqq i\leqq 2M_r^2} \Big|\sum_{s=1}^{i} b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x)\Big| \quad (x\in(0, 2)).$$

Auf Grund des Hilfssatzes I, aus der Definition der Folge b und aus (62) folgt,

$$\int_0^1 \delta_r^2(x)\,dx \leqq \int_0^2 \delta_r^2(x)\,dx \leqq c_1 \log^2(2M_r^2+1) \sum_{s=1}^{2M_r^2} b_{2^s,2^{N_r}}^2 \leqq$$

$$\leqq c_1 (\log^2 M_r)(1/r^2 \lambda_{2^{2M_r^2}} \mu_{2^{N_r+1}}^2) \leqq 4c_{16} r^2 \quad (r = 1, 2, \ldots).$$

Da $N_r \geqq r$ $(r=1, 2, \ldots)$ offensichtlich gilt, erhalten wir daraus:

$$\sum_{r=1}^{\infty} (1/2^{2N_r}) \int_0^1 \delta_r^2(x)\,dx \leqq 4c_{16} \sum_{r=1}^{\infty} (1/2^{2N_r}) r^2 \leqq 4c_{16} \sum_{r=1}^{\infty} r^2/2^{2r} < \infty;$$

folglich gilt

$$\lim_{r\to\infty} (1/2^{N_r})\delta_r(x) = 0 \quad \text{fast überall in } (0, 1). \tag{72}$$

Es sei r eine positive ganze Zahl, und $x \in (0, 1)$. Dann besteht

$$\begin{aligned}&\sum_{s=1}^{\min(2M_r^2, m)} \left(1-(2^s-1)/2^m\right) b_{2^s,2^{N_r}} \psi_{2^s,2^{N_r}}(x) = \\ &= -(1/2^m) \sum_{s=1}^{m-1} (2^s - 2^{s+1}) \sum_{\sigma=1}^{s} b_{2^\sigma,2^{N_r}} \psi_{2^\sigma,2^{N_r}}(x) + \left(1-(2^m-1)/2^m\right) \sum_{\sigma=1}^{m} b_{2^\sigma,2^{N_r}} \psi_{2^\sigma,2^{N_r}}(x)\end{aligned} \tag{73}$$

im Falle $m \leqq 2M_r^2$, und

$$\begin{aligned}&\sum_{s=1}^{\min(2M_r^2, m)} \left(1-(2^s-1)/2^m\right) b_{2^s,2^{N_r}} \psi_{2^s,2^{N_r}}(x) = \\ &= -(1/2^m) \sum_{s=1}^{2M_r^2-1} (2^s - 2^{s+1}) \sum_{\sigma=1}^{s} b_{2^\sigma,2^{N_r}} \psi_{2^\sigma,2^{N_r}}(x) + \\ &+ \left(1-(2^{2M_r^2-1}-1)/2^m\right) \sum_{\sigma=1}^{2M_r^2} b_{2^\sigma,2^{N_r}} \psi_{2^\sigma,2^{N_r}}(x)\end{aligned} \tag{74}$$

im Falle $m > 2M_r^2$. Aus (73) und (74) folgt

$$\begin{aligned}&\left|\sum_{s=1}^{\min(2M_r^2, m)} \left(1-(2^s-1)/2^m\right) b_{2^s,2^{N_r}} \psi_{2^s,2^{N_r}}(x)\right| \leqq \\ &\leqq \left(1+1-(2^m-1)/2^m\right)\delta_r(x) \leqq 2\delta_r(x)\end{aligned} \tag{75}$$

im Falle $m \leqq 2M_r^2$, bzw.

$$\begin{aligned}&\left|\sum_{s=1}^{\min(2M_r^2, m)} \left(1-(2^s-1)/2^m\right) b_{2^s,2^{N_r}} \psi_{2^s,2^{N_r}}(x)\right| \leqq \\ &\leqq \left(2^{2M_r^2}/2^m + 1 - (2^{2M_r^2}-1)/2^m\right)\delta_r(x) \leqq 2\delta_r(x)\end{aligned} \tag{76}$$

im Falle $m > 2M_r^2$, für $x \in (0, 1)$. Aus (72), (75) und (76) bekommen wir, daß in $(0, 1)$ fast überall

$$(77) \qquad \lim_{r\to\infty} (1/2^{N_r}) \sum_{s=1}^{\min(2M_r^2, m)} \left(1-(2^s-1)/2^m\right) b_{2^s, 2^{N_r}} \psi_{2^s, 2^{N_r}}(x) = 0$$

gleichmäßig in m besteht. Aus (70), (71) und (77) erhalten wir endlich, daß

$$\lim_{\min(m,n)\to\infty} \sigma'_{\overline{2^m, 2^n}}(b, \psi;\ x) = 0$$

in $(0, 1)$ fast überall ist. Nach dem Obigen gilt also $\lim_{\min(m,n)\to\infty} \sigma_{2^m, 2^n}(b, \psi;\ x) = 0$ in $(0, 2)$ fast überall.

Endlich setzen wir

$$\Psi_{kl}(x) = \sqrt{2}\, \psi_{kl}(2x) \quad (x \in (0, 1);\ k, l = 1, 2, \ldots).$$

Für das System $\Psi = \{\Psi_{kl}(x)\}_{k,l=1}^{\infty}$ und für die Folge b sind also alle Erforderungen des Falles 2 erfüllt.

Damit haben wir Satz II bewiesen.

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On approximation by arbitrary systems in L^2-spaces

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Dedicated to Professor László Leindler on his 50th birthday

1. Introduction. Let $-\infty<a<b<\infty$, $p=b-a$. Let $L^2=L^2[p]$ be the space of all square integrable functions defined on $(-\infty, \infty)$ which are p-periodic. The norm in $L^2[p]$ is defined by

$$\|f\|_2=\Big\{\int_a^b |f(x)|^2\,dx\Big\}^{1/2}, \quad f\in L^2[p].$$

Let $\Phi=\{\varphi_k\}_{k=0}^{\infty}$ be a complete orthonormal system in $L^2[p]$. For $f_1, f_2, \ldots$ $\ldots, f_n\in L^2[p]$ let us denote by $[f_1, f_2, \ldots, f_n]$ the linear span of $f_1, f_2, \ldots, f_n$. For any $f\in L^2[p]$ let

(1) $$E_n = E_n^\Phi(f) = \inf_{q\in[\varphi_0, \varphi_1, \ldots, \varphi_n]} \|f-q\|_2, \quad n=0,1,2,\ldots$$

be the n-th best approximation of f with respect to the system Φ. We know that $E_n^\Phi(f)$ can be given by the generalized Fourier coefficients of f with respect to the system Φ, more precisely,

$$E_n^\Phi(f) = \Big[\sum_{k=n+1}^{\infty} c_k^2(f)\Big]^{1/2}, \quad n = 0, 1, 2, \ldots$$

where

$$c_k(f) = \int_a^b f(x)\varphi_k(x)\,dx, \quad k = 0, 1, 2, \ldots.$$

In this paper we give an answer to the following question due to Prof. L. Leindler: Characterize those orthonormal systems Φ for which

$$E_n^\Phi(f) \leqq c\omega(f, 1/n), \quad \forall f\in L^2[p], \quad n = 1, 2, \ldots$$

where $\omega(f, \delta)$ denotes the L^2-modulus of continuity of f, i.e.

$$\omega(f, \delta) = \sup_{|h|\leqq\delta} \|f(x+h)-f(x)\|_2.$$

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2. Lemmas. We need the following lemmas.

Lemma 1. *Let* $\varrho_n>0$ $(n=1,2,\dots)$. *Suppose that the system* $\Phi=\{\varphi_k\}_{k=0}^{\infty}$ *contains a constant function, say:* $\varphi_0\equiv C$. *The following statements are equivalent:*

a) *There exists an absolute constant* C_1 *such that*

$$E_n^{\Phi}(f)\leqq C_1\omega(f,\varrho_n),\quad \forall f\in L^2[p]. \tag{3}$$

b) *There exists an absolute constant* C_2 *such that*

$$E_n^{\Phi}(F)\leqq C_2\varrho_n\|f\|_2,\quad \forall f\in L^2[p] \tag{4}$$

where $F(x)=\int\limits_a^x f(t)\,dt$.

Proof. 1. a)$\to$b): Let $h>0$. By the formula

$$F(x+h)-F(x)=\int\limits_0^h f(x+t)\,dt$$

we have

$$\|F(x+h)-F(x)\|_2=\Big\|\int\limits_0^h f(x+t)\,dt\Big\|_2\leqq\int\limits_0^h\|f(\cdot+t)\|_2\,dt=\int\limits_0^h\|f\|_2\,dt=h\|f\|_2$$

hence $\omega(F,\delta)\leqq\delta\|f\|_2$. So, from a) we obtain

$$E_n(F)\leqq C_1\omega(F,\varrho_n)\leqq C_1\varrho_n\|f\|_2.$$

This proves (4).

2. b)$\to$a): We apply the transform of Steklov: Let

$$f_n(x)=\varrho_n^{-1}\int\limits_0^{\varrho_n} f(x+t)\,dt,\quad x\in[a,b].$$

Then $f_n(x)$ is absolute continuous, therefore $f_n(x)$ is an integral function of f_n':

$$f_n(x)=\int\limits_a^x f_n'(t)\,dt+f_n(a)=\tilde f_n(x)+f_n(a).$$

Since the system Φ contains the constant function we have $E_n(f_n)=E_n(\tilde f_n)$. On the other hand, we have

$$\|f-\tilde f_n\|_2=\Big\|\varrho_n^{-1}\int\limits_0^{\varrho_n}[f(x+t)-f(x)]\,dt\Big\|_2\leqq\omega(f,\varrho_n),$$

$$\|\tilde f_n'\|_2=\varrho_n^{-1}\|f(x+\varrho_n)-f(x)\|_2\leqq\varrho_n^{-1}\omega(f,\varrho_n).$$

Hence we obtain by (4):

$$E_n(f)=E_n(\tilde f_n)+\|f-\tilde f_n\|_2\leqq C_2\varrho_n\|\tilde f_n'\|_2+\omega(f,\varrho_n)\leqq(1+C_2)\omega(f,\varrho_n).$$

This proves (3).

Now, we introduce the following class of functions:

$$L_n = [\varphi_0, \varphi_1, \ldots, \varphi_n], \quad L_n^\perp = \{g \in L^2[p]: (g, q) = 0, \quad \forall q \in L_n\}, \quad n = 0, 1, 2, \ldots$$

where $(g, q) = \int_a^b g(x)q(x)\,dx$. If the system Φ is complete, then this definition is equivalent to the following:

$$L_n^\perp = \{g = \sum_{k=n+1}^\infty c_k\varphi_k : \sum_{k=n+1}^\infty c_k^2 < \infty\}, \quad n = 0, 1, 2, \ldots . \tag{5}$$

We notice that L_n and $L_n^\perp$ are (linear and closed) subspaces of $L^2[p]$.

Lemma 2. (4) *is equivalent to the following:*

$$\|G\|_2 \leqq C_2 \varrho_n \|g\|_2, \quad \forall g \in L_n^\perp, \quad n = 0, 1, 2, \ldots \tag{6}$$

where $G(x) = \int_a^x g(t)\,dt$.

Proof. Let $f \in L^2[p]$ and let $S(f)$ be the generalized Fourier series of f with respect to the system Φ, that is

$$S(f) = \sum_{k=0}^\infty c_k(f)\varphi_k$$

where

$$c_k(f) = \int_a^b f(x)\varphi_k(x)\,dx, \quad k = 0, 1, 2, \ldots .$$

We have by the minimum property of an orthonormal system:

$$E_n^\Phi(f) = \|\sum_{k=n+1}^\infty C_k(f)\varphi_k\|_2,$$

or, equivalently,

$$E_n(f) = \sup_{\substack{g \in L_n^\perp \\ \|g\|_2 \leqq 1}} \int_a^b f(x)g(x)\,dx, \quad n = 0, 1, \ldots . \tag{7}$$

Now, we apply this formula for the proof of Lemma 2.

a) (6)$\to$(4): Let $f \in L_n$, $g \in L_n^\perp$, $\|g\|_2 \leqq 1$, and let

$$G(x) = \int_a^x g(t)\,dt, \quad F(x) = \int_a^x f(t)\,dt.$$

We have by integration by parts and (6):

$$\int_a^b F(x)g(x)\,dx = FG|_a^b - \int_a^b f(x)G(x)\,dx =$$

$$= \int_a^b f(x)G(x)\,dx \leqq \|f\|_2\|G\|_2 \leqq C_2\varrho_n\|f\|_2$$

(we notice that since $g\in L_n^\perp$ and $\varphi_0\equiv C$, we have $G(a)=G(b)=0$). From the last inequality we obtain (4) by an application of (7).

b) (4)$\to$(6): Let $f\in L^2$, $g\in L_n^\perp$, $\|g\|_2\leqq 1$. Since

$$\int_a^b F(x)g(x)\,dx = \int_a^b G(x)f(x)\,dx,$$

from (4) and (7) we have

$$\int_a^b f(x)G(x)\,dx \leqq C_2\varrho_n\|f\|_2. \tag{8}$$

Now, let $0\neq g\in L_n^\perp$ be fixed. Let $g^*=g/\|g\|_2$; then $g^*\in L_n^\perp$ and $\|g^*\|_2=1$. Let

$$G^*(x) = \int_a^x g^*(t)\,dt.$$

From (8) we obtain:

$$\int_a^b f(x)G^*(x)\,dx \leqq C_2\varrho_n\|f\|_2, \quad \forall f\in L^2[p].$$

Hence, $\|G^*\|_2\leqq C_2\varrho_n$ from which it follows that $\|G\|_2\leqq C_2\varrho_n\|g\|_2$. This proves (6).

Now let us denote by I the integral operator, that is,

$$If(x)=\int_a^x f(t)\,dt,\ f\in L^2[p],\ x\in[a,b],$$

and let $If(x)$ be a p-periodic function. We know that the operator I is a bounded linear operator of the space L^2 to L^2. Let $I_n: L_n^\perp \to L^2[p]$ be the restriction of I to the space $L_n^\perp$, and let $|||I_n|||$ denote the norm of I_n, that is,

$$|||I_n||| = \sup_{\substack{g\in L_n^\perp\\ \|g\|_2\leqq 1}} \|I_n g\|_2 = \sup_{\substack{g\in L_n^\perp\\ \|g\|_2\leqq 1}} \|Ig\|_2. \tag{9}$$

Then we have

$$\|Ig\|_2 \leqq |||I_n|||\,\|g\|_2, \quad g\in L_n^\perp, \tag{10}$$

so that (6) is always true for $C_2\varrho_n=|||I_n|||$.

Therefore we have:

Lemma 3. *Let* $\lambda_n = |||I_n|||$ $(n=0, 1, 2, \ldots)$.

a) *We have*

$$E_n(F) \leqq \lambda_n \|f\|_2, \quad \forall f \in L^2[p], \tag{11}$$

where $F(x) = If(x)$.

b) *The order* λ_n *is best possible, this means that if for* $\lambda'_n > 0$:

$$E_n(F) \leqq \lambda_n \|f\|_2, \quad \forall f \in L^2[p],$$

then $\lambda'_n \geqq \lambda_n$ $(n=0, 1, 2, \ldots)$.

Proof. a) is proved above. Claim b) follows from the fact that if $E_n(F) \leqq \leqq \lambda'_n \|f\|_2$, $\forall f \in L^2[p]$, then by Lemma 2 we have $\|G\|_2 \leqq \lambda'_n \|g\|_2$, $\forall g \in L_n^\perp$, hence we obtain by the definition of the norm $|||I_n|||$ that $\lambda'_n \geqq |||I_n||| = \lambda_n$.

In the following we consider only a complete orthonormal system $\Phi = \{\varphi_0, \varphi_1, \ldots\}$ which satisfies the following conditions:

$$\varphi_0(t) \equiv C \text{ (constant)}, \tag{12}$$

$$\text{for} \quad n = 0, 1, 2, \ldots, \quad I\varphi_{n+1} \in L_n^\perp. \tag{13}$$

We remark that the condition (13) is equivalent to the following: for $n=0, 1, 2, \ldots$, if $g \in L_n^\perp$ then $Ig \in L_n^\perp$.

Lemma 4. *Let* $\Phi = \{\varphi_0, \varphi_1, \ldots\}$ *be the complete orthonormal system satisfying* (12) *and* (13). *Let* $\psi_k = I\varphi_k$, $k=0, 1, 2, \ldots$, *where* I *denotes the integral operator. Then for* $n=0, 1, 2, \ldots$ *the system* $\{\psi_k\}_{k=n+1}^\infty$ *is complete, linearly independent in the subspace* $L_n^\perp$.

Proof. a) $\{\psi_k\}_{k=n+1}^\infty$ is linearly independent. Suppose that α_k $(k = n+1, n+2, \ldots, n+m)$ are real numbers satisfying

$$\sum_{k=n+1}^{n+m} \alpha_k \psi_k = 0.$$

Then by differentiation we have

$$\sum_{k=n+1}^{n+m} \alpha_k \varphi_k = 0$$

hence $\alpha_k = 0$ $(k=n+1, \ldots, n+m)$, since $\{\varphi_k\}_{k=0}^\infty$ is independent.

b) $\{\psi_k\}_{k=n+1}^\infty$ is complete in $L_n^\perp$. Suppose that $g \in L_n^\perp$ satisfies

$$\int_a^a g(x)\psi_k(x)\,dx = 0 \quad (k \geqq n+1).$$

Let $Ig=G(x)$. Integrating by parts we obtain (by (12) we have $\psi_k(a)=\psi_k(b)=0$ for $k\geqq n+1>0$):

$$0=\int_a^b g(x)\psi_k(x)\,dx=\int_a^b G(x)\varphi_k(x)\,dx\quad(k\geqq n+1). \tag{14}$$

Since $g\in L_n^\perp$, by (13) we have $G\in L_n^\perp$, that is

$$\int_a^b G(x)\varphi_k(x)\,dx=0\quad(k\leqq n)$$

and so (14) is valid for every $k=0, 1, 2, \dots$ from which it follows by the completeness of the system $\Phi=\{\varphi_k\}_{k=0}^\infty$ that $G(x)\equiv 0$, therefore $g(x)\equiv C$ (constant). But $g\in L_n^\perp$, so by (12) we have $g(x)\equiv C=0$.

Let now $n\geqq 0$ and fixed. Let $\Phi_n=\{\psi_{n+1}, \psi_{n+2}, \dots\}$. Since Φ_n is linearly independent (Lemma 4), by the process of Gram—Schmidt we obtain the orthonormal system $H=(h_1, h_2, \dots)\subset L_n^\perp$ as follows. For $m=1, 2, \dots$ let

$$\Delta_m(\Phi_n)=|a_{lk}^{(n)}|_{l,k=1}^m=\begin{vmatrix}(\psi_{n+1}, \psi_{n+1})(\psi_{n+1}, \psi_{n+2})\dots(\psi_{n+1}, \psi_{n+m})\\ (\psi_{n+2}, \psi_{n+1})(\psi_{n+2}, \psi_{n+2})\dots(\psi_{n+2}, \psi_{n+m})\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ (\psi_{n+m}, \psi_{n+1})(\psi_{n+m}, \psi_{n+2})\dots(\psi_{n+m}, \psi_{n+m})\end{vmatrix} \tag{15}$$

be the m-th Gram—Schmidt's determinant of the system Φ_n. Let $D_{m,l}^n=D_{m,l}(\Phi_n)$ be the cofactor of an element $a_{lm}^{(n)}$ $(l=1, 2, \dots, m)$. We define the following infinite matrix:

$$A(\Phi_n)=(\alpha_{lk}^{(n)})_{l,k=1}^\infty=\begin{pmatrix}\dfrac{1}{\sqrt{\Delta_1(\Phi_n)}} & \dfrac{D_{12}(\Phi_n)}{\sqrt{\Delta_1(\Phi_n)\Delta_2(\Phi_n)}} & \dfrac{D_{13}(\Phi_n)}{\sqrt{\Delta_2(\Phi_n)\Delta_3(\Phi_n)}}\dots\\ 0 & \dfrac{D_{22}(\Phi_n)}{\sqrt{\Delta_1(\Phi_n)\Delta_2(\Phi_n)}} & \dfrac{D_{33}(\Phi_n)}{\sqrt{\Delta_2(\Phi_n)\Delta_3(\Phi_n)}}\dots\\ \vdots & \vdots & \vdots\end{pmatrix}. \tag{16}$$

From the matrix $A(\Phi_n)$ we define the matrix $A_m(\Phi_n)$:

$$A_m(\Phi_n)=\begin{pmatrix}\alpha_{11}^{(n)} & \alpha_{12}^{(n)} & \dots & \alpha_{1m}^{(n)}\\ 0 & \alpha_{22}^{(n)} & \dots & \alpha_{2m}^{(n)}\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \dots & \alpha_{mm}^{(n)}\end{pmatrix}=(\alpha_{lk}^{(n)})_{l,k=1}^m. \tag{17}$$

Let $B_m(\Phi_n)=A_m^{-1}(\Phi_n)$ be the inverse matrix of $A_m(\Phi_n)$:

$$B_m(\Phi_n)=(\beta_{lk}^{(n)})_{l,k=1}^m=\begin{pmatrix}\beta_{11}^{(n)} & \beta_{12}^{(n)} & \dots & \beta_{1m}^{(n)}\\ \beta_{21}^{(n)} & \beta_{22}^{(n)} & \dots & \beta_{2m}^{(n)}\\ \beta_{m1}^{(n)} & \beta_{m2}^{(n)} & \dots & \beta_{mm}^{(n)}\end{pmatrix}. \tag{18}$$

From the the matrix $B_m(\Phi_n)$ $(m=1, 2, \dots)$ we define the infinite matrix:

(19) $$B(\Phi_n) = (\beta_{lk}^{(n)})_{l,k=1}^{\infty}.$$

The process of Gram—Schmidt gives the following formula:

(20) $$\Phi_n A(\Phi_n) = H, \quad HB(\Phi_n) = \Phi_n$$

where $\Phi_n A(\Phi_n)$ and $HB(\Phi_n)$ denote the usual products of matrices (infinite matrices).

Now we return to the determination of the exact value of $|||I_n|||$. Let $g \in L_n^{\perp}$. Then we have

$$g = \sum_{k=n+1}^{\infty} C_k \varphi_k, \quad \|g\|_2 = (\sum_{k=n+1}^{\infty} C_k^2)^{1/2}.$$

Since the operator I is linear and continuous (in the metric of L^2), we have

$$Ig = \sum_{k=n+1}^{\infty} C_k I\varphi_k = \sum_{k=n+1}^{\infty} C_k \psi_k = \sum_{l=1}^{\infty} d_l h_l$$

where

(21) $$d = CB(\Phi_n)$$

with $C=(C_{n+1}, C_{n+2}, \dots)$, $d=(d_1, d_2, \dots)$. By Parseval's formula we have

(22) $$\|Ig\|_2 = (\sum_{l=1}^{\infty} d_l^2)^{1/2}.$$

Let l^2 denote the Hilbert space of all sequences $c=(c_1, c_2, \dots)$ for which $\|c\|_{l^2} = (\sum_{k=1}^{\infty} c_k^2)^{1/2} < \infty$. Now, from (21), (22) we obtain

(23) $$|||I_n||| = \sup_{\substack{g \in L_n^{\perp} \\ \|g\|_2 \leqq 1}} \|Ig\|_2 = \sup_{\substack{c \in l^2 \\ \|c\|_{l^2} \leqq 1}} \|CB(\Phi_n)\|_{l^2}.$$

Finally, from (23), by a known theorem of functional analysis (see e.g. Л. В. Канторович—Г. П. Акилов [1], p. 193) we have

(24) $$|||I_n||| = \sup_{m \geqq 1} \max_{1 \leqq j \leqq m} \sqrt{\lambda_j[B_m^*(\Phi_n) B_m(\Phi_m)]}$$

where $B_m^*(\Phi_n)$ denotes the adjoint matrix of $B_m(\Phi_n)$ and $\lambda_j[B_m^*(\Phi_n)B_m(\Phi_n)]$ denotes an eigenvalue of the matrix $B_m^*(\Phi_n)B_m(\Phi_n)$.

3. So, the formula (24), and Lemmas 1, 3 prove the following theorem.

Theorem. *Let $\Phi = \{\varphi_k\}_{k=0}^{\infty}$ be a complete orthonormal system in $L^2[p]$, which satisfies the conditions* (12) *and* (13). *Let $B_m(\Phi_n)$ be the matrix defined by* (15), (16), (17), (18), *and let $\lambda_j^{(n,m)}$ be the eigenvalues of the self-adjoint matrix $B_m^*(\Phi_n)B_m(\Phi_n)$. Let*

(25) $$\varrho_n = \varrho_n(\Phi) = \sup_{m \geqq 1} \max_{1 \leqq j \leqq m} \sqrt{\lambda_j^{(n,m)}}, \quad n = 0, 1, 2, \dots.$$

Then we have

a) $$E_n^\Phi(f) \leqq C_3\omega(f, \varrho_n), \quad \forall f \in L^2[p], \quad n = 1, 2, \ldots$$

where C_3 is an absolute constant (we can select $C_3=2$; see the proof of Lemma 2);

b) *ϱ_n is best possible, that is if $E_n(f) \leqq C_4\omega(f, \varrho_n')$, $\forall f \in L^2$, $n=1, 2, \ldots$, then $\varrho_n = O(\varrho_n')$.*

Remark 1. Let $\Omega(p)$ be the set of all functions f, which are absolute continuous in $[a, b]$ and for which $f' \in L^2[p]$, $\|f'\|_2 \leqq 1$. Let

$$E_n^\Phi(\Omega) = \sup_{f\in\Omega} E_n^\Phi(f) \quad \text{and} \quad d_n(\Omega) = \inf_{\Phi\in\mathcal{S}} E_n^\Phi(\Omega), \quad n = 0, 1, 2, \ldots,$$

where $\mathcal{S}$ denotes the class of orthonormal systems in $L^2[p]$; $d_n(\Omega)$ is called the n-th width of the set Ω. If for some $\Phi^* \in \mathcal{S}$ we have $d_n(\Omega) = E_n^{\Phi^*}(\Omega)$, $n=0, 1, 2, \ldots$, then we say that Φ^* is an extremal system for the set Ω.

Let now T be the trigonometric system

$$T = \left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots\right\}.$$

We know that for a set $\Omega = \Omega(2\pi) \subset L^2[2\pi]$, the system T is an extremal system in $L^2[2\pi]$, and

$$d_n[\Omega(2\pi)] = E_n^T[\Omega(2\pi)] = 1/(n+1), \quad n = 0, 1, 2, \ldots.$$

(See e.g. G. G. Lorentz [2] p. 140.) So the system

$$T_p = \left\{\frac{1}{\sqrt{2p}}, \frac{2\sqrt{\pi}}{p} \sin\left(\frac{p}{2\pi} t + a\right), \frac{2\sqrt{\pi}}{p} \cos\left(\frac{p}{2\pi} t + a\right), \ldots\right\}$$

is orthonormal in $L^2[p]$; it is an extremal system for the set $\Omega = \Omega(p) \subset L^2[p]$ and

(26) $$d_n[\Omega(p)] = E_n^{T_p}[\Omega(p)] = \big(1/(n+1)\big)(2\pi/p), \quad n = 0, 1, 2, \ldots.$$

We return to the definition of $\varrho_n(\Phi)$. We have

(27) $$\varrho_n(\Phi) = |||I_n||| = \sup_{\substack{g\in L_n^\perp \\ \|g\|_2 \leqq 1}} \|Ig\|_2 \geqq \sup_{If\in\Omega} E_n^\Phi(If) = E_n^\Phi(\Omega), \quad n = 0, 1, 2, \ldots.$$

From (26) and (27) we obtain that

(28) $$\varrho_n(\Phi) \geqq (2\pi/p)\big(1/(n+1)\big), \quad n = 0, 1, 2, \ldots.$$

Remark 2. From the above theorem and (28) it follows that for some orthonormal system Φ satisfying (12) and (13), the following statements are equivalent:

a) $$E_n^\Phi(f) \leqq C_5\omega(f, 1/n), \quad f \in L^2[p], \quad n = 1, 2, \ldots,$$

b) $$(2\pi/p)\big(1/(n+1)\big) \leqq \varrho_n(\Phi) \leqq C_6(1/n), \quad n = 1, 2, \ldots,$$

where $\varrho_n(\Phi)$ is defined by (25); C_5 and C_6 denote absolute constants.

Remark 3. For the trigonometric system T, the following inequalities are valid (for $\varrho_n(T)=1/(n+1)$):

$$(29)\qquad \begin{gathered} E_n^T(f) \leqq C_7\varrho_n(T)\|f'\|, \ \forall f\in L^2[2\pi],\ f'\in L^2[2\pi], \\ \|t_n'\| \leqq C_8\varrho_n^{-1}(T)\|t_n\|, \quad \forall t_n\in T_n \end{gathered}$$

where T_n denotes the set of all trigonometric polynomials of order at most n, and $C_7=C_8=1$. The two inequalities in (29) play an important role in the proofs of the direct and converse approximation theorems.

We can ask: is (29) true for an arbitrary system? The answer is that in general (29) is not true. Indeed, let us consider the following system. Let $n_0\geqq 1$ be a fixed integer. Let

$$T=\left\{\frac{1}{\sqrt{2\pi}},\ \frac{\cos kx}{\sqrt{\pi}},\ \frac{\sin kx}{\sqrt{\pi}}\right\}_{k=1}^{\infty}=\left\{\frac{1}{\sqrt{2\pi}};\ C_k(x),\ S_k(x)\right\}_{k=1}^{\infty}.$$

We consider the following system:

$$T^*=\{1/\sqrt{2\pi},\ C_1, S_1, C_2, S_2, \ldots, C_{n_0-1}, S_{n_0-1}, C_{n_0+1}, S_{n_0+1},$$

$$C_{n_0+2}, S_{n_0+2}, \ldots, C_{n_0^2-1}, S_{n_0^2-1}, C_{n_0}, S_{n_0}, C_{n_0^2+1}, S_{n_0^2+1},$$

$$C_{n_0^2+2}, S_{n_0^2+2}, \ldots, C_{n_0^4-1}, S_{n_0^4-1}, C_{n_0^2}, S_{n_0^2}, C_{n_0^4+1}, S_{n_0^4+1}, \ldots\}.$$

We have $\varrho_n(T^*)\sim 1/\sqrt{n}$. So the second inequality in (29) is not true for $\varrho_n^{-1}(T^*)\sim\sim\sqrt{n}$.

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On the restricted convergence and $(C, 1, 1)$-summability of double orthogonal series

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Dedicated to Professor László Leindler on his 50th birthday

1. Introduction. Let $(X, \mathcal{F}, \mu)$ be an arbitrary positive measure space and $\{\varphi_{ik}(x)\colon i, k=1, 2, \ldots\}$ an orthonormal system (in abbreviation: ONS) on X. We shall consider the double orthogonal series

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}\varphi_{ik}(x), \tag{1.1}$$

where $\{a_{ik} : i, k=1, 2, \ldots\}$ is a double sequence of real numbers (coefficients), for which

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 < \infty. \tag{1.2}$$

By the extended Riesz—Fischer theorem there exists a function $f(x)\in L^2= =L^2(X, \mathcal{F}, \mu)$ such that (1.1) is the Fourier series of $f(x)$ with respect to $\{\varphi_{ik}(x)\}$ and the rectangular partial sums

$$s_{mn}(x) = \sum_{i=1}^{m}\sum_{k=1}^{n} a_{ik}\varphi_{ik}(x) \quad (m, n = 1, 2, \ldots)$$

converge to $f(x)$ in the L^2-metric:

$$\int [s_{mn}(x)-f(x)]^2\, d\mu(x) \to 0 \quad \text{as} \quad \min(m, n) \to \infty.$$

Here and in the sequel, the integrals are taken over the whole space X.

Beside $s_{mn}(x)$ we consider the first arithmetic means, the so-called $(C, 1, 1)$-means $\sigma_{mn}(x)$ of series (1.1) defined by

$$\sigma_{mn}(x) = \frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n} s_{ik}(x) =$$

$$= \sum_{i=1}^{m}\sum_{k=1}^{n}\left(1-\frac{i-1}{m}\right)\left(1-\frac{k-1}{n}\right) a_{ik}\varphi_{ik}(x) \quad (m, n = 1, 2, \ldots).$$

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2. Unrestricted convergence. It is well-known that condition (1.2) itself does not ensure the pointwise convergence of $s_{mn}(x)$ or $\sigma_{mn}(x)$. The extension of the famous Rademacher—Menšov theorem proved by a number of authors (see [1], [7], etc.) reads as follows.

Theorem A. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log (i+1)]^2 [\log (k+1)]^2 < \infty, \tag{2.1}$$

then

$$s_{mn}(x) \to f(x) \quad a.e. \text{ as } \quad \min (m, n) \to \infty$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{m,n \geqq 1} |s_{mn}(x)| \leqq F(x) \quad a.e.$$

In this paper the logarithms are to the base 2.

The following theorem (see, e.g. [8]) gives information on the order of magnitude of $s_{mn}(x)$ in the more general setting of (1.2).

Theorem B. *Under condition* (1.2),

$$s_{mn}(x) = o_x\{\log (m+1) \log (n+1)\} \quad a.e. \text{ as } \quad \max (m, n) \to \infty \tag{2.2}$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{m,n \geqq 1} \frac{|s_{mn}(x)|}{\log (m+1) \log (n+1)} \leqq F(x) \quad a.e.$$

Similarly to the case of the single orthogonal series, the convergence properties improve when the first arithmetic means $\sigma_{mn}(x)$ are considered instead of the rectangular partial sums $s_{mn}(x)$. The following extension of the summation theorem of Menšov and Kaczmarz was proved in [10]. We note that it was stated earlier in [5] and [4], but the proofs are not complete in them.

Theorem C. *If*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 [\log \log (i+3)]^2 [\log \log (k+3)]^2 < \infty, \tag{2.3}$$

then

$$\sigma_{mn}(x) \to f(x) \quad a.e. \text{ as } \quad \min (m, n) \to \infty$$

and there exists a function $F(x) \in L^2$ *such that*

$$\sup_{m,n \geqq 1} |\sigma_{mn}(x)| \leqq F(x) \quad a.e.$$

The order of magnitude of $\sigma_{mn}(x)$, under condition (1.2), is also better in general than that of $s_{mn}(x)$. In contrast to the theory of single orthogonal series, the performance of an Abel transformation is avoided in the proof given in [11].

Theorem D. *Under condition* (1.2),

(2.4) $\quad \sigma_{mn}(x) = o_x\{\log\log(m+3)\log\log(n+3)\}$ *a.e. as* $\max(m, n)\to\infty$

and there exists a function $F(x)\in L^2$ *such that*

$$\sup_{m,n\geqq 1}\frac{|\sigma_{mn}(x)|}{\log\log(m+3)\log\log(n+3)}\leqq F(x) \quad a.e.$$

3. Restricted convergence. In the statements of Theorems A and C both m and n tend to ∞ independently of each other.

We say that m and n tend restrictedly to ∞ if $\min(m, n)\to\infty$ in such a way that the ratios m/n and n/m remain bounded, i.e., there exists a real number $\theta\geqq 1$ such that $\theta^{-1}\leqq n/m\leqq\theta$ while both m and n tend to ∞. We say that $s_{mn}(x)$ or $\sigma_{mn}(x)$ restrictedly converges to $f(x)$ a.e. if $s_{mn}(x)$ or $\sigma_{mn}(x)$ tends to $f(x)$ a.e., respectively, whenever m and n tend restrictedly to ∞. In the case of $\sigma_{mn}(x)$, we may say that series (1.1) is restrictedly $(C, 1, 1)$-summable to $f(x)$ a.e.

The first remarkable fact is that the a.e. restricted convergence of $s_{mn}(x)$ cannot be ensured in general by any weaker condition than (2.1). This means that, in terms of coefficient tests, there is no difference between the a.e. unrestricted convergence and the a.e. restricted convergence of the rectangular partial sums of double orthogonal series.

Theorem E. *For every nonincreasing sequence* $\{\varepsilon(m)\colon m=1, 2, \ldots\}$ *of positive numbers tending to* 0 *as* $m\to\infty$, *there exist a double ONS* $\{\varphi_{ik}(x)\}$ *on the unit square* $I^2=[0, 1]\times[0, 1]$ *and a double sequence* $\{a_{ik}\}$ *of coefficients such that*

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2\varepsilon(\min(i, k))[\log(\max(i, k)+1)]^4<\infty$$

and

$$\limsup_{m,n\to\infty:\,1/2\leqq n/m\leqq 2}|s_{mn}(x)|=\infty \quad a.e. \text{ on } \quad I^2.$$

The order of magnitude of $s_{mn}(x)$, under condition (1.2), exhibits the same phenomenon. Relation (2.2) is also the best possible when m and n restrictedly tend to ∞.

Theorem F. *For every* $\{\varepsilon(m)\}$ *occurring in Theorem E, there exist a double ONS* $\{\varphi_{ik}(x)\}$ *on* I^2 *and a double sequence* $\{a_{ik}\}$ *of coefficients such that condition* (1.2) *is satisfied and*

$$\limsup_{m,n\to\infty:\,1/2\leqq n/m\leqq 2}\frac{|s_{mn}(x)|}{\varepsilon(\min(m, n))[\log(\max(m, n)+1)]^2}=\infty \quad a.e. \text{ on } \quad I^2.$$

Both Theorem E and Theorem F were actually proved in [12] (though the fulfilment of the condition $1/2 \leqq n/m \leqq 2$ is not stated explicitly there).

Now, the main results of the present paper say that the situation is quite different for the first arithmetic means $\sigma_{mn}(x)$. The a.e. restricted convergence of $\sigma_{mn}(x)$ can be ensured under a weaker condition than (2.3).

Theorem 1. *If*

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} a_{ik}^2 [\log\log(\max(i, k)+3)]^2 < \infty, \tag{3.1}$$

then $\sigma_{mn}(x)$ restrictedly converges to $f(x)$ a.e. and for every $\theta \geqq 1$ there exists a function $F_\theta(x) \in L^2$ such that

$$\sup_{m, n \geqq 1: \theta^{-1} \leqq n/m \leqq \theta} |\sigma_{mn}(x)| \leqq F_\theta(x) \quad a.e. \tag{3.2}$$

Assuming only (1.2), the order of magnitude of $\sigma_{mn}(x)$ becomes smaller in comparison with (2.4) in the case when m and n tend restrictedly to ∞.

Theorem 2. *Under condition* (1.2), *for every* $\theta \geqq 1$

$$\max_{n: \theta^{-1} \leqq n/m \leqq \theta} |\sigma_{mn}(x)| = o_x\{\log\log(m+3)\} \quad a.e. \text{ as } \quad m \to \infty \tag{3.3}$$

and there exists a function $F_\theta(x) \in L^2$ such that

$$\sup_{m, n \geqq 1: \theta^{-1} \leqq n/m \leqq \theta} \frac{|\sigma_{mn}(x)|}{\log\log(m+3)} \leqq F_\theta(x) \quad a.e.$$

It is worth including two interesting consequences of Theorems 1 and 2. The following Theorem 3 extends a theorem of Borgen [3] from single orthogonal series to double ones. We remark that the possibility of this extension was already indicated in [9].

Theorem 3. *If condition* (1.2) *is satisfied and series* (1.1) *is restrictedly* $(C, 1, 1)$-*summable to $f(x)$ a.e., then for every* $\theta \geqq 1$

$$\frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1} i}^{\theta i} [s_{ik}(x) - f(x)]^2 \to 0 \quad a.e. \text{ as } \quad m \to \infty. \tag{3.4}$$

If, in addition, for every $\theta \geqq 1$ there exists a function $F_\theta(x) \in L^2$ such that (3.2) *is satisfied, then there exists a function $G_\theta(x) \in L^2$ such that*

$$\sup_{m \geqq 1} \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1} i}^{\theta i} [s_{ik}(x) - f(x)]^2 \right\}^{1/2} \leqq G_\theta(x) \quad a.e.$$

Here by $\sum_{k=\theta^{-1} i}^{\theta i}$ we mean that the summation is extended over those integers k for which $\theta^{-1} \leqq k/i \leqq \theta$.

Via the Cauchy inequality, relation (3.4) implies that

$$\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}|s_{ik}(x)-f(x)| = o_x\{1\} \quad \text{a.e. as} \quad m\to\infty.$$

Our last theorem in this Section shows that, under condition (1.2), a certain average of $s_{ik}^2(x)$ is essentially less than it would be expected on the basis of (2.2).

Theorem 4. *Under condition* (1.2), *for every* $\theta\geqq 1$

$$\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i} s_{ik}^2(x) = o_x\{\log\log(m+3)\}^2 \quad \textit{a.e. as} \quad m\to\infty \tag{3.5}$$

and there exists a function $F_\theta(x)\in L^2$ *such that*

$$\sup_{m\geqq 1}\frac{1}{\log\log(m+3)}\left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i} s_{ik}^2(x)\right\}^{1/2} \leqq F_\theta(x) \quad \textit{a.e.}$$

By (3.5) and the Cauchy inequality, we have again

$$\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}|s_{ik}(x)| = o_x\{\log\log(m+3)\} \quad \text{a.e. as} \quad m\to\infty.$$

Finally, we raise two open questions: Under what conditions can we conclude that

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}[s_{ik}(x)-f(x)]^2 = o_x\{1\} \quad \text{a.e.} \tag{3.6}$$

and

$$\frac{1}{mn}\sum_{i=1}^{m}\sum_{k=1}^{n}s_{ik}^2(x) = o_x\{\log\log(m+3)\log\log(n+3)\}^2 \quad \text{a.e.} \tag{3.7}$$

as $\min(m, n)\to\infty$ (while m and n run to ∞ independently of each other)?

4. Proofs of Theorems 1 and 2. For the sake of brevity, we introduce the following notation. Given a system $\{f_p(x)\colon p=0, 1, \ldots\}$ of functions in L^2 and a sequence $\{\lambda(p)\}$ of positive numbers, we write

$$f_p(x) = o_x\{\lambda(p)\} \quad \text{a.e.}$$

if

$$f_p(x)/\lambda(p)\to 0 \quad \text{a.e.} \quad \text{as} \quad p\to\infty$$

and there exists a function $F(x)\in L^2$ such that

$$\sup_{p\geqq 0}|f_p(x)|/\lambda(p) \leqq F(x) \quad \textit{a.e.}$$

First we present five lemmas.

Lemma 1. *Under condition* (3.1),

$$s_{2^p,2^p}(x)-f(x) = o_x\{1\} \quad a.e. \tag{4.1}$$

This is an immediate consequence of the following Theorem G proved in [7]. Let $Q_0 \subset Q_1 \subset Q_2 \subset \ldots$ be an arbitrary sequence of finite regions in $\mathbf{N}^2 = \{(i,k): i,k=1,2,\ldots\}$ such that $\bigcup_{p=0}^{\infty} Q_p = \mathbf{N}^2$. Set

$$s_p(Q;\, x) = \sum_{(i,k)\in Q_p} a_{ik}\varphi_{ik}(x) \quad (p=0,1,\ldots).$$

Theorem G. *If*

$$\sum_{p=0}^{\infty}\Big(\sum_{(i,k)\in Q_p\setminus Q_{p-1}} a_{ik}^2\Big)[\log(p+2)]^2 < \infty \quad (Q_{-1}=\emptyset), \tag{4.2}$$

then

$$s_p(Q;\, x)-f(x) = o_x\{1\} \quad a.e. \tag{4.3}$$

Now, it is not hard to verify that in the special case when $Q_p = \{(i,k): i,k=1,2,\ldots,2^p\}$ $(p=0,1,\ldots)$ the conditions (3.1) and (4.2) are equivalent, while the statements (4.1) and (4.3) coincide.

Lemma 2. *Under condition* (1.2),

$$s_{2^p,2^p}(x)-\sigma_{2^p,2^p}(x) = o_x\{1\} \quad a.e. \tag{4.4}$$

Proof. Using the representation

$$s_{2^p,2^p}(x)-\sigma_{2^p,2^p}(x) = \sum_{i=1}^{2^p}\sum_{k=1}^{2^p}\left(\frac{i-1}{2^p}+\frac{k-1}{2^p}-\frac{(i-1)(k-1)}{2^{2p}}\right)a_{ik}\varphi_{ik}(x),$$

we can simply estimate as follows

$$\int [s_{2^p,2^p}(x)-\sigma_{2^p,2^p}(x)]^2\, d\mu(x) \leqq$$

$$\leqq 3\sum_{i=1}^{2^p}\sum_{k=1}^{2^p}\left(\frac{(i-1)^2}{2^{2p}}+\frac{(k-1)^2}{2^{2p}}+\frac{(i-1)^2(k-1)^2}{2^{4p}}\right)a_{ik}^2 = 3(I_1+I_2+I_3), \quad \text{say.}$$

By (1.2),

$$I_1 = \sum_{i=2}^{\infty}\sum_{k=1}^{\infty}(i-1)^2 a_{ik}^2 \sum_{p:\, 2^p \geqq \max(i,k)} \frac{1}{2^{2p}} < \infty.$$

A similar inequality holds true for I_2. Finally, $I_3 \leqq I_1$. The application of B. Levi's theorem completes the proof of (4.4).

Lemma 3. *Under condition* (1.2), *for every* $\theta \geqq 1$

$$M_{p,\theta}^{(3)}(x) = \max_{\theta^{-1}2^p < m \leqq \theta 2^{p+1}} |\sigma_{m,2^p}(x)-\sigma_{2^p,2^p}(x)| = o_x\{1\} \quad a.e. \tag{4.5}$$

The symmetric counterpart of Lemma 3 is the following: Under condition (1.2), for every $\theta \geqq 1$

$$M_{p,\theta}^{(4)}(x) = \max_{\theta^{-1}2^p < n \leqq \theta 2^{p+1}} |\sigma_{2^p,n}(x) - \sigma_{2^p,2^p}(x)| = o_x\{1\} \quad \text{a.e.} \tag{4.6}$$

Proof of Lemma 3. It is clear that

$$\begin{aligned} M_{p,\theta}^{(3)}(x) \leqq & \max_{\theta^{-1}2^p < m \leqq 2^p} |\sigma_{m,2^p}(x) - \sigma_{2^p,2^p}(x)| + \\ & + \max_{2^p < m \leqq \theta 2^{p+1}} |\sigma_{m,2^p}(x) - \sigma_{2^p,2^p}(x)| = M_{p,\theta}^{(5)}(x) + M_{p,\theta}^{(6)}(x), \quad \text{say.} \end{aligned} \tag{4.7}$$

(If $\theta = 1$, then $M_{p,\theta}^{(5)}(x) \equiv 0$.) For example, we prove that

$$M_{p,\theta}^{(6)}(x) = o_x\{1\} \quad \text{a.e.} \tag{4.8}$$

We begin with the obvious estimate

$$M_{p,\theta}^{(6)}(x) \leqq \sum_{m=2^p+1}^{\theta 2^{p+1}} |\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x)|,$$

whence, via the Cauchy inequality,

$$[M_{p,\theta}^{(6)}(x)]^2 \leqq (2\theta - 1) \sum_{m=2^p+1}^{\theta 2^{p+1}} m [\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x)]^2.$$

Using the representation

$$\sigma_{m,2^p}(x) - \sigma_{m-1,2^p}(x) = \sum_{i=1}^{m} \sum_{k=1}^{2^p} \frac{i-1}{m(m-1)} \left(1 - \frac{k-1}{2^p}\right) a_{ik} \varphi_{ik}(x),$$

we can easily see that

$$\begin{aligned} & \sum_{p=0}^{\infty} \int [M_{p,\theta}^{(6)}(x)]^2 \, d\mu(x) \leqq \\ & \leqq (2\theta - 1) \sum_{p=0}^{\infty} \sum_{m=2^p+1}^{\theta 2^{p+1}} m \sum_{i=1}^{m} \sum_{k=1}^{2^p} \frac{(i-1)^2}{m^2(m-1)^2} \left(1 - \frac{k-1}{2^p}\right)^2 a_{ik}^2 \leqq \\ & \leqq (2\theta - 1)^2 \sum_{p=0}^{\infty} \frac{1}{2^{2p}} \sum_{i=2}^{\theta 2^{p+1}} \sum_{k=1}^{2^p} (i-1)^2 a_{ik}^2 = \\ & \leqq (2\theta - 1)^2 \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 \sum_{p:\, 2^{p+1} \geqq \max(i/\theta,\, 2k)} \frac{1}{2^{2p}} < \infty. \end{aligned}$$

Applying B. Levi's theorem, we get (4.8).

Similarly, we can prove that

$$M_{p,\theta}^{(5)}(x) = o_x\{1\} \quad \text{a.e.} \tag{4.9}$$

The combination of (4.7), (4.8) and (4.9) provides (4.5) to be proved.

Lemma 4. *Under condition* (1.2), *for every* $\theta \geqq 1$

$$M^{(7)}_{p,\theta}(x) =$$

(4.10)
$$= \max_{2^p < m \leqq 2^{p+1}} \max_{\theta^{-1} 2^p < n \leqq \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)| = o_x\{1\} \quad a.e.$$

Proof. It is enough again to prove that

(4.11)
$$M^{(8)}_{p,\theta}(x) = \max_{2^p < m \leqq 2^{p+1}} \max_{2^p < n \leqq \theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^p}(x)| = o_x\{1\} \quad \text{a.e.}$$

We use the trivial estimate

$$M^{(8)}_{p,\theta}(x) \leqq \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} |\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)|,$$

whence, by the Cauchy inequality,

$$[M^{(8)}_{p,\theta}(x)]^2 \leqq (2\theta - 1) \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} mn[\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x)]^2.$$

On the basis of the representation

$$\sigma_{mn}(x) - \sigma_{m-1,n}(x) - \sigma_{m,n-1}(x) + \sigma_{m-1,n-1}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)(k-1)}{m(m-1)n(n-1)} a_{ik} \varphi_{ik}(x),$$

we can conclude that

$$\sum_{p=0}^{\infty} \int [M^{(8)}_{p,\theta}(x)]^2 \, d\mu(x) \leqq$$

$$\leqq (2\theta - 1) \sum_{p=0}^{\infty} \sum_{m=2^p+1}^{2^{p+1}} \sum_{n=2^p+1}^{\theta 2^{p+1}} mn \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{(i-1)^2(k-1)^2}{m^2(m-1)^2 n^2(n-1)^2} a_{ik}^2 \leqq$$

$$\leqq (2\theta - 1)^2 \sum_{p=0}^{\infty} \frac{1}{2^{4p}} \sum_{i=1}^{2^{p+1}} \sum_{k=1}^{\theta 2^{p+1}} (i-1)^2 (k-1)^2 a_{ik}^2 =$$

$$= (2\theta - 1)^2 \sum_{i=2}^{\infty} \sum_{k=2}^{\infty} (i-1)^2 (k-1)^2 a_{ik}^2 \sum_{p:\, 2^{p+1} \geqq \max(i, k/\theta)} \frac{1}{2^{4p}} < \infty.$$

Applying B. Levi's theorem, we get (4.11).

Since the relation

$$\max_{2^p < m \leqq 2^{p+1}} \max_{\theta^{-1} 2^p < n \leqq 2^p} |\sigma_{mn}(x) - \sigma_{m,2^p}(x) - \sigma_{2^p,n}(x) + \sigma_{2^p,2^q}(x)| = o_x\{1\} \quad \text{a.e.}$$

can be similarly proved, this completes the proof of Lemma 4.

Proof of Theorem 1. We can estimate in the following way: for $2^p < m \leqq 2^{p+1}$ and $\theta^{-1} \leqq n/m \leqq \theta$ $(p=0, 1, \ldots)$ we have

$$|\sigma_{mn}(x)-f(x)| \leqq |s_{2^p,2^p}(x)-f(x)|+|\sigma_{2^p,2^p}(x)-s_{2^p,2^p}(x)|+ \\ +M_{p,1}^{(3)}(x)+M_{p,\theta}^{(4)}(x)+M_{p,\theta}^{(7)}(x).$$

Now, we have to collect (4.1), (4.4), (4.5), (4.6) and (4.10) in order to obtain the statement of Theorem 1.

Proof of Theorem 2. It is quite similar to that of Theorem 1. Relying on Lemmas 2, 3 and 4, it is enough to prove the next

Lemma 5. *Under condition* (1.2),

$$s_{2^p,2^p}(x) = \sigma_x\{\log(p+2)\} \quad a.e.$$

Proof of Lemma 5. We will prove the following more general proposition: Whatsoever the monotonic sequence $\{Q_p\colon p=0, 1,\}$ of finite regions in $\mathbf{N}^2$ is, under condition (1.2) we have

$$s_p(Q;\, x) = o_x\{\log(p+2)\} \quad \text{a.e.} \tag{4.12}$$

(cf. the notation before Theorem G above).

To this effect, let us set

$$A_r = \Big(\sum_{(i,k)\in Q_r\setminus Q_{r-1}} a_{ik}^2\Big)^{1/2} \quad (r=0, 1, \ldots;\ Q_{-1}=\emptyset)$$

and

$$\Phi_r(x) = \begin{cases} \dfrac{1}{A_r}\displaystyle\sum_{(i,k)\in Q_r\setminus Q_{r-1}} a_{ik}\varphi_{ik}(x) & \text{if } A_r \neq 0, \\ \dfrac{1}{|Q_r\setminus Q_{r-1}|^{1/2}}\displaystyle\sum_{(i,k)\in Q_r\setminus Q_{r-1}} \varphi_{ik}(x) & \text{if } A_r = 0, \end{cases}$$

where by $|Q_r\setminus Q_{r-1}|$ we denote the number of the lattice points of $\mathbf{N}^2$ contained in $Q_r\setminus Q_{r-1}$.

It is clear that $\{\Phi_r(x)\colon r=0, 1, \ldots\}$ is an ONS, and by (1.2)

$$\sum_{r=0}^{\infty} A_r^2 = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} a_{ik}^2 < \infty.$$

By the classical Rademacher estimate (see, e.g. [2, p. 83]),

$$\sum_{r=0}^{p} A_r\Phi_r(x) = o_x\{\log(p+2)\} \quad \text{a.e.}$$

But this is equivalent to (4.12) since

$$s_p(Q;\, x) = \sum_{r=0}^{p} A_r\Phi_r(x) \quad (p=0, 1, \ldots).$$

5. Proofs of Theorems 3 and 4. We begin with the following

Lemma 6. *Under condition* (1.2), *for every* $\theta \geqq 1$

$$A_{m,\theta}(x) = \left\{\frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} [s_{ik}(x) - \sigma_{ik}(x)]^2\right\}^{1/2} = o_x\{1\} \quad a.e. \tag{5.1}$$

Proof. Our first aim is to show that the function $F_\theta(x)$ defined by

$$F_\theta(x) = \left\{\sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} [s_{mn}(x) - \sigma_{mn}(x)]^2\right\}^{1/2}$$

belongs to L^2. To this end, we use the representation

$$s_{mn}(x) - \sigma_{mn}(x) = \sum_{i=1}^{m} \sum_{k=1}^{n} \left(\frac{i-1}{m} + \frac{k-1}{n} - \frac{(i-1)(k-1)}{mn}\right) a_{ik}\varphi_{ik}(x)$$

and estimate the termwise integrated series from above as follows

$$\int F_\theta^2(x)\, d\mu(x) \leqq 3 \sum_{m=1}^{m} \frac{1}{m^2} \sum_{n=\theta^{-1}m}^{\theta m} \sum_{i=1}^{m} \sum_{k=1}^{n} \left(\frac{(i-1)^2}{m^2} + \frac{(k-1)^2}{n^2} + \frac{(i-1)^2(k-1)^2}{m^2 n^2}\right) a_{ik}^2 =$$

$$= 3(I_4 + I_5 + I_6), \quad \text{say.}$$

Performing elementary steps, by (1.2)

$$I_4 \leqq (\theta - \theta^{-1} + 1) \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{i=2}^{m} \sum_{k=1}^{\theta m} (i-1)^2 a_{ik}^2 =$$

$$= (\theta - \theta^{-1} + 1) \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} (i-1)^2 a_{ik}^2 \sum_{m:\, m \geqq \max(i,\, k/\theta)} \frac{1}{m^3} < \infty.$$

Similarly,

$$I_5 \leqq \theta^2(\theta - \theta^{-1} + 1) \sum_{i=1}^{\infty} \sum_{k=2}^{\infty} (k-1)^2 a_{ik}^2 \sum_{m:\, m \geqq \max(i,\, k/\theta)} \frac{1}{m^3} < \infty.$$

And, finally, $I_6 \leqq I_4$.

Now, if we apply the well-known Kronecker lemma (see, e.g. [2, p. 72]) we come to (5.1).

Proof of Theorem 3. By assumption, for every $\theta \geqq 1$, $\sigma_{mn}(x)$ converges to $f(x)$ a.e. as $m, n \to \infty$ and $\theta^{-1} \leqq n/m \leqq \theta$. Consequently,

$$B_{m,\theta}(x) = \left\{\frac{1}{m} \sum_{i=1}^{m} \frac{1}{i} \sum_{k=\theta^{-1}i}^{\theta i} [\sigma_{ik}(x) - f(x)]^2\right\}^{1/2} \to 0 \quad \text{a.e.} \tag{5.2}$$

(Here we cannot guarantee the existence of a function $F_\theta(x) \in L^2$ such that $B_{m,\theta}(x) \leqq$ $\leqq F_\theta(x)$ a.e. for every $m = 1, 2, \ldots$.)

If we take into consideration the triangle inequality

$$(5.3)\qquad \left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}[s_{ik}(x)-f(x)]^2\right\}^{1/2} \leqq A_{m,\theta}(x)+B_{m,\theta}(x),$$

then (5.1) and (5.2) imply (3.4) to be proved. The additional statement in Theorem 3 also easily follows from (5.3).

Proof of Theorem 4. This time we rely on the following inequality:

$$(5.4)\qquad \left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}s_{ik}^2(x)\right\}^{1/2} \leqq A_{m,\theta}(x)+C_{m,\theta}(x),$$

where

$$C_{m,\theta}(x)=\left\{\frac{1}{m}\sum_{i=1}^{m}\frac{1}{i}\sum_{k=\theta^{-1}i}^{\theta i}\sigma_{ik}^2(x)\right\}^{1/2}.$$

By Theorem 2,

$$C_{m,\theta}(x)=o_x\{\log\log(m+3)\}\quad \text{a.e.}$$

Referring again to (5.1), (5.4) implies both statements of Theorem 4.

6. On the sharpness of Theorems 1 and 2. FEDULOV [5] showed that Theorem C is the best possible in the following sense. Let $\{\varepsilon(m)\colon m=1,2,\ldots\}$ be a nonincreasing sequence of positive numbers tending to 0 as $m\to\infty$. Then there exist a double ONS $\{\varphi_{ik}(x)\}$ on the unit square I^2 and a double sequence $\{a_{ik}\}$ of coefficients such that

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}a_{ik}^2\varepsilon(\min(i,k))[\log\log(i+3)]^2[\log\log(k+3)]^2<\infty$$

and

$$\limsup_{m,n\to\infty}|\sigma_{mn}(x)|=\infty\quad\text{a.e. on}\quad I^2.$$

(In [5] the formulation is somewhat different from ours.)

Theorem D is also exact in general. It was pointed out in [11] that, given any sequence $\{\varepsilon(m)\}$ with the properties indicated just above, there exist a double ONS $\{\varphi_{ik}(x)\}$ on I^2 and a double sequence $\{a_{ik}\}$ of coefficients such that condition (1.2) is satisfied and

$$\limsup_{m,n\to\infty}\frac{|\sigma_{mn}(x)|}{\varepsilon(\min(m,n))\log\log(m+3)\log\log(n+3)}=\infty\quad\text{a.e. on}\quad I^2.$$

Now we are going to add the following supplement. Theorems 1 and 2 are the best possible even in the very special case $\theta=1$ (i.e. $m=n$). Indeed, given a sequence $\{\varepsilon(m)\}$ with the above properties, there exist a double ONS $\{\varphi_{ik}(x)\}$ on the unit interval I and a sequence $\{a_{ik}: a_{ik}=0 \text{ for } i\neq k\}$ such that

$$\sum_{i=1}^{\infty} a_{ii}^2 \varepsilon(i)[\log\log(i+3)]^2 < \infty$$

and

$$\limsup_{m\to\infty} |\sigma_{mm}(x)| = \infty \quad \text{a.e. on} \quad I.$$

Similarly, there exist possible another double ONS $\{\varphi_{ik}(x)\}$ on I and a double sequence $\{a_{ik}: a_{ik}=0 \text{ for } i\neq k\}$ such that condition (1.2) is satisfied and

$$\limsup_{m\to\infty} \frac{|\sigma_{mm}(x)|}{\varepsilon(m)\log\log(m+3)} = \infty \quad \text{a.e. on} \quad I.$$

The last two counterexamples can be constructed with the help of the "one-dimensional" counterexamples of MENŠOV [6] and TANDORI [13, Theorem 8], respectively. The only important modification is that now we need an infinite number of "indifferent" orthonormal functions at our disposal in order to place them for $\varphi_{ik}(x)$ with $i\neq k$ $(i, k=1, 2, \dots)$ (and these functions do not play any role later on because for different i and k all the coefficients a_{ik} are chosen to equal 0). On the other hand, the orthonormal functions themselves occurring in the corresponding counterexamples of Menšov and Tandori are used in the capacity of $\varphi_{ii}(x)$ $(i=1, 2, \dots)$. Since the latter functions are step ones, this contruction can be carried out without any difficulty. We do not enter into further details.

On closing we remark that the results of the present paper can be extended, without any essential modification, to the case of d-multiple orthogonal series as well $(d=3, 4, \dots)$.

Note added in proof. Questions (3.6) and (3.7) are studied in another paper of mine: On the strong summability of double orthogonal series, *Michigan Math. J.*, **31** (1984), 241—255.

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Representation of functionals via summability methods. II

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Dedicated to Professor L. Leindler on his 50th birthday

1. Introduction

This article is a direct continuation of the paper [4]. There we showed that if K is a metrizable compact space and $C(K)$ is the sup-normed Banach-space of all real valued continuous functions on K, then to every $L\in C^*(K)$ there are sequences $\{c_k\}\in l^\infty$ and $\{x_k\}\subseteq K$ such that for every $f\in C(K)$

$$Lf = \lim_{n\to\infty} (1/n)(c_1 f(x_1) + \ldots + c_n f(x_n)) \tag{1.1}$$

holds. We proved also that every positive linear functional L with norm 1 (shortly $PL1$ functional) has the form

$$Lf = \lim_{n\to\infty} (1/n)(f(x_1) + \ldots + f(x_n)) \tag{1.2}$$

with a suitable sequence $\{x_k\}\subseteq K$.

Extensions to subadditive functionals by replacing lim with limsup were also treated. Using the language of [4] we call a functional on a certain space which has the form (1.1) or (1.2) a weighted $(C, 1)$-functional or a $(C, 1)$-functional, respectively.

Here, in Section 2, we show that these results can be extended to $Q[0, 1]$, the space of functions having discontinuities only of the first kind, and that $Q[0, 1]$ is maximal, in a certain sense, among spaces having this representability property. In Section 3 we determine those functionals of $R[0, 1]$, the space of Riemann-integrable functions, which have the form (1.1) and Section 5 contains an application of this result to density measures: we give all finitely additive measures which can be obtained as the density of a certain sequence in $\mathbf{R}^n$. Finally, in Section 4 we solve the problem: by which summability methods can we replace the arithmetical mean method (i.e. the $(C, 1)$-method) in (1.1) and (1.2)?

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2. The space $Q[0, 1]$

Let $Q[0, 1]$ be the sup-normed real Banach space of bounded functions defined on $[0, 1]$ having discontinuities only of the first kind, i.e. $f \in Q[0, 1]$ if and only if

$$f(x+0) = \lim_{y \to x+0} f(y), \quad f(x-0) = \lim_{y \to x-0} f(y),$$
$$f(1+0) \stackrel{\text{def}}{=} f(1), \quad f(0-0) \stackrel{\text{def}}{=} f(0)$$

exist at every point $x \in [0, 1]$. It is an easy task to prove that $Q[0, 1]$ is exactly the uniform closure of the set of step functions. KALTENBORN [1] determined the dual space $Q^*[0, 1]$ by the aid of a certain generalized Stieltjes-integral.

Now we shall show that on $Q[0, 1]$ every $PL1$ functional is a $(C, 1)$-functional and that there is no larger "natural" space with this property.

Theorem 1. *On* $Q[0, 1]$ *every* $PL1$ *functional* L *has the form* (1.2) *with a suitable sequence* $\{x_k\}$.

This yields at once

Corollary 1. *Every* $L \in Q^*[0, 1]$ *has the form* (1.1).

Note that $Q[0, 1]$ is far from being separable.

Now let B be a sup-normed space of bounded functions defined on $[0, 1]$ which is closed under substitution of continuously differentiable homeomorphisms of $[0, 1]$, i.e. if $\varphi\colon [0, 1] \to [0, 1]$ is a strictly increasing continuously differentiable function with $\varphi(0)=0$, $\varphi(1)=1$ and $f \in B$ then $f \circ \varphi \in B$. Such spaces are $C[0, 1]$; $Q[0, 1]$; $R[0, 1]$ — the set of all Riemann-integrable functions; the space of left continuous functions, etc. We shall show that $Q[0, 1]$ is maximal among such spaces having the $(C, 1)$-representability property.

Theorem 2. *Let* B *be as above. If* $Q \subset B$ *then for some* $x_0 \in [0, 1]$ *no extension of the functional*

$$L_{x_0} f = (f(x_0-0)+f(x_0+0))/2 \quad (f \in Q[0, 1])$$

to B *is a weighted* $(C, 1)$*-functional.*

Proof of Theorem 1. Let $L \in Q^*[0, 1]$ be a $PL1$ functional and set

$$H_1 = \{x \mid L\chi_{\{x\}} > 0\},$$
$$H_2 = \left\{x \mid \lim_{\varepsilon \to 0+0} L\chi_{(x, x+\varepsilon)} > 0\right\}, \quad H_3 = \left\{x \mid \lim_{\varepsilon \to 0+0} L\chi_{(x-\varepsilon, x)} > 0\right\}.$$

(χ_A denotes the characteristic function of the set A.) Since L is bounded, these sets are countable say, $H_1 = \{y_k^{(1)}\}$, $H_2 = \{y_k^{(2)}\}$, $H_3 = \{y_2^{(3)}\}$, and if

$$\tau_k^{(1)} = L\chi_{\{y_k^{(1)}\}}, \quad \tau_k^{(2)} = \lim_{\varepsilon \to 0} L\chi_{(y_k^{(2)}, y_k^{(2)}+\varepsilon)}, \quad \tau_k^{(3)} = \lim_{\varepsilon \to \infty} L\chi_{(y_k^{(3)}-\varepsilon, y_k^{(3)})},$$

then for the numbers

$$\mu_1 = \sum_k \tau_k^{(1)}, \quad \mu_2 = \sum_k \tau_k^{(2)}, \quad \mu_3 = \sum_k \tau_k^{(3)}$$

we have $\mu_1+\mu_2+\mu_3 \leqq \|L\| = 1$. An easy consideration shows that for

$$L_1 f = (1/\mu_1) \sum_k \tau_k^{(1)} f(y_k^{(1)}), \quad L_2 f = (1/\mu_2) \sum_k \tau_k^{(2)} f(y_k^{(2)}+0),$$

$$L_3 f = (1/\mu_3) \sum_k \tau_k^{(3)} f(y_k^{(3)}-0) \quad (f \in Q[0, 1])$$

the functional $L^* = L - \mu_1 L_1 - \mu_2 L_2 - \mu_3 L_3$ is a positive functional with norm $\mu_4 \overset{\text{def}}{=} 1-\mu_1-\mu_2-\mu_3$. Let

$$L_4 f = (1/\mu_4) L^* f \quad (f \in Q[0, 1]).$$

By our construction

$$L_4 \chi_{\{x\}} = \lim_{\varepsilon \to 0} L_4 \chi_{(x \pm \varepsilon, x)} = 0 \quad (x \in [0, 1]),$$

therefore the function

$$\alpha(x) = L_4 \chi_{[0, x]} \tag{2.1}$$

is a continuous and increasing function. Exactly as in the proof of [4, Corollary 3] it can be proved that if $\{z_k\} \subseteqq [0, 1]$ is an arbitrary dense sequence then there exists a sequence $\{x_i\}$ such that with the notation

$$\sigma_n(\{x_i\}, f) = (1/n)(f(x_1)+\dots+f(x_n))$$

we have

$$\alpha(z_k) = \lim_{n \to \infty} \sigma_n(\{x_i\}, \chi_{[0, z_k]})$$

for every k. By the monotonicity and continuity of α,

$$\alpha(x) = \lim_{n \to \infty} \sigma_n(\{x_i\}, \chi_{[0, x]}) \tag{2.2}$$

also holds for every $x \in [0, 1]$, and since the set of step functions is dense in $Q[0, 1]$ we can conclude by (2.1) and (2.2) that

$$L_n f = \lim_{n \to \infty} \sigma_n(\{x_i\}, f)$$

for every $f \in Q[0, 1]$, i.e. L_4 is a $(C, 1)$-functional.

Since it is easy to verify that L_1, L_2 and L_3 are also $(C, 1)$-functionals and since $L = \mu_1 L_1 + \dots + \mu_4 L_4$, $\mu_1 + \dots + \mu_4 = 1$, the theorem follows by a familiar argument (cf. [4]).

Also, the proof of Corollary 1 is standard (cf. [4]).

Proof of Theorem 2. If $Q \subset B$ then there exists a function f which does not have e.g. right hand limit at a certain point x_0. Let $\tilde{L}$ be any extension of L_{x_0}

to B, and let us suppose on the contrary that $\tilde{L}$ is represented in the sense of (1.1) by the sequences $\{c_k\}$, $\{x_k\}$. The idea is to construct a function in B by the aid of f for which the limit in (1.1) does not exist.

We shall only sketch the proof. By linearity we may suppose that there are sequences $1=u_1>v_1>u_2>v_2>\ldots>x_0$ converging to x_0 with

$$\lim_{k\to\infty} f(u_k) = 1, \quad \lim_{k\to\infty} f(v_k) = 0.$$

For the sake of convenience we shall use the notation

$$\sigma_n(g) = (1/n)\big(c_1 g(x_1)+\ldots+c_n g(x_n)\big)$$

in the rest of the proof.

Since for every $\varepsilon>0$ $(\varepsilon<1-x_0)$ we have $\tilde{L}\chi_{(x_0,x_0+\varepsilon)}=1/2$, $\tilde{L}\chi_{(x_0+\varepsilon,1]}=0$, the sequences $\{n_j\}$, $\{\varepsilon_j\}$, $\{x_i^{(j)}\}_{i=1}^{k_j}$, $\{c_i^{(j)}\}_{i=1}^{k_j}$ and $\eta_j\to 0$ can be determined successively according to the requirements:

$$\sigma_{n_1}(\chi_{(x_0,1)}) = 1/2+\eta_1, \quad |\eta_1| < 1/2, \quad \varepsilon_1 = \min_{\substack{1\leqq k\leqq n_1\\ x_k>x_0}} (x_k-x_0),$$

$$\{x_i^{(1)}\}_{i=1}^{k_1} = \{x_k |\, 1 \leqq k \leqq n_1,\ x_k > x_0\},$$

and let $c_i^{(1)}$ $(1\leqq i\leqq k_1)$ be the corresponding constants (i.e. if $x_i^{(1)}=x_\nu$ then let $c_i^{(1)}=c_\nu$);

$$\sigma_{n_2}(\chi_{(x_0,\varepsilon_1)}) = 1/2+\eta_2, \quad |\eta_2| < 1/4, \quad \varepsilon_2 = \min_{\substack{1\leqq k\leqq n_2\\ x_k>x_0}} (x_k-x_0),$$

$$\{x_i^{(2)}\}_{i=1}^{k_2} = \{x_k \,|\, 1 \leqq k \leqq n_2,\quad x_0 < x_k < x_0+\varepsilon_1\}$$

and $\{c_i^{(2)}\}_{i=1}^{k_2}$ the set of the corresponding constants, and so on. We may assume as well that $(k_1+\ldots+k_i)/n_{i+1}\to 0$ as $i\to\infty$.

Now let

$$\varphi_1, \varphi_2\colon \bigcup_{j=1}^{\infty} \{x_i^{(j)}\}_{i=1}^{k_j} \to \{u_k\}_{k=1}^{\infty} \cup \{v_k\}_{k=1}^{\infty}$$

be $1-1$, monotonically increasing mappings with the properties:

$$\left.\begin{aligned} &\varphi_1(x_i^{(2j-1)}) = \varphi_2(x_i^{(2j-1)}) \in \{u_k\}_{k=1}^{\infty}, \quad 1 \leqq i \leqq k_{2j-1}, \\ &\varphi_1(x_i^{(2j)}) \in \{u_k\}_{k=1}^{\infty}, \quad \varphi_2\{x_i^{(2j)}\} \in \{v_k\}_{k=1}^{\infty}, \quad 1 \leqq i \leqq k_{2j}, \end{aligned}\right\} \quad j = 1, 2, \ldots,$$

$$\big(\varphi_\tau(x_i^{(j)})-x_0\big)/(x_i^{(j)}-x_0) = o(1) \quad (\tau = 1, 2)$$

as $j\to\infty$ uniformly in $1\leqq i\leqq k_j$. φ_1 and φ_2 can be extended to continuously differentiable homeomorphisms of $[0, 1]$ with $\varphi_1'(x_0)\equiv\varphi_2'(x_0)$ and $\varphi_1(x)\equiv\varphi_2(x)$ for $x\in[0, x_0]$.

The construction gives

$$\sigma_{n_{2j+1}}(f\circ\varphi_1-f\circ\varphi_2)=\sigma_{n_{2j+1}}(\chi_{(x_0,1)}f\circ\varphi_1-\chi_{(x_0,1)}f\circ\varphi_2)=$$

$$=o(1)+1/n_{2j+1}\sum_{i=1}^{k_{2j+1}}c_i^{(2j+1)}\big(f(\varphi_1(x_i^{(2j+1)}))-f(\varphi_2(x_i^{(2j+1)}))\big)=o(1)+0=o(1)$$

and

$$\sigma_{n_{2j}}(f\circ\varphi_1-f\circ\varphi_2)=o(1)+\big(1+o(1)\big)\sigma_{n_{2j}}\big(\chi_{(x_0,x_0+\varepsilon_{2j-1})}(f\circ\varphi_1-1)\big)-$$

$$-\big(1+o(1)\big)\sigma_{n_{2j}}(\chi_{(x_0,x_0+\varepsilon_{2j-1})}f\circ\varphi_2)+\big(1+o(1)\big)\sigma_{n_{2j}}(\chi_{(x_0,x_0+\varepsilon_{2j-1})})=$$

$$=o(1)+o(1)+o(1)+\big(1+o(1)\big)\big(1/2+o(1)\big),$$

i.e. either for $f\circ\varphi_1\in B$ or for $f\circ\varphi_2\in B$ the limit on the right of (1.1) does not exist, which contradicts our assumption concerning the sequences $\{c_k\}$, $\{x_k\}$.

3. The space $\mathscr{R}[0, 1]$

Let $\mathscr{R}[0, 1]$ denote the space of Riemann-integrable bounded functions defined on $[0, 1]$. We equip $\mathscr{R}[0, 1]$ with the sup norm. By Theorem 2 $\mathscr{R}[0, 1]$ has bounded linear functionals which are not weighted $(C, 1)$-functionals. In the present section we characterize the (weighted) $(C, 1)$-functionals of $\mathscr{R}[0, 1]$. An application to density measures will be given in the last section.

Theorem 3. *A functional $L\in\mathscr{R}^*[0, 1]$ is a weighted $(C, 1)$-functional (i.e. it has form* (1.1)*) if and only if L is of the form*

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i)+\int_0^1 f(t)g(t)\,dt\quad (f\in\mathscr{R}[0, 1]),$$

where $\tau_i\in[0, 1]$ $(1\leqq i)$, $\sum_{i=1}^{\infty}|\mu_i|<\infty$ *and* $g\in L^1[0, 1]$.

Corollary 2. *A PL1 functional $L\in\mathscr{R}^*[0, 1]$ is a $(C, 1)$-functional (i.e. it has form* (1.2)*) if and only if there are* $\tau_i\in[0, 1]$, $\mu_i\geqq 0$ $(1\leqq i)$, $g\in L^1[0, 1]$, $g\geqq 0$ *such that*

$$\int_0^1 g(t)\,dt=1,\quad \sum_{i=1}^{\infty}\mu_i\leqq 1,$$

and for every $f\in\mathscr{R}[0, 1]$

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i)+\big(1-\sum_{i=1}^{\infty}\mu_i\big)\int_0^1 f(t)g(t)\,dt.$$

Proof. First we prove the necessity part of Theorem 3. Let us call a point $x\in[0, 1]$ a singular point of L if $L\chi_{\{x\}}\neq 0$, and a functional having the form

$$Lf=\sum_{i=1}^{\infty}\mu_i f(\tau_i),\quad \sum|\mu_i|<\infty,$$

will be called a discrete functional. First we show

Lemma 1. *For every $L\in\mathscr{R}^*[0, 1]$ the set of singular points is countable and $L=L_1+L_2$ where L_1 is a discrete functional and L_2 is without singular points.*

Proof. Since for arbitrary points $x_1, \dots, x_n$ we have

$$\Big|\sum_{i=1}^{n}\pm L\chi_{\{x_i\}}\Big|=\Big|L\sum_{i=1}^{n}\pm\chi_{\{x_i\}}\Big|\leqq\|L\|,$$

there are at most countably many singular points of L. Let they be $\tau_1, \tau_2, \dots$. The previous inequality shows that the numbers $\mu_i=L_{\{\tau_i\}}$ satisfy $\sum_{i=1}^{\infty}|\mu_i|\leqq$ $\leqq\|L\|$. Now

$$L_1 f=\sum_{i=1}^{\infty}\mu_i f(\tau_i)\quad\text{and}\quad L_2=L-L_1$$

clearly satisfy the requirements of the lemma.

We need also another lemma.

Lemma 2. *If $L\in\mathscr{R}^*[0, 1]$ is a weighted $(C, 1)$-functional without singular points then the function $\alpha(x)=L\chi_{[0,x]}$ $(x\in[0, 1])$ is absolutely continuous.*

Proof. If $0=w_0<w_1<\dots<w_n=1$ are arbitrary points then for certain signs $+, -$ we have

$$\sum_{i=0}^{n-1}|\alpha(w_{i+1})-\alpha(w_i)|=L\Big(\pm\chi_{[w_0, w_1]}+\sum_{i=1}^{n-1}\pm\chi_{(w_i, w_{i+1}]}\Big)\leqq\|L\|,$$

i.e. α is of bounded variation. We show first that α is continuous.

Let us suppose on the contrary that α is not continuous at the point x. Then either $\alpha(x+0)\neq\alpha(x)$ or $\alpha(x-0)\neq\alpha(x)$, let us consider e.g. the former case. If e.g. $\alpha(x+0)>\alpha(x)$ then there are constants $\varepsilon>0$, $\delta>0$ such that for $x<y<x+\delta$ we have $\alpha(y)-\alpha(x)>\varepsilon$. Since L is a weighted $(C, 1)$-functional, there are sequences $\{c_i\}$, $\{x_i\}$ such that

$$\lim_{n\to\infty}\sigma_n(f)\overset{\text{def}}{=}\lim_{n\to\infty}(1/n)\big(c_1 f(x_1)+\dots+c_n f(x_n)\big)=Lf \tag{3.1}$$

holds for every $f\in\mathscr{R}[0, 1]$.

Let $x<y_1<x+\delta$ be arbitrary. By our assumptions there is an n_1 such that $\sigma_{n_1}(\chi_{(x,y_1]})>\varepsilon$. Let $z_1, \dots, z_{k_1}$ be those x_i's for which $x_i \in (x, y_1]$, $1 \leqq i \leqq n_1$. (For the sake of easier printing, in subscripts we shall write $\{z_i; \nu, \mu\}$ for $\{z_i\}_{i=\nu}^{\mu}$.) Then we have also $\sigma_{n_1}(\chi_{\{z_i;1,k_1\}})>\varepsilon$. L is without singular points, therefore there exists an N_1 with

$$\sigma_n(\chi_{\{z_i;1,k_1\}}) < \varepsilon/4 \quad \text{for} \quad n \geqq N_1.$$

After this let $x<y_2<x+\delta/2$ be such that

$$y_2 - x < \min_{\substack{1 \leqq i \leqq N_1 \\ x_i > x}} (x_i - x)$$

is satisfied. Again there is an $n_2>N_1$ with

$$\sigma_{n_2}(\chi_{(x,y_2]}) > \varepsilon$$

and if $z_{k_1+1}, \dots, z_{k_2}$ are the x_i's for which $x_i \in (x, y_2]$, $1 \leqq i \leqq n_2$ then there is an N_2 such that for $n \geqq N_2$ we have

$$\sigma_n(\chi_{\{z_i;1,k_2\}}) < \varepsilon/4.$$

Repeating this argument we obtain a sequence $\{z_i\}_{i=1}^{\infty}$ converging to x and sequences $\{n_j\}_{j=1}^{\infty}$, $\{N_j\}_{j=1}^{\infty}$ such that

$$\sigma_{n_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{n_j}(\chi_{\{z_i;k_{j-1}+1,k_j\}}) + \sigma_{n_j}(\chi_{\{z_i;1,k_{j-1}\}}) \geqq \varepsilon - \varepsilon/4 = 3\varepsilon/4$$

while

$$\sigma_{N_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{N_j}(\chi_{\{z_i;1,k_j\}}) < \varepsilon/4$$

i.e.

$$\lim_{n\to\infty} \sigma_n(\chi_{\{z_i,1,\infty\}})$$

does not exist, which is a contradiction since $\chi_{\{z_i;1,\infty\}}$ is Riemann-integrable.

The absolute continuity of $\alpha(x)$ will be proved by a similar argument. Let α be the signed Borel measure associated with $\alpha(x)$ (cf. [3, p. 173]), in the sense

$$\alpha([0, x]) = \alpha(x),$$

and let α^+, α^- and $|\alpha|=\alpha^+ + \alpha^-$ be the positive and negative parts and the total variation of α, respectively (cf. [3, pp. 134, 125]). We have to prove that α is absolutely continuous with respect to the Lebesgue measure. Suppose not. Then either α^+ or α^- is not absolutely continuous, let us consider e.g. the first case. Since α^+ and α^- have disjoint (not necessarily compact) supports and the singular part of α^+ does not vanish, the regularity of α^+ and α^- yields a closed set $H_0 \subseteqq [0, 1]$ with Lebesgue measure zero and constants ε, $\delta_0, \delta_1, \ldots > 0$, $\delta_n \to 0$, such that

$$\alpha^+(H_0) > \varepsilon, \quad \alpha^-(H_0^{\delta_n}) < \varepsilon/4^{n+1} \quad (n = 0, 1, \ldots)$$

are satisfied where

$$H_0^{\delta} = \{x \mid \operatorname{dist}(x, H_0) < \delta\}.$$

Let $\eta_0=\delta_0$. We have $\alpha(H_0^{\eta_0})>\varepsilon-\varepsilon/4$ and since $H_0^{\eta_0}$ is the union of finitely many intervals we obtain together with this also that

$$L\chi_{H_0^{\eta_0}} > \varepsilon-\varepsilon/4,$$

by which

$$\sigma_{n_0}(\chi_{H_0^{\eta_0}}) > \varepsilon-\varepsilon/4$$

for some n_0. Let $z_1, ..., z_{k_0}$ be those x_i's for which $1\leqq i\leqq n_0$ and $x_i\in H_0^{\eta_0}$ are satisfied. Since L has no singular point, there exists N_0 such that for every $n\geqq N_0$ we have

$$\sigma_n(\chi_{\{z_i;\,1,k_0\}}) < \varepsilon/8.$$

Since α is continuous we have $|\alpha|(\{x\})=0$ for every x and the regularity of the measure $|\alpha|$ yields that we can choose disjoint closed intervals $U_1, ..., U_{N_0}$ around the points $x_1, ..., x_{N_0}$ in such a way that $|\alpha|(\bigcup_{i=1}^{N_0} U_i)<\varepsilon/16$ is satisfied. Let $x_i\in U_i'\subseteqq U_i$ be open intervals without common endpoints with U_i and

$$H_1 = H_0/(\bigcup_{i=1}^{N_0} U_i').$$

If $\eta_1>0$ is less than δ_1 and less than the distances between the endpoints of the U_i''s and U_i's and also less than the distances of the endpoints of U_i''s from x_i's then we have

$$H_1^{\eta_1} \supseteqq H_0^{\eta_1}/[H_0\cap(\bigcup_{i=1}^{N_0} U_i')]^{\eta_1} \supseteqq H_0^{\eta_1}/\bigcup_{i=1}^{N_1} U_i,$$

$$\alpha^+(H_1^{\eta_1}) \geqq \alpha^+(H_0^{\eta_1})-\alpha^+(\bigcup_{i=1}^{N_1} U_i) \geqq \varepsilon-\varepsilon/16, \quad \alpha^-(H_1^{\eta_1}) \leqq \alpha^-(H_1^{\eta_1}) \leqq \varepsilon/16,$$

and hence

$$\alpha(H_1^{\eta_1}) = L\chi_{H_1^{\eta_1}} > \varepsilon-\varepsilon/8.$$

There exists an $n_1>N_0$ with

$$\sigma_{n_1}(\chi_{H_1^{\eta_1}}) > \varepsilon-\varepsilon/8,$$

and if $z_{k_0+1}, ..., z_{k_1}$ are the points x_i for which $x_i\in H_1^{\eta_1}$, $1\leqq i\leqq n_1$ then we have $z_i\notin H_0$, $z_i\neq z_j$ for $1\leqq j\leqq k_0$, $k_0+1\leqq i\leqq k_1$;

$$\sigma_{n_1}(\chi_{\{z_i;\,1,k_1\}}) = \sigma_{n_1}(\chi_{\{z_i;\,k_0+1,k_1\}})+\sigma_{n_1}(\chi_{\{z_i;\,1,k_0\}}) \geqq \varepsilon-\varepsilon/8-\varepsilon/8$$

and

$$\sigma_n(\chi_{\{z_i;\,1,k_1\}}) < \varepsilon/8 \quad \text{for} \quad n\geqq N_1$$

for some N_1. If $U_1^*, ..., U_{N_1}^*$ are disjoint closed intervals around $x_1, ..., x_{N_1}$ with $|\alpha|(\bigcup_{i=1}^{N_1} U_i^*)<\varepsilon/32$ and $x_i\in U_i''\subseteqq U_i^*$, U_i'' open, $H_2=H_1\setminus(\bigcup_{i=1}^{N_1} U_i'')$ then exactly as above we obtain for small $\eta_2>0$,

$$\alpha(H_2^{\eta_2}) = L\chi_{H_2^{\eta_2}} > \varepsilon-\varepsilon/8-\varepsilon/16.$$

Repeating this argument we obtain sequences $\{z_i\}_{i=1}^{\infty}$, $\{H_j\}_{j=1}^{\infty}$, $\{n_j\}_{j=1}^{\infty}$, $\{N_j\}_{j=1}^{\infty}$, $\{k_j\}_{j=1}^{\infty}$ such that $H_{j+1}\subseteqq H_j$ closed, $z_i\notin H_j$ for $i\leqq k_{j-1}$, $z_i\neq z_j$ for $i\neq j$, the sequence $\{z_i\}$ may have limit points only in $H=\bigcap_{j=1}^{\infty} H_j$ and

$$\sigma_{n_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{n_j}(\chi_{\{z_i;k_{j-1}+1,k_j\}})-\sigma_{n_j}(\chi_{\{z_i;1,k_{j-1}\}}) \geqq \geqq (\varepsilon-\varepsilon/8-\varepsilon/16-\ldots)-\varepsilon/8 \geqq \varepsilon/2 \tag{3.2}$$

but

$$\sigma_{N_j}(\chi_{\{z_i;1,\infty\}}) = \sigma_{N_j}(\chi_{\{z_i;1,k_j\}}) < \varepsilon/8. \tag{3.3}$$

Since

$$\chi_{\{z_i;1,\infty\}} = \chi_{H\cup\{z_i;1,\infty\}}-\chi_H$$

and $H, H\cup\{z_i\}_{i=1}^{\infty}$ are closed and have Lebesgue measure zero, we obtain that $\chi_{\{z_i;1,\infty\}}$ is Riemann-integrable and (3.2)—(3.3) contradict our assumption concerning the convergence of $\{\sigma_n(g)\}$ for every $g\in\mathscr{R}[0, 1]$. This contradiction proves Lemma 2.

Let us return to the proof of Theorem 3. By Lemma 1 $L=L_1+L_2$ where L_1 is discrete and L_2 has no singular point. An easy argument gives that every discrete functional is a weighted $(C, 1)$-functional, so if L is assumed to be weighted $(C, 1)$-functional then L_2 is also a weighted $(C, 1)$-functional. By Lemma 2 the function $\alpha(x)=L_2\chi_{[0,x]}$ is absolutely continuous, let $g(x)=\alpha'(x)$ (a.e.). Then $g\in L^1[0, 1]$ and

$$L_2\chi_{[0,x]} = \alpha(x) = \int_0^x g(t)\,dt\,.$$

by which

$$L_2h = \int_0^1 hg \tag{3.4}$$

for every step-function h. Let $f\in\mathscr{R}^*[0, 1]$ be arbitrary, and let

$$L_2^*h = L_2(hf) \quad (h\in\mathscr{R}[0, 1]).$$

It is obvious that $L_2^*\in\mathscr{R}^*[0, 1]$ and together with L_2, L_2^* is a weighted $(C, 1)$-functional without singular points. By Lemma 2 the function

$$\alpha^*(x) = L_2^*\chi_{[0,x]} = L_2(f\chi_{[0,x]})$$

is absolutely continuous and hence to every $\varepsilon>0$ there exists a $\delta>0$ such that if H is disjoint union of finitely many intervals and m denotes the Lebesgue measure, then $m(H)<\delta$ implies

$$|L_2(f\chi_H)| < \varepsilon. \tag{3.5}$$

We may assume that

$$|L_2\chi_H| < \varepsilon, \quad \int_H |g| < \varepsilon \quad \text{for} \quad m(H) < \delta \tag{3.6}$$

are also satisfied. Since f is Riemann-integrable there are step-functions φ and Φ such that

$$\varphi \leqq f \leqq \Phi, \quad |\varphi|, |\Phi| \leqq \sup |f|, \quad \int_0^1 (\Phi-\varphi) < \varepsilon\delta.$$

Thus, if

$$H = \{x \mid \Phi(x)-\varphi(x) > \varepsilon\}$$

then $m(H)<\delta$. Let H be the disjoint union of the closed, half-closed or open intervals $\{u_1, v_1\}, \ldots, \{u_n, v_n\}$ and let $w_i \in \{u_i, v_i\}$. We may assume that φ is constant on each interval $\{u_i, v_i\}$. By (3.4)—(3.6)

$$\left|L_2 f - \int_0^1 fg\right| \leqq \left|L_2((f-\varphi)\chi_{[0,1]\setminus H})\right| + |L_2(f\chi_H)| +$$

$$+\left|\sum_{i=1}^n \varphi(w_i)\big(\alpha(v_i)-\alpha(u_i)\big)\right| + \left|L_2\varphi - \int_0^1 \varphi g\right| + \left|\,\left|\int_{[0,1]\setminus H} (f-\varphi)g\right| + \left|\int_H (f-\varphi)g\right| \leqq$$

$$\leqq \varepsilon\|L_2\| + \varepsilon + \sup|\varphi| \sum_{i=1}^n |\alpha(v_i)-\alpha(u_i)| + 0 + \varepsilon\|g\|_{L^1} + 2\sup|f| \int_H |g| \leqq$$

$$\leqq \varepsilon(\|L_2\| + \|g\|_{L^1} + 1) + 2\varepsilon \sup|f| + 2\varepsilon \sup|f| \leqq K\varepsilon$$

with a K independent of ε, by which the equality

$$L_2 f = \int_0^1 fg \quad (f \in \mathscr{R}[0,1])$$

is verified, and the necessity of our condition is proved.

The necessity of the condition in Corollary 2 follows easily from the above consideration, all what we have to mention is that, by the positivity of L and by $\|L\|=1$, we have $\mu_i \geqq 0$, $\sum_{i=1}^{\infty} \mu_i \leqq 1$, and in the case $\sum_{i=1}^{\infty} \mu_i < 1$ the derivative of $\alpha(x) = L_2\chi_{[0,x]}$ is positive because L_2 is also a positive functional (notice that for every n and $f \geqq 0$

$$Lf \geqq L\Big(\sum_{i=1}^n f(\tau_i)\chi_{\{\tau_i\}}\Big) = \sum_{i=1}^n \mu_i f(\tau_i)\Big).$$

After these let us turn to the sufficiency part of our proof. Obviously it is sufficient to prove this for Corollary 2, and since a functional of the form

$$Lf = \sum_{i=1}^{\infty} \mu_i f(\tau_i), \quad \mu_i \geqq 0, \quad \sum_{i=1}^{\infty} \mu_i = 1$$

is easily seen to be a $(C, 1)$-functional our task has reduced to the verification of the following: if

$$Lf = \int_0^1 fg, \quad (f \in \mathcal{R}[0, 1])$$

where $g \in L^1[0, 1]$, $g \geqq 0$ and $\int_0^1 g(t)\,dt = 1$ then L is a $(C, 1)$-functional.

Exactly as in the proof of [4, Corollary 3] one can give a sequence $\{x_k\}$ such that

$$\lim_{n\to\infty} \sigma_n(\{x_k\}, \chi_{[0, z_j]}) = \int_0^{z_j} g(t)\,dt \quad \left(\sigma_n(\{x_k\}, g) = (1/n) \sum_{k=1}^{n} g(x_k)\right)$$

is satisfied for a sequence $\{z_j\}$ dense in $[0, 1]$. By monotonicity and by the continuity of $\int_0^z g(t)\,dt$ we obtain the same relation for every $z \in [0, 1]$ and hence

$$\lim_{n\to\infty} \sigma(\{x_k\}, h) = \int_0^1 hg$$

for every step function h. If $f \in \mathcal{R}[0, 1]$, $\varepsilon > 0$, are arbitrary then there are step functions φ, Φ with the properties:

$$\varphi \leqq f \leqq \Phi, \quad |\varphi|, |\Phi| \leqq \sup |f|,$$

$m(H) < \varepsilon$, where $H = \{x \mid \Phi(x) - \varphi(x) \geqq \varepsilon\}$ (see above) and these yield

$$L\varphi = \int_0^1 \varphi g = \lim_{n\to\infty} \sigma_n(\{x_k\}, \varphi) \leqq \liminf_{n\to\infty} \sigma_n(\{x_k\}, f) \leqq$$

$$\leqq \limsup_{n\to\infty} \sigma_n(\{x_k\}, f) \leqq \lim_{n\to\infty} \sigma_n(\{x_k\}, \Phi) = \int_0^1 \Phi g = L\Phi,$$

$$L\varphi \leqq Lf \leqq L\Phi, \quad L\Phi - L\varphi = \int_0^1 (\Phi - \varphi) g \leqq \varepsilon \int_0^1 g + 2 \sup |f| \int_H |g|.$$

Since here the right hand side can be made arbitrary small by appropriate choice of ε, these formulas prove the convergence

$$\lim_{n\to\infty} \sigma_n(\{x_k\}, f) = Lf$$

for every $f \in \mathcal{R}[0, 1]$ and the theorem is proved.

4. Other summability methods

In this section we characterize those matrix summability methods which can be substituted in [4, Corollary 3] for the $(C, 1)$-method.

Let thus $T=(t_{n,k})_{n,k=1}^{\infty}$ be an infinite matrix. We say that T sums the sequence $\{s_k\}$ to the limit $T\text{-}\lim\limits_k s_k$ if

$$T\text{-}\lim_k s_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} s_k$$

holds. If for every convergent $\{s_k\}$ we have

$$T\text{-}\lim_k s_k = \lim_{k\to\infty} s_k.$$

then T is said to be regular. By the well known Toeplitz theorem T is regular if and only if

(i) $\lim\limits_{n\to\infty} t_{nk}=0$ for every k,

(ii) $\sum\limits_{k=1}^{\infty} |t_{nk}|=O(1)$,

(iii) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} t_{nk}=1$

hold.

By analogy to $(C, 1)$-functionals let us call a functional $L\in C[0, 1]$ a T-functional if there exists a sequence $\{x_k\}_{k=1}^{\infty}\subseteq[0, 1]$ such that

$$Lf = T\text{-}\lim_k f(x_k)$$

holds for every $f\in C[0, 1]$. In order to avoid unnecessary technical difficulties we assume T to be non-negative. Our matrices $T=(t_{nk})$ will have the property that

$$S(T) := \lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk}$$

exists. We say that T is decomposed into the matrices $T_1, \dots, T_n, \dots$ (in abbreviation $T=\bigcup\limits_n T_n$) if the columns of each T_n are columns of T, they follow each other in T_n in the same order as in T, and each column of T belongs to exactly one of the matrices T_n. Now let us call T completely regular if T can be decomposed into the matrices T_1, T_2 such that $S(T_1)=S(T_2)=S(T)/2$ is satisfied, furthermore T_1 and T_2 can be decomposed into T_{11}, T_{12} and T_{21}, T_{22}, respectively such that $S(T_{11})= =\dots=S(T_{22})=S(T)/4$ is satisfied, $T_{11}, \dots, T_{22}$ can further be decomposed into $T_{111}, \dots$ and so on. E.g. the complete regularity of the $(C, 1)$ matrix is a trivial fact.

We shall prove

Theorem 4. *Let T be a non-negative matrix. The following assertions are equivalent:*

(i) *every $PL1$ functional on $C[0, 1]$ is a T-functional,*

(ii) *there exists a sequence $\{x_k\} \subseteq [0, 1]$ such that*

$$T\text{-}\lim_k f(x_k) = \int_0^1 f(t)\, dt$$

for every $f \in C[0, 1]$,

(iii) *T is regular and completely regular.*

Corollary 3. *Any of* (i)—(iii) *implies that to every $L \in C^*[0, 1]$ there are bounded sequences $\{c_k\}_{k=1}^\infty$ and $\{x_k\}_{k=1}^\infty \subseteq [0, 1]$ such that*

$$T\text{-}\lim_k c_k f(x_k) = Lf \tag{4.1}$$

holds for every $f \in C[0, 1]$.

Remarks 1. In (ii) the functional $f \to \int f$ can be exchanged for every functional $f \to \int f\, d\mu$ with continuous μ, but it cannot be exchanged for one with discontinuous μ.

2. In Theorem 4 we characterized the matrices by which every $PL1$ functional can be represented rather than those by which every $L \in C^*[0, 1]$ can be represented in the form (4.1). Clearly, if (4.1) holds and if we multiply the columns of T by certain numbers and at the same time we divide the c_k's by the same numbers then the T' and $\{c_k'\}$ obtained still satisfy (4.1); therefore the characterization of the T's with the (4.1) representability property is rather hopeless.

Proof. (i)$\Rightarrow$(ii) is obvious. First we show that (iii) implies (i). To this end we need the following definition and lemma. For $x \in [0, 1]$ and $\varepsilon > 0$ let

$$f_{x,\varepsilon}(t) = \begin{cases} 0 & \text{if} \quad |t-x| > \varepsilon \\ 1 - |t-x|/\varepsilon & \text{if} \quad |t-x| \leqq \varepsilon \end{cases} \quad t \in [0, 1].$$

We say that x is a singular point of the $PL1$ functional L if

$$\mu_x = \liminf_{\varepsilon \to 0+0} Lf_{x,\varepsilon} = \lim_{\varepsilon \to 0+0} Lf_{x,\varepsilon} > 0.$$

(Note that this notion differs from that used in the proof of Theorem 3). A functional L having the form $Lf = \sum \mu_i f(\tau_i)$, $\sum \mu_i = 1$, $i \geqq 0$, will be called discrete.

Lemma 3. *Let $L \in C^*[0, 1]$ be a $PL1$ functional.*

(α) *The set of the singular points of L is countable and $\sum_x \mu_x \leqq 1$.*

(β) *$L = \mu L_1 + (1-\mu) L_2$ where $0 \leqq \mu \leqq 1$, L_1 and L_2 are $PL1$ functionals, L_1 is discrete and L_2 does not have singular points.*

(γ) *If the PL1 functional L has no singular point then to every* $\eta>0$ *there exists an* $\varepsilon>0$ *such that* $Lf_{x,\varepsilon}<\eta$ *for every* $x\in[0, 1]$.

Proof. If $\tau_1, \ldots, \tau_k\in[0, 1]$ are distinct points and $\varepsilon_1, \ldots, \varepsilon_k$ are so small that

$$\sum_{i=1}^{k} f_{\tau_i,\varepsilon_i} \leqq 1$$

is satisfied then

$$0 \leqq \sum_{i=1}^{k} Lf_{\tau_i,\varepsilon_i} \leqq 1$$

which proves (α).

Let $\tau_1, \tau_2, \ldots$ be the singular points of L and let

$$L_1 f = (1/\mu) \sum_{i=1}^{\infty} \mu_{\tau_i} f(\tau_i), \quad \mu = \sum_{i=1}^{\infty} \mu_{\tau_i} \quad (f\in C[0, 1]).$$

If $\mu<1$ then $L_2=(1/(1-\mu))(L-\mu L_1)$ is without singular points. For every $f\geqq 0$, $\delta>0$ and $k\geqq 1$ there are $\varepsilon_1, \ldots, \varepsilon_k>0$ with

$$f \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i) f_{\tau_i,\varepsilon_i},$$

by which

$$Lf \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i) Lf_{\tau_i,\varepsilon_i} \geqq (1-\delta) \sum_{i=1}^{k} f(\tau_i)\mu_{\tau_i}.$$

Since here $\delta>0$ and $k\geqq 1$ are arbitrary we can deduce that L_2 is again a *PL*1 functional which proves (β).

Finally, if (γ) were not true then there would be an $\eta>0$ and a sequence $x_1, \ldots, x_n, \ldots$ with $Lf_{x_n,1/n}\geqq\eta$. If x is a cluster point of $\{x_n\}$ then to every $\varepsilon>0$ there would be an n with $(1/2)f_{x_n,1/n}\leqq f_{x,\varepsilon}$ by which $Lf_{x,\varepsilon}\geqq(1/2)\eta$ $(\varepsilon>0)$ contradicting the assumption that L does not have any singular point.

Now in the proof of (iii)$\Rightarrow$(i) we prove first that every *PL*1 functional L without singular points is a T-functional. An easy argument gives that T can be converted into a triangle-matrix $T^*=(t^*_{nk})$ (i.e. $t^*_{nk}=0$ for $k>n$) which is also regular and completely regular and the limits $T\text{-}\lim_k s_k$ and $T^*\text{-}\lim_k s_k$ exist at the same time and they are equal for every bounded sequence $\{s_k\}$ (first make T to be row-finite and then repeat the rows of T sufficiently many times). Thus, from the point of view of our problem T and T^* are equivalent, so we may assume without loss of generality T to be a triangle-matrix.

Also one can show easily that the complete regularity of T implies the following: if $0\leqq s\leqq 1$ then T can be decomposed into T_1 and T_2 so that $S(T_1)=s$, $S(T_2)=1-s$ are satisfied, furthermore, to every $0\leqq r_1\leqq s$ and $0\leqq r_2\leqq 1-s$ the obtained T_1 and

T_2 can be decomposed into T_{11}, T_{12}, T_{21} and T_{22} so that $S(T_{11})=r_1$, $S(T_{12})=s-r_1$, $S(T_{21})=r_2$, $s(T_{22})=(1-s)-r_2$ are satisfied, etc. We shall call such decompositions completely regular.

Let us consider the functions

$$g_k^{(m)}(x)=\begin{cases}1 & \text{if} \quad k/2^m \leqq x \leqq 1,\\ 0 & \text{if} \quad 0\leqq x\leqq (k-1)/2^m, \quad m=1,2,\ldots, \quad 1\leqq k\leqq 2^m,\\ \text{linear on} & [(k-1)/2^m, k/2^m],\end{cases}$$

$$g_0^{(m)}(x)\equiv 1.$$

Let $q_k^{(m)}=Lg_k^{(m)}$ and $q_k^{(m)}=: \sum_{j=k}^{2^m} p_j^{(m)}$ $(0\leqq k\leqq 2^m,\ m=1,2,\ldots)$. By positivity we have $p_k^{(m)}\geqq 0$ and $\sum_{k=0}^{2^m} p_k^{(m)}=1$.

Let $T=T_0^{(1)}\cup T_1^{(1)}\cup T_2^{(1)}$ be a completely regular decomposition of T such that

$$S(T_0^{(1)})=p_0^{(1)},\quad S(T_1^{(1)})=p_1^{(1)},\quad S(T_2^{(1)})=p_2^{(1)}$$

are satisfied, and let

$$x_n^{(1,0)}=\begin{cases}1 & \text{if} \quad n\in \operatorname{ind} T_2^{(1)},\\ 1/2 & \text{if} \quad n\in \operatorname{ind} T_1^{(1)},\\ 0 & \text{if} \quad n\in\operatorname{ind} T_0^{(1)}\end{cases}$$

where ind T' denotes the set of those natural numbers j for which the j-th column of T belongs to T'. It is clear, that there exists a number $N^{(1,0)}$ such that for $s\geqq N^{(1,0)}$ we have

$$|t_s(\{x_n^{(1,0)}\}, g_i^{(1)})-Lg_i^{(1)}|<1/2 \quad (i=0,1,2),$$

where the notation

$$t_s(\{x_k\}, g):=\sum_{k=1}^{s} t_{sk}\, g(x_k)$$

is used.

For a given m and $0\leqq k\leqq 2^m$ let us consider the functions

$$g_0^{(m+1)},\ g_1^{(m+1)},\ \ldots,\ g_{2k}^{(m+1)},\ g_{k+1}^{(m)},\ g_{k+2}^{(m)},\ \ldots,\ g_{2^m}^{(m)}. \tag{4.2}$$

In the following φ will denote any of these functions. Let $p_{2k}^{*(m+1)}$ be defined by

$$q_{2k}^{(m+1)}=q_{k+1}^{(m)}+p_{2k}^{*(m+1)} \quad \text{if} \quad 2k<2^{m+1} \quad \text{and} \quad p_{2^{m+1}}^{*(m+1)}=p_{2^{m+1}}^{(m+1)}.$$

We suppose that for the pair (m,k) we have already defined the completely regular decomposition of T into the matrices

$$T_0^{(m+1)},\ T_1^{(m+1)},\ \ldots,\ T_{2k-1}^{(m+1)},\ T_{2k}^{*(m+1)},\ T_{k+1}^{(m)},\ \ldots,\ T_{2^m}^{(m)},$$

the sequence $\{x_n^{(m,k)}\}_{n=0}^{\infty}$ and the number $N^{(m,k)}$ so that

$$S(T_0^{(m+1)}) = p_0^{(m+1)}, \ldots, S(T_{2k-1}^{(m+1)}) = p_{2k-1}^{(m+1)}, S(T_{2k}^{*(m+1)}) = p_{2k}^{*(m+1)},$$
$$S(T_{k+1}^{(m)}) = p_{k+1}^{(m)}, \ldots, S(T_{2^m}^{(m)}) = p_{2^m}^{(m)},$$

for $n > N^{(m,k)}$ we have

$$x_n^{(m,k)} = \begin{cases} i/2^{m+1} & \text{if} \quad n \in \operatorname{ind} T_i^{(m+1)} \quad (i = 0, \ldots, 2k-1), \\ 2k/2^{m+1} & \text{if} \quad n \in \operatorname{ind} T_{2k}^{*(m+1)}, \\ i/2^m & \text{if} \quad n \in \operatorname{ind} T_i^{(m)} \quad (i = k+1, \ldots, 2), \end{cases}$$

and for $s \geqq N^{(m,k)}$

$$|t_s(\{x_n^{(m,k)}\}, \varphi) - L\varphi| < 1/2^m$$

are satisfied for every φ from (4.2). We want to go over to the pair $(m, k+1)$ (if $k=2^m$ then to the pair $(m+1, 0)$; this case can be treated similarly as the following one).

The regularity of T implies that if we cancel those columns of the matrices $T_{2k}^{*(m+1)}$, $T_{k+1}^{(m)}$ which belong to the first $N^{(m,k)}$ columns of T then the obtained matrices $W_{2k}^{*(m+1)}$, $W_{k+1}^{(m)}$ are still completely regular. Let us unite $W_{2k}^{*(m+1)}$ and $W_k^{(m)}$ into the matrix $V_k^{(m)}$ (the columns in $V_k^{(m)}$ follow each other in the same order as in T), and then decompose $V_k^{(m)}$ into the completely regular matrices $T_{2k}^{(m+1)}, T_{2k+1}^{(m+1)}, T_{2k+2}^{*(m+1)}$ so that

$$S(T_{2k}^{(m+1)}) = p_{2k}^{(m+1)}, \quad S(T_{2k+1}^{(m+1)}) = p_{2k+1}^{(m+1)}, \quad S(T_{2k+2}^{*(m+1)}) = p_{2k+2}^{*(m+1)}$$

be satisfied. This is possible because

$$p_{2k+2}^{*(m+1)} + q_{k+2}^{(m)} + p_{2k+1}^{(m+1)} + p_{2k}^{(m+1)} = q_{2k}^{(m+1)} = p_{2k}^{*(m+1)} + p_{k+1}^{(m)} + q_{k+2}^{(m)}.$$

Now we set

$$x_n^{(m,k+1)} = \begin{cases} 2k/2^{m+1} & \text{if} \quad n \in \operatorname{ind} T_{2k}^{(m+1)}, \\ (2k+1)/2^{m+1} & \text{if} \quad n \in \operatorname{ind} T_{2k+1}^{(m+1)}, \\ (2k+2)/2^{m+1} & \text{if} \quad n \in \operatorname{ind} T_{2k+2}^{*(m+1)}, \\ x_n^{(m,k)} & \text{otherwise.} \end{cases}$$

It follows easily that for $0 \leqq r \leqq 2k+2$

$$\lim_{s \to \infty} t_s(\{x_n^{(m,k+1)}\}, g_r^{(m+1)}) = \sum_{j=k+2}^{2^m} S(T_j^{(m)}) + S(T_{2k+2}^{*(m+1)}) + \sum_{j=r}^{2k+1} S(T_j^{(m+1)}) =$$
$$= \sum_{j=k+2}^{2^m} p_j^{(m)} + p_{2k+2}^{*(m+1)} + \sum_{j=r}^{2k+1} p_j^{(m+1)} = q_r^{(m+1)} = Lg_r^{(m+1)}$$

and similarly for $k+2 \leqq r \leqq 2^m$

$$\lim_{s \to \infty} t_s(\{x_n^{(m,k+1)}\}, g_r^{(m)}) = Lg_r^{(m)}.$$

Therefore, there exists a constant $N^{(m,k+1)} > N^{(m,k)}$ such that for $s \geqq N^{(m,k+1)}$ we have

$$|t_s(\{x_n^{(m,k+1)}\}, \psi) - L\psi| < 1/2^m$$

where ψ denotes any of the functions

$$g_r^{(m+1)}, \quad 0 \leqq r \leqq 2k+2, \quad g_r^{(m)}, \quad k+2 \leqq r \leqq 2^m.$$

(If we adjoin the omitted columns to $T_{2k+2}^{*(m+1)}$ we obtain again a completely regular decomposition of T and the prescribed properties hold for the pair $(m, k+1)$.) Thus, for all m and $0 \leqq k \leqq 2^m$ we can define the sequences $\{x_n^{(m,k)}\}_n$ which have also the property that for $0 \leqq n \leqq N^{(m,k)}$ and for $m' > m$ or $m' = m$ and $k' > k$, $x_n^{(m,k)}$ coincides with $x_n^{(m',k')}$, i.e. $x_n^{(m,k)} = x_n^{(m',k')}$. Hence the limit

$$x_n = \lim_{N^{(m,k)} \to \infty} x_n^{(m,k)}$$

exists for every n and $x_n \in [0, 1]$. We show that

$$T\text{-}\lim_n g_k^{(m)}(x_n) = L g_k^{(m)}$$

for every m and k which already implies

$$T\text{-}\lim_n f(x_n) = Lf$$

for every $f \in C[0, 1]$ because the linear combinations of the $g_k^{(m)}$'s constitute a dense set in $C[0, 1]$.

Let

$$I_1^{(m,k)} = \{n \mid n \in \operatorname{ind} T_{2k}^{*(m+1)}, \ N^{(m,k)} \leqq n \leqq N^{(m,k+1)}\},$$

$$I_2^{(m,k)} = \{n \mid n \in \operatorname{ind} T_{k+1}^{(m)}, \ N^{(m,k)} \leqq n \leqq N^{(m,k+1)}\},$$

and

$$K_1^{(m,k)} = \max_{N^{(m,k)} \leqq s \leqq N^{(m,k+1)}} \sum_{j \in I_1^{(m,k)}} t_{s,j},$$

$$K_2^{(m,k)} = \max_{N^{(m,k)} \leqq s \leqq N^{(m,k+1)}} \sum_{j \in I_2^{(m,k)}} t_{s,j}.$$

We claim that $K_1^{(m,k)} \to 0$, $K_2^{(m,k)} \to 0$ as $N^{(m,k)} \to \infty$. Suppose not, e.g. $K_1^{(m,k)} \geqq \geqq \varepsilon > 0$ for infinitely many pairs (m, k). For each such (m, k) we have by our construction

$$|t_s(\{x_n^{(m,k)}\}, \ g_{2k}^{(m+1)}) - t_s(\{x_n^{(m,k)}\}, \ g_{k+1}^{(m)})| \geqq \varepsilon,$$

which together with the estimates

$$|t_s(\{x_n^{(m,k)}\}, \ g_{2k}^{(m+1)}) - L g_{2k}^{(m+1)}| < 1/2^m,$$

$$|t_s(\{x_n^{(m,k)}\}, \ g_{k+1}^{(m)}) - L g_{k+1}^{(m)}| < 1/2^m \quad N^{(m,k)} \leqq s \leqq N^{(m,k+1)}$$

yield for infinitely many (m, k)

$$|Lg_{2k}^{(m)}-Lg_{2k+1}^{(m)}|>\varepsilon-2/2^m,$$

but this contradicts Lemma 3 (γ) (L is assumed to have no singular point).

Now if $|f|\leqq 1$ is *arbitrary* then for $N^{(m,k)}<s\leqq N^{(m,k+1)}$

$$|t_s(\{x_n\},f)-t_s(\{x_n^{(m,k)}\},f)|\leqq 2(K_1^{(m,k)}+K_2^{(m,k)})$$

and so, according to what we have just proved, to every $\varepsilon>0$ there exists an N such that if $s\geqq N$ then

$$|t_s(\{x_n\},\varphi)-L\varphi|<\varepsilon$$

for an arbitrary function φ from (4.2) with (m, k) satisfying $N^{(m,k)}<s\leqq N^{(m,k+1)}$. But for $m_1<m$ every one of the $g_k^{(m_1)}$'s is a convex linear combination of such φ's by which

$$\lim_{s\to\infty} t_s((\{x_n\}, g_k^{(m)}) = Lg_k^{(m)}$$

for every m and $0\leqq k\leqq 2^m$, and the proof is complete.

Now let L be discrete:

$$Lf=\sum_i \mu_i f(\tau_i),\quad \mu_i\geqq 0,\quad \sum_i \mu_i = 1.$$

By assumption we have matrices T_i with $T=\bigcup_i T_i$, $S(T_i)=\mu_i$, hence putting $x_n=\tau_i$ if $n\in \operatorname{ind} T_i$ we obtain

$$Lf=T\text{-}\lim_n f(x_n).$$

Finally, if $L=\mu L_1+(1-\mu)L_2$, $0<\mu<1$ where L_1 is discrete and L_2 is without singular points then there are completely regular matrices T_1, T_2 such that $T=T_1\cup T_2$, $S(T_1)=\mu$, $S(T_2)=1-\mu$. Above we proved that there are sequences $\{x_k^{(1)}\}$ and $\{x_k^{(2)}\}$ with

$$((1/\mu)T_1)\text{-}\lim_k f(x_k^{(1)}) = L_1 f,\quad ((1/(1-\mu)T_2)\text{-}\lim_k f(x_k^{(2)}) = L_2 f,$$

and hence putting

$$x_n = x_k^{(j)}\quad \text{if the } n\text{-th column of } T \text{ is the } k\text{-th column of } T_j;\ j=1 \text{ or } 2,$$

we obtain

$$T\text{-}\lim_n f(x_n) = T_1\text{-}\lim_k f(x_k^{(1)})+T_2\text{-}\lim_k f(x_k^{(2)}) = \mu L_1 f+(1-\mu)L_2 f = Lf$$

and the proof of the implication (iii)$\Rightarrow$(i) is complete.

Finally we prove that (ii) implies (iii). Let

$$\int_0^1 f = T\text{-}\lim_n f(x_n)$$

for all $f \in C[0, 1]$. Putting $f \equiv 1$ we obtain

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = T\text{-}\lim 1 = \int_0^1 1 = 1.$$

Thus, T is regular if $\lim_{n\to\infty} t_{nk} = 0$ is also satisfied for every n, but this obviously follows from the complete regularity of T which we show in a moment.

If $I \subseteqq [0, 1]$ is an interval let $|I|$ denote its length, and let the matrix T_I be determined by

$$\text{ind } T_I = \{n \mid x_n \in I\}.$$

We claim that $S(T_I) = |I|$ which already implies the complete regularity of T. For any functions $f_1, f_2 \in C[0, 1]$ satisfying $0 \leqq f_1, f_2 \leqq 1$, $f_1(x) = 0$ for $x \notin I$, $f_2(x) = 1$ for $x \in I$ we have, if l_n denotes the sum of the elements of T_I in the n-th row,

$$\int_0^1 f_1 \leqq \liminf_{n\to\infty} l_n \leqq \limsup_{n\to\infty} l_n \leqq \int_0^1 f_2$$

and since to every $\varepsilon > 0$ there are functions f_1 and f_2 of the above kind satisfying

$$|I| - \varepsilon \leqq \int_0^1 f_1 \leqq |I| \leqq \int_0^1 f_2 \leqq |I| + \varepsilon$$

we have indeed

$$S(T_I) = \lim_{n\to\infty} l_n = |I|.$$

The proof is complete.

Corollary 3 can be proved easily using Theorem 4.

5. Density measures

Let $X = \{x_k\}_{k=1}^{\infty}$ be a sequence from the n dimensional Euclidean space. We say that X has density α in the set $A \subseteqq \mathbf{R}^n$ if the number of the first n points belonging to A divided by n tends to α i.e. if

$$|\{k \mid 1 \leqq k \leqq n, x_k \in A\}|/n = (1/n) \sum_{k=1}^{n} \chi_A(x_k) \to \alpha \quad (n \to \infty).$$

Let J^n be the ring of the Jordan measurable sets of $\mathbf{R}^n$. A set function $\mu: J^n \to \mathbf{R}_+$ is said to be density measure if there is a sequence $X = \{x_k\}_{k=1}^{\infty}$ such that X has density $\mu(A)$ in every Jordan measurable set A. In this section we characterize these density measures. It will follow among others that they are really measures, i.e. they are countably additive and they can be extended to a Borel measure.

Theorem 5. *A $\mu\colon J^n \to \mathbf{R}_+$ is a density measure if and only if it has the form*

$$\mu(A) = \sum_{\tau_i \in A} \mu_i + \Big(1 - \sum_{i=1}^{\infty} \mu_i\Big) \int_A g \tag{5.1}$$

with suitable $\mu_i \geqq 0, \tau_i \in \mathbf{R}^n$ $(i=1, 2, \ldots), g \geqq 0, g \in L^1(\mathbf{R}^n)$ satisfying $\sum_{i=1}^{\infty} \mu_i \leqq 1$, and $\int_{\mathbf{R}^n} g = 1$.

In other words the density measures are the convex combinations of the discrete and (with respect to the Lebesgue measure of $\mathbf{R}^n$) absolutely continuous measures.

Corollary 4. *Every density measure is σ-additive and hence it can be extended to a Borel measure of $\mathbf{R}^n$.*

Remark 1. One could try to extend the notion of the density measure to other domains than the Jordan measurable sets but such extensions can result in that the only "density measures" are the discrete ones. This happens e.g. if we require μ to be defined for all open (and hence for all closed) sets of $\mathbf{R}^n$.

2. One could also use other summability methods, i.e. μ can be defined as

$$\mu(A) = T\text{-}\lim_k \chi_A(x_k)$$

for some $X = \{x_k\} \subseteq \mathbf{R}^n$ and a non-negative regular matrix T. A similar argument that will follow proves that Theorem 5 holds word for word for this "modified density measure".

Proof. We shall only sketch the proof. The details are very similar to those of Section 3.

I. Necessity. Let $X = \{x_k\}$ be the sequence generating μ. A standard argument yields that the number of those points x for which $\mu(\{x\}) > 0$ is countable. Let these be $\tau_1, \tau_2, \ldots$ and let $\mu_i = \mu(\{\tau_i\})$. If N_i is that subsequence of the natural numbers N for which $k \in N_i$ iff $x_k = \tau_i$ then N_i has density μ_i in N. N_i contains a subsequence N_i' such that N_i' has density μ_i in N, too, but it is sufficiently sparse, namely if $N_i' = \{n_1^{(i)}, n_2^{(i)}, \ldots\}$ then $k/n_k^{(i)} \leqq \mu_i$ is satisfied for every k. One can easily verify that if $X' = \{x_j'\}$ is that subsequence of X for which $x_k \in X'$ iff $k \notin \bigcup_{i=1}^{\infty} N_i'$ then X' satisfies the property that if $\mu^* = \sum_{i=1}^{\infty} \mu_i < 1$ then X' has density

$$\mu_1(A) = \big(1/(1-\mu^*)\big)\Big(\mu(A) - \sum_{\tau_i \in A} \mu_i\Big)$$

in every set $A \in J^n$ and hence

$$\mu(A) = \sum_{\tau_i \in A} \mu_i + \Big(1 - \sum_{i=1}^{\infty} \mu_i\Big) \mu_1(A).$$

If $\sum_{i=1}^{\infty} \mu_i = 1$ then, clearly,

$$\mu(A) = \sum_{r_i \in A} \mu_i$$

and our task has been reduced to show that there is a function $g \in L^1(\mathbf{R}^n)$, $g \geqq 0$, $\int_{\mathbf{R}^n} g = 1$ such that

$$\mu_1(A) = \int_A g$$

is satisfied.

Let us consider the following mapping $\varphi\colon \mathbf{R}^1 \to \mathbf{R}^n$: if $x \in \mathbf{R}^1$ has binary expansion

$$x = \ldots \alpha_{-2}\alpha_{-1}\alpha_0, \ \alpha_1\alpha_2\alpha_3 \ldots$$

where infinitely many of the $\alpha_1, \alpha_2, \ldots$ vanish then putting

$$x^j = \ldots \alpha_{j-n}, \alpha_j \alpha_{j+n} \ldots, \quad j = 1, 2, \ldots n$$

let

$$\varphi(x) = (x^1, \ldots, x^n).$$

Then $\varphi(\mathbf{R}^1) = \mathbf{R}^n$ and for each $x \in \mathbf{R}^n$, $\varphi^{-1}(x)$ consists of at most 2^n points. $\varphi([m/2^k, (m+1)/2^k])$ is a closed rectangular parallelepiped without one vertex with volume $1/2^k$ and

$$\varphi^{-1}(\{(x_1, \ldots, x_n) \mid m_1/2^k \leqq x_1 < (m_1+1)/2^k, \ldots, m_n/2^k \leqq x_n < (m_n+1)/2^k\})$$

is an interval of length $1/2^{kn}$ plus a nowhere dense set with zero Lebesgue (and hence Jordan) measure. It follows that $\varphi(A) \in J^n$ for every $A \in J^1$ and $\varphi^{-1}(A) \in J^1$ for every $A \in J^n$, furthermore, φ is a measure preserving transformation (cf. also [2, pp. 81—83]). Let x^* be an element of $\varphi^{-1}(x)$ and let $x_j^* = (x_j')^*$ i.e. $x_j^* \in \varphi^{-1}(x_j')$. We claim that $X^* = \{x_j^*\}_{j=1}^{\infty}$ has density in every Jordan measurable set of $\mathbf{R}^1$.

Let $A \in J^1$. If $x_j^* \in A$ then $x_j' \in \varphi(A)$. Conversely, if $x_j' \in \varphi(A)$ but $x_j^* \notin A$ then $x_j^* \in \varphi^{-1}(\varphi(A)) \setminus A =: B$. By what we have said above B has Jordan measure 0 and hence $\varphi(B)$ also has Jordan measure 0. But then every $B' \subseteqq \varphi(B)$ is Jordan measurable and X' has density $\mu_1(B')$ in every such B', furthermore, X' has density 0 in every point which imply, by an argument similar to that used in the proof of Lemma 2, that X' has density 0 in $\varphi(B)$. By this X^* has density 0 in $\varphi^{-1}(\varphi(B)) \supseteqq B$ and we obtain that for all j but a sequence j_n with $j_n/n \to \infty$ the conditions $x_j^* \in A$ and $x_j' \in \varphi(A)$ are equivalent, which proves our assertion.

It follows also that X^* has density $\mu^*(A) := \mu_1(\varphi(A))$ in a set $A \in J^1$. Since $\mu_1(\{\varphi(x)\}) = 0$ for every x, the consideration of Lemma 2 yields that the function

$$\alpha(x) = \mu^*((-\infty, x]) = \mu_1(\varphi(-\infty, x]) = \{\text{the density of } X^* \text{ in } (-\infty, x]\}$$

is absolutely continuous and increasing, furthermore $\alpha(-\infty)=0$, $\alpha(\infty)=1$. If

$$g(x)=\alpha'(x^*)\quad (x\in\mathbf{R}^n)$$

then

$$\mu_1(A)=\mu^*(\varphi^{-1}(A))=\int_{\varphi^{-1}(A)}\alpha'=\int_A g\quad (A\in J^n)$$

because φ is measure preserving, and the proof is complete.

II. Sufficiency. Let μ have the form (5.1). We put

$$Lf=\sum_i \mu_i f(\tau_i)+\Big(1-\sum_i \mu_i\Big)\int fg$$

for every Riemann-integrable function f. Either by the method of Section 3 or by using the above transformation φ and applying Theorem 3 one can prove that there is a sequence $X=\{x_n\}\subseteqq\mathbf{R}^n$ with

$$Lf=\lim_{n\to\infty}(1/n)\big(f(x_1)+\ldots+f(x_n)\big)$$

and an application of this to the characteristic function χ_A of A yields the desired representation.

We have completed our proof.

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Approximation by modified Szász operators

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1. Introduction

I. J. L. DURRMEYER [5] defined the approximation process

$$D_n f(x) = \sum_{k=0}^{n} \Big(\int_0^1 b_{nk}(t) f(t)\, dt \Big) b_{nk}(x), \quad b_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

which can be used for restoring f if its moments $\int_0^1 f(t) t^k dt$ are given. In a recent paper M. M. DERRIENNIC [3] proved several results concerning these operators that have certain analogues with the corresponding results for Bernstein polynomials from which the operators D_n originate.

Now we shall similarly modify the Szász—Mirakian operators [7, 8]

$$S_n(f;\ x) = \sum_{k=0}^{\infty} f(k/n) p_{n,k}(x), \quad p_{n,k}(x) = e^{-nx}(nx)^k/k! \quad (x \geqq 0)$$

and prove exact estimates and saturation results for the modified operator. Actually, we have two modifications in mind:

$$L_n(f;\ x) = f(0) p_{n,0}(x) + n \sum_{k=1}^{\infty} \Big(\int_0^{\infty} f(t) p_{n,k-1}(t)\, dt \Big) p_{n,k}(x)$$

and

$$\mathsf{L}_n(f;\ x) = n \sum_{k=0}^{\infty} \Big(\int_0^{\infty} f(t) p_{n,k}(t)\, dt \Big) p_{n,k}(x).$$

Clearly L_n is the perfect analogue of D_n but it will turn out that L_n has much nicer properties than L_n; the main difference between them is that L_n reproduces every linear function while L_n reproduces only the constant ones.

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Since we are interested in uniform approximation and the transforms $L_n(f)$ and $\mathcal{L}_n(f)$ are continuous on $[0, \infty)$ (if they exist at all), in what follows we assume the continuity of f on $[0, \infty)$. Besides, when treating uniform approximation we shall always assume the boundedness of f as well.

We shall prove global results, i.e. the whole interval $[0, \infty)$ will be considered, but because of the strong localization valid both for L_n and $\mathcal{L}_n$, these also solve the corresponding local problems.

In what follows let $\|\cdot\|$ denote the supremum norm and $\varphi(x)=\sqrt{x}$.

2. Weighted estimates

I. Results. Let

$$\Delta_h^2(f;\ x) = f(x-h)-2f(x)+f(x+h) \quad (x \geqq h)$$

be the usual symmetric second difference of f and

$$\omega^2(f;\ \delta) = \sup_{0 \leqq h \leqq \delta,\, x \geqq h} |\Delta_h^2(f;\ x)|$$

the modulus of smoothness of f.

Theorem 1. *For every function* $f \in C[0, \infty)$ *we have*

$$|L_n(f;\ x)-f(x)| \leqq 11\omega^2(f;\ \sqrt{x/n}). \tag{2.1}$$

Corollary. *Let* $f \in C[0, \infty)$ *be bounded. Then there exists a non-negative and continuous function* $\psi\colon [0, \infty) \to R_+$, $\psi(0)=0$, *such that*

$$|L_n(f;\ x)-f(x)| \leqq K\psi(x/n) \quad (x \geqq 0, \quad n = 1, 2, \ldots)$$

holds if and only if f is uniformly continuous on $[0, \infty)$.

The proof of Theorem 1 follows that of [1, Theorem 8] and it gives somewhat more: if $\{L_n\}$ is an arbitrary sequence of positive operators reproducing the linear functions and $(1/2)L_n((t-x)^2; x)=\mu_n^2(x)$, then

$$|L_n(f;\ x)-f(x)| \leqq 11\omega^2(f;\ \mu_n(x)) \quad (x \geqq 0).$$

The saturation result is as follows:

Theorem 2. *Let* $f \in C[0, \infty)$, $f(x)=O(e^{Ax})$ $(A>0)$. *If* $L_n(f; x)-f(x)=o_x(x/n)$ $(x \geqq 0,\ n \to \infty)$, *then f is a linear function; furthermore*

$$|L_n(f;\ x)-f(x)| \leqq Kx/n \quad (x \geqq 0,\ n = 1, 2, \ldots)$$

holds if and only if f has a derivative belonging to Lip 1, *where*

$$\mathrm{Lip}\ 1 = \{f\colon\ |f(x+h)-f(x)| \leqq K_f h,\ x \geqq 0,\ h>0\}.$$

The next result solves the so-called non-optimal approximation problem. Let

$$\mathrm{Lip}^2\alpha = \{f\in C[0, \infty):\ \omega^2(f;\ \delta) \leqq K_f\delta^\alpha,\quad \delta > 0\}.$$

With this notation we have:

Theorem 3. *Let $f\in C[0, \infty)$ be bounded. Then with $0<\alpha<1$,*

$$|L_n(f;\ x)-f(x)| \leqq K(x/n)^\alpha \quad (x \geqq 0,\ n = 1, 2, \ldots)$$

holds if and only if $f\in \mathrm{Lip}^2\, 2\alpha$.

For L_n the situation is much more complex. We mention only that if $\omega^1(f; \delta)$ denotes the ordinary modulus of continuity of f, then

$$|\mathsf{L}_n(f;\ x)-f(x)| \leqq K\omega^1(f;\ \sqrt{(x+1/n)/n}) \quad (x > 0,\ n = 1, 2, \ldots)$$

follows from a result of SHISHA and MOND [4, p. 28] and by well known properties of ω^1 this implies that

$$(2.2)\qquad (1/(1+\sqrt{x}))|\mathsf{L}_n(f;\ x)-f(x)| \leqq K\omega^1(f;\ 1/\sqrt{n}) \quad (x > 0,\quad n = 1, 2, \ldots).$$

It can be shown that in general (2.2) cannot be improved since neither the weight $\{1+\sqrt{x}\}^{-1}$ nor the rate $\omega^1(f; 1/\sqrt{n})$ can be replaced by a smaller quantity. An analogue of Theorem 1 holds for L_n as well. However it is far less obvious how the analogues of Theorems 2—3 look like in the case of L_n.

II. Proofs. Proof of Theorem 1. We follow the argument of [1, Theorem 8]. Using the relations

$$\int_0^\infty p_{n,k}(x)\,dx = 1/n, \quad (k/n)p_{n,k}(x) = xp_{n,k-1}(x) \quad (n = 1, 2, \ldots, k \geqq 0)$$

one can easily verify that $L_n(g; x)\equiv g(x)$ for every linear function g, furthermore

$$(2.3)\qquad L_n((t-x)^2, x) = x/n \quad (x > 0,\quad n = 1, 2, \ldots).$$

First let us suppose that f is twice continuously differentiable on $[0, \infty)$. By Taylor's formula

$$(2.4)\qquad \begin{aligned} f(t) &= f(x)+(t-x)f'(x)+\int_x^t\int_x^s f''(u)\,du\,ds = \\ &= f(x)+(t-x)f'(x)+\int_x^t (t-\tau)f''(\tau)\,d\tau. \end{aligned}$$

Here

$$\Big|\int_x^t\int_x^s f''(u)\,du\,ds\Big| \leqq \|f''\|(t-x)^2/2,$$

hence

(2.5) $$|L_n(f;\ x)-f(x)| \leqq \|f''\| L_n((t-x)^2/2;\ x) = \|f''\| x/n.$$

Now let

(2.6) $$f_h(x) = (2/h)^2 \int_0^{h/2} \int_0^{h/2} [2f(x+s+t)-f(x+2(s+t))]\, ds\, dt.$$

By [1, (21)], we have $\|f-f_h\| \leqq \omega^2(f; h)$, $\|f_h''\| \leqq 9h^{-2}\omega^2(f; h)$ and so using (2.5) for f_h, we obtain

$$|L_n(f;\ x)-f(x)| \leqq |L_n(f-f_h;\ x)|+|(f-f_h)(x)|+|L_n(f_h;\ x)-f_h(x)| =$$
$$\leqq 2\omega^2(f;\ h)+9h^{-2}\omega^2(f, h)x/n,$$

and putting here $h=\sqrt{x/n}$ we obtain (2.1).

Proof of Corollary. We have to prove only the necessity part. Let $x\geqq 1$ and $n=M[x]$. Exactly as in the proof of Theorem 4 below we have $|L_n'(f; x)| \leqq \leqq \|f\|\sqrt{n/x}$ and so if we assume

$$|L_n(f;\ x)-f(x)| \leqq K\psi(\sqrt{x/n})$$

we obtain that

$$|f(x)-f(x+h)| \leqq 2K\psi(\sqrt{x+h}/\sqrt{M[x]})+|L_n(f;\ x+h)-L_n(f;\ x)| \leqq$$
$$\leqq 2K\psi(3/\sqrt{M})+h\|f\|\sqrt{M}$$

which can be made arbitrarily small by choosing first a large M and then sufficiently small h.

Proof of Theorem 2. First we prove the following strong localization result: if $f(x)=O(e^{Ax})$ and $f(x)$ vanishes on an interval $(a-\varepsilon, b+\varepsilon)$ $(\varepsilon>0)$, then $L_n(f; x)= =O(n^{-2})$ uniformly on $[a, b]$. In fact one can easily see that if A is an integer, then

$$e^{At}p_{n,k}(t) \leqq Ke^{A_1k/n}p_{n-A,k}(t)$$

with a constant A_1 and so

$$|L_n(f;\ x)| \leqq Kn \sum_{k=0}^{\infty} e^{A_1k/n}p_{n,k}(x)\Big(\int_0^{a-\varepsilon} + \int_{b+\varepsilon}^{\infty}\Big)p_{n-A,k}(t)\,dt = I_1+I_2.$$

$$I_2 \leqq nK\Big(\sum_{k\leqq(b+\varepsilon/2)n} p_{n,k}(x) \int_{b+\varepsilon}^{\infty} p_{n-A,k}(t)\,dt + \sum_{k>(b+\varepsilon/2)n} p_{n,k}(x)e^{A_1k/n} \int_0^{\infty} p_{n-A,k}(t)\,dt\Big) \leqq$$

$$\leqq \frac{K}{n^2} \sum_{k\leqq(b+\varepsilon/2)n} p_{n,k}(x)+K \sum_{k>(b+\varepsilon/2)n} p_{n,k}(b)e^{A_1k/n}n \leqq$$
$$\leqq Kn^{-2}+Knp_{n,[(b+\varepsilon/2)n]}(b)\exp[A_1(b+\varepsilon/2)] \leqq Kn^{-2},$$

where to obtain the second inequality we used the estimate (see [6, p. 212])

$$\int_{|u-k|>\eta n} (e^{-ku} u^k/k!)\, du \leqq Kk^3/(\eta n)^6,$$

for the second and third ones we used the facts that $p_{n,k}(x)$ increases on the interval $(0, k/n)$ and

$$p_{n,k+1}(b)/p_{n,k}(b) \leqq \big(b/(b+\varepsilon/2)\big) \exp(A_1/n) \leqq 1-\varepsilon/2b$$

for sufficiently large n and $k>(b+\varepsilon/2)n$, and to obtain the last one we used Stirling's formula.

One can get similar estimates for I_1 and thus our assertion concerning the localization of L_n has been proved. For later applications let us mention that in view of our proof, the same holds true for the operators L_n.

Using the above localization, one can see that the proofs of [4, Theorem 5.1] and [4, Theorem 5.4] hold for L_n on every finite interval $[a, b]\subseteq[0, \infty)$ and we obtain that

$$L_n(f, x)-f(x) = o_x(1)$$

implies that f is locally and hence globally linear, furthermore

$$L_n(f;\ x)-f(x) = O(x/n)$$

implies that f has a derivative which is absolutely continuous on every interval $(a, b)\subseteq[0, \infty)$. But using again our localization, the proof of [4, Lemma 5.5] yields that

$$\lim_{n\to\infty} (n/x)\big(L_n(f;\ x)-f(x)\big) = f''(x)/2$$

at every point x, where $f''(x)$ exists. So

$$L_n(f;\ x)-f(x) = O(x/n) \quad \text{implies} \quad f''(x) = O(1)$$

and this is the same as $f'\in\operatorname{Lip} 1$.

The sufficiency of the condition $f'\in\operatorname{Lip} 1$ follows from Theorem 1 since it implies $\omega^2(f;\delta)=O(\delta^2)$.

Proof of Theorem 3. Here, again, the sufficiency directly follows from Theorem 1.

To prove the necessity of the condition $f\in\operatorname{Lip}^2 2\alpha$, first we verify the inequality

(2.7) $$|L_n''(f, x)| \leqq K\omega^2(f;\ \delta)(n/x+\delta^{-2}) \quad (\delta, x>0,\ n = 1, 2, \ldots)$$

(cf. also [1, (23)]). Let

(2.8) $$F_{n0}(f) = f(0), \quad F_{nk}(f) = n\int_0^\infty f(t)\, p_{n,k-1}(t)\, dt \quad (k = 1, 2, \ldots).$$

Since

$$p'_{n,k}(x) = n\big(p_{n,k-1}(x)-p_{n,k}(x)\big) = (k/x-n)\, p_{n,k}(x),$$

simple computation gives that with $\gamma_{n,k}=(k/n-x)^2-k/n^2$,

$$L_n''(f;\ x) = (n/x)^2 \sum_{k=0}^{\infty} \gamma_{n,k}(x) F_{n,k}(f) p_{n,k}(x) = \tag{2.9}$$
$$= n^2 \sum_{k=0}^{\infty} \big(F_{n,k}(f) - 2F_{n,k+1}(f) + F_{n,k+2}(f)\big) p_{n,k}(x)$$

and so

$$|L_n''(f;\ x)| \leqq (n/x)^2 \sum_{k=0}^{\infty} |\gamma_{n,k}(x)|\,|F_{n,k}(f) - F_{n,k}(f_\delta)|\, p_{n,k}(x) +$$
$$+ n^2 \sum_{k=0}^{\infty} |F_{n,k}(f_\delta) - 2F_{n,k+1}(f_\delta) + F_{n,k+2}(f_\delta)|\, p_{n,k}(x) = I_1 + I_2,$$

where the function f_δ is the same as defined in (2.6).

Since

$$|F_{n,k}(f) - F_{n,k}(f_\delta)| \leqq F_{n,k}(|f-f_\delta|) \leqq \|f-f_\delta\| \leqq \omega^2(f;\delta)$$

we have by [1, (27)]

$$I_1 \leqq \omega^2(f;\ \delta)(n/x)^2 \sum_{k=0}^{\infty} |\gamma_{n,k}(x)|\, p_{n,k}(x) \leqq K\omega^2(f;\ \delta)\, n/x.$$

In I_2 we use the Taylor expansion (2.4) for f_δ. Let first $k \geqq 1$. Then integrations by parts give that

$$n^2 |F_{n,k}(f_\delta) - 2F_{n,k+1}(f_\delta) + F_{n,k+2}(f_\delta)| =$$
$$= n \left| \int_0^{\infty} n^2 p_{n,k-1}(t) \left[1 - \frac{2nt}{k} + \frac{(nt)^2}{k(k+1)}\right] \Big(\int_x^t \int_x^s f_\delta''(u)\, du\, ds\Big) dt \right| =$$
$$= n \left| \int_0^{\infty} (p_{n,k+1}(t))'' \Big(\int_x^t \int_x^s f_\delta''(u)\, du\, ds\Big) dt \right| = n \left| \int_0^{\infty} p_{n,k+1}(t) f_\delta''(t)\, dt \right| \leqq$$
$$\leqq \|f_\delta''\| \leqq 9\delta^{-2}\omega^2(f;\ \delta).$$

A direct but rather long computation verifies the identity

$$f_\delta(0) - 2n \int_0^{\infty} e^{-nt} f_\delta(t)\, dt + n \int_0^{\infty} e^{-nt}(nt) f_\delta(t)\, dt = \int_0^{\infty} t e^{-nt} f_\delta''(t)\, dt, \tag{2.10}$$

by which

$$n^2 |F_{n,0}(f_\delta) - 2F_{n,1}(f_\delta) + F_{n,2}(f_\delta)| \leqq \|f_\delta''\|\, n \int_0^{\infty} (nt) e^{-nt}\, dt = \|f_\delta''\| \leqq 9\delta^{-2}\omega^2(f;\ \delta).$$

Collecting our estimates we obtain (2.7).

Now using (2.7) and $|f(x)-L_n(f;x)|=O\big((x/n)^\alpha\big)$ we get

$$|\Delta_h^2(f;\ x)| \leqq |f(x)-L_n(f;\ x)|+2|f(x+h)-L_n(f;\ x+h)|+$$

$$+|f(x+2h)-L_n(f;\ x+2h)|+\Big|\int_0^h\int_0^h L_n''(f,x+s+t)\,ds\,dt\Big| \leqq$$

$$\leqq K\big((x/n)^\alpha+h^2(n/x+\delta^{-2})\omega^2(f;\ \delta)\big).$$

Putting here $\delta=\sqrt{x/n}$ we obtain

$$\omega^2(f;\ h) \leqq K(\delta^{2\alpha}+(h/\delta)^2\omega^2(f;\ \delta)) \quad (h,\delta>0)$$

and it is well-known (see [2, Lemma]) that this and the boundedness of f imply $\omega^2(f;h)=O(h^{2\alpha})$, i.e. $f\in\mathrm{Lip}^2\,2\alpha$.

Our proofs are complete.

3. Uniform approximation

I. Results. Let C_B be the set of bounded and continuous functions defined on $[0,\infty)$. A very natural question is the following: For which functions f do the transforms $L_n(f)$ $\big(\mathsf{L}_n(f)\big)$ approximate f *uniformly* on the whole interval $[0,\infty)$? The answer is given by

Theorem 4. *If $f\in C_B$, then $L_n(f)-f=o(1)$ $(n\to\infty)$ is satisfied uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

The saturation result sounds as

Theorem 5. *Let $f\in C_B$. Then $L_n(f)-f=o(1/n)$ $(n\to\infty)$ uniformly on $[0,\infty)$ iff f is constant, furthermore $L_n(f)-f=O(1/n)$ iff f has a locally absolutely continuous derivative with $|xf''(x)|\leqq K_f$ $(x>0)$.*

Finally concerning the non-optimal approximation we have

Theorem 6. *Let $f\in C_B$ and $0<\alpha<1$. Then $L_n(f)-f=O(n^{-\alpha})$ holds uniformly on $[0,\infty)$ iff*

(3.1) $$x^\alpha|\Delta_h^2(f;\ x)| \leqq Kh^{2\alpha} \quad (x>h,\ h>0)$$

is satisfied.

The analogues of Theorems 4 and 5 for L_n are:

Theorem 7. *If $f\in C_B$, then $\mathsf{L}_n(f)-f=o(1)$ $(n\to\infty)$ holds uniformly on $[0,\infty)$ iff $f(x^2)$ is uniformly continuous on $[0,\infty)$.*

Theorem 8. *Let* $f\in C_B$. *Then* $\mathrm{L}_n(f)-f=o(1/n)$ $(n\to\infty)$ *iff* f *is constant, furthermore* $\mathrm{L}_n(f)-f=O(1/n)$ *iff* f *has a continuous derivative with* $xf'(x)\in\mathrm{Lip}\,1$.

We can see that $\{L_n\}$ and $\{\mathrm{L}_n\}$ do not differ from the point of view of uniform approximation but they do differ from the point of view of global saturation, e.g. if $f\in C_B$ is twice continuously differentiable on $(0, \infty)$ and

$$f(x)=\begin{cases} x\log x & x\in[0,1] \\ 0 & x>2, \end{cases}$$

then f belongs to the saturation class of $\{L_n\}$ but not to that of $\{\mathrm{L}_n\}$ (it is easy to see that the former includes the latter).

In the proofs of Theorems 4—6 we shall use the general results of [9]. Theorems 7 and 8 do not come so easily, they require special considerations.

II. Proofs. Proof of Theorem 4. Since L_n reproduces the linear functions and satisfies (2.3) we can apply [9, Theorem 2] and the remark made after it, according to which $L_n(f)-f=o(1)$ and the uniform continuity of

$$f\circ g^{-1}(x)=f(x^2)\quad \Big(g(x)=(1/4)\int_0^x (1/\varphi(t))\,dt,\quad \varphi(x)=\sqrt{x}\Big)$$

are equivalent provided we can verify that

$$L_n'(f;\ x)\leqq K_n/\varphi(x)\quad \big(x>0,\quad \varphi(x)=\sqrt{x}\big).$$

With the notation (2.8) we get by an application of the Schwartz inequality

$$|L_n'(f;\ x)|=(n/x)\Big|\sum_{k=0}^{\infty}(k/n-x)F_{n,k}(f)p_{n,k}(x)\Big|\leqq (n/x)\,\|f\|\sum_{k=0}^{\infty}|k/n-x|\,p_{n,k}(x)\leqq$$

$$\leqq (n/x)\,\|f\|\Big(\sum_{k=0}^{\infty}(k/n-x)^2p_{n,k}(x)\Big)^{1/2}=\|f\|\sqrt{n}/\sqrt{x},$$

where at the last step we used the fact that $\sum_{k=0}^{\infty}(x-k/n)^2p_{n,k}(x)=x/n$. This proves the theorem.

Proof of Theorem 5. The first part of the statement of the theorem follows from Theorem 2 and the boundedness of f. Let $\varepsilon>0$ and

$$h_{x,\varepsilon}(t)=\begin{cases}(t-x)^2 & \text{for}\quad |t-x|\geqq\varepsilon, \\ 0 & \text{for}\quad |t-x|<\varepsilon.\end{cases}$$

By [9, Proposition 1] the second part will also follow if

$$L_n(h_{x,\varepsilon};\ x)=o_{x,\varepsilon}(1/n)\quad (n\to\infty)$$

is satisfied for every $x>0$ and $\varepsilon>0$. Since a stronger result was proved in the proof of Theorem 2, the proof is over.

Proof of Theorem 6. Let

$$\omega_\varphi(f;\ \delta) = \sup_{0\leqq h\leqq\delta,\, x>0} |\Delta^2_{h\varphi(x)}(f;\ x)| \quad (\varphi(x) = \sqrt{x})$$

be the modified modulus of smoothness of f. Putting $h=\delta\sqrt{x}$ into (3.1) we can see that (3.1) is equivalent of $\omega_\varphi(f;\delta)=O(\delta^{2\alpha})$. Now by [9, Proposition 2, Corollary] the theorem will be proved if we verify

$$|xL_n''(f;\ x)| \leqq Kn\|f\| \quad (f\in C_B), \tag{3.2}$$

and for any $f\in C_B$ having an absolutely continuous derivative

$$|xL_n''(f;\ x)| \leqq K\|\varphi^2 f''\|. \tag{3.3}$$

The proof of (3.2) is easy. Using (2.9) we have

$$|xL_n''(f;\ x)| \leqq (n^2/x)\sum_{k=0}^{\infty} |F_{n,k}(f)|\,|\gamma_{n,k}(x)|\,p_{n,k}(x) \leqq$$

$$\leqq n\|f\|\Big((n/x)\sum_{k=0}^{\infty}(k/n-x)^2 p_{n,k}(x)+(n/x)\sum_{k=0}^{\infty}(k/n^2)p_{n,k}(x)\Big) = 2n\|f\|.$$

In the proof of (3.3) we use again (2.9) so that

$$|xL_n''(f;\ x)| \leqq n^2x\sum_{k=0}^{\infty}|F_{n,k}(f)-2F_{n,k+1}(f)+F_{n,k+2}(f)|p_{n,k}(x).$$

By (2.10)

$$\begin{aligned} n^2x|F_{n,0}(f)-2F_{n,1}(f)+F_{n,2}(f)|p_{n,0}(x) &\leqq \\ \leqq \|\varphi^2 f''\|\Big(n\int_0^\infty e^{-nt}\,dt\Big)\,nxe^{-nx} &\leqq \|\varphi^2 f''\|. \end{aligned} \tag{3.4}$$

For $k\geqq 1$ let us apply Taylor's formula (2.4) by which

$$\begin{aligned} x|F_{n,k}(f)-2F_{n,k+1}(f)+F_{n,k+2}(f)| &= \\ = \left|nx\int_0^\infty p_{n,k}(t)\left(1-\frac{2nt}{k+1}+\frac{(nt)^2}{(k+1)(k+2)}\right)\int_0^t (t-\tau)f''(\tau)\,d\tau\,dt\right|&. \end{aligned} \tag{3.5}$$

Here

$$\left|1-\frac{2nt}{k+1}+\frac{(nt)^2}{(k+1)(k+2)}-\left(1-\frac{nt}{k+1}\right)^2\right| \leqq \frac{(nt)^2}{(k+1)^2(k+2)}$$

and by

$$\left|\int_x^t ((t-\tau)/\tau)\,d\tau\right| \leqq \left|\int_x^t ((t-x)/x)\,d\tau\right| \leqq (t-x)^2/x$$

we also have

$$\Big|\int_x^t (t-\tau)f''(\tau)\,d\tau\Big| \leqq \|\varphi^2 f''\|(t-x)^2/x.$$

Hence we can continue (3.5) as

$$\leqq \|\varphi^2 f''\| n \int_0^\infty p_{n,k}(t)\left(\left(1-\frac{nt}{k+1}\right)^2+\frac{(nt)^2}{(k+1)^2(k+2)}\right)(t-x)^2\,dt \leqq$$

$$\leqq \|\varphi^2 f''\| n \int_0^\infty p_{n,k}(t)\left[\left(1-\frac{2nt}{k+1}+\frac{(nt)^2}{(k+1)(k+2)}\right)+\frac{2(nt)^2}{(k+1)^2(k+2)}\right](t-x)^2\,dt =$$

$$= \|\varphi^2 f''\|(E_k-2E_{k+1}+E_{k+2})+2E_{k+2}/(k+1),$$

where

$$E_k = n\int_0^\infty e^{-nt}\big((nt)^k/k!\big)(t-x)^2\,dt = (k+1)(k+2)/n^2-2(k+1)x/n+x^2.$$

Now

$$E_k-2E_{k+1}+E_{k+2} = 2/n^2, \quad 2E_{k+2}/(k+1) \leqq 3\big((k+4)/n^2-2x/n+x^2/(k+1)\big)$$

and so

$$n^2x\sum_{k=1}^\infty |F_{n,k}(f)-2F_{n,k+1}(f)+F_{n,k+2}(f)|\,p_{n,k}(x) \leqq$$

$$\leqq \|\varphi^2 f''\| n^2\Big(\sum_{k=1}^\infty (2/n^2)\,p_{n,k}(x)+3n^2\sum_{k=1}^\infty \big((k+4)/n^2\big)\,p_{n,k}(x)-$$

$$-2(x/n)\sum_{k=1}^\infty p_{n,k}(x)+x^2\sum_{k=1}^\infty p_{n,k}(x)/(k+1)\Big)\Big) \leqq$$

$$\leqq \|\varphi^2 f''\|\Big(14+3n^2\big(-x/n+2xe^{-nx}/n+(x/n)\sum_{k=1}^\infty p_{n,k+1}(x)\big)\Big) \leqq$$

$$\leqq \|\varphi^2 f''\|(14+6nxe^{-nx}) \leqq 20\|\varphi^2 f''\|$$

and this together with (3.4) prove (3.3), and thus the proof is complete.

Proof of Theorem 7. First let us suppose that $f(x^2)$ is uniformly continuous on $[0,\infty)$. It is easy to see that then $f(x\pm\delta\sqrt{x})-f(x)\to 0$ as $\delta\to 0$ uniformly in $x\geqq 0$ and so to a given $\varepsilon>0$ there is a $\delta>0$ such that $|f(x\pm h\sqrt{x})-f(x)|\leqq\varepsilon$ whenever $0<h\leqq\delta$. This implies that

$$f(x)-\varepsilon-2\|f\|\delta^{-2}(t-x)^2/x \leqq f(t) \leqq f(x)+\varepsilon+2\|f\|\delta^{-2}(t-x)^2/x$$

and hence for $x\geqq 1$

$$|L_n(f;\,x)-f(x)| \leqq \varepsilon+2\|f\|\delta^{-2}L_n\big((t-x)^2,x\big)/x =$$

$$= \varepsilon+2\|f\|\delta^{-2}(2x+2/n)/nx \leqq \varepsilon+8\|f\|/\delta^2 n < 2\varepsilon$$

provided $n > 8\delta^{-2}\varepsilon^{-1}$. Since $L_n(f; x) - f(x) = o(1)$ $(n \to \infty)$, $0 \leqq x \leqq 1$, follows from the analogue of the localization theorem proved in the proof of Theorem 2 and from (2.2), the sufficiency part of the theorem has been verified.

Now let us suppose that $L_n(f; x) - f(x) = o(1)$ $(n \to \infty)$ uniformly on $[0, \infty)$. Exactly as in the proof of Theorem 4 one can show that $|L_n'(f; x)| \leqq K_n/\sqrt{x}$ and so

$$|f(x+\delta\sqrt{x}) - f(x)| \leqq 2\,\|L_N(f) - f\| + K_N \left| \int_x^{x+\delta\sqrt{x}} \tau^{-1/2}\, d\tau \right| < 2\varepsilon$$

if N is sufficiently large and δ is small. Thus $f(x+\delta\sqrt{x}) - f(x)$ tends to zero uniformly on $[0, \infty)$ as $\delta \to 0$ and this again is equivalent to the uniform continuity of $f(x^2)$. Thus the proof is complete.

Proof of Theorem 8. We shall constantly use the identities

$$L_n(1;\, x) = 1, \quad L_n(t, x) = x + 1/n \quad \text{and} \quad L_n\big((t-x)^2, x\big) = 2x/n + 2/n^2.$$

We separately prove the sufficiency and necessity of the given conditions.

(1) Sufficiency. Let us suppose that $xf'(x) \in \mathrm{Lip}\, 1$. This yields the absolute continuity of f' and the boundedness of $g(x) = (xf'(x))'$. We may suppose that $|g| \leqq 1$. Since then

$$f(x) = \int_0^x (1/\tau) \int_0^\tau g(u)\, du\, d\tau + c \log x + d,$$

with some constants c and d, and $f \in C_B$ implies $c = 0$, it follows that $|f'| \leqq 1$ and so $|xf''(x)| = |g(x) - f'(x)| \leqq 2$, i.e. $|f''(x)| \leqq 2/x$. For

$$h_x(t) = \int_x^t \int_x^\tau (du/u)\, d\tau,$$

we have

$$h_x(t) = t \log\big(1 + (t-x)/x\big) - t + x \leqq t(t-x)/x - (t-x) = (t-x)^2/x, \quad t \geqq 0$$

and $h_x''(t) = 1/t$, and the latter implies (see above) that the functions $\gamma_\pm = 2h_x \pm f$ are convex ($\gamma_\pm'' \geqq 0$). Now if γ is convex and differentiable, then, because of the inequality $\gamma(t) - \gamma(x) \geqq \gamma'(x)(t-x)$, we have

$$L_n(\gamma;\, x) - \gamma(x) \geqq \gamma'(x) L_n(t-x, x) = \gamma'(x)(1/n).$$

Putting here $\gamma = \gamma_\pm$ and taking into account that

$$|\gamma_\pm'(x)| = \left| \int_x^t (du/u)|_{t=x} \pm f'(x) \right| = |\pm f'(x)| \leqq 1$$

we get

$$|L_n(f;\ x)-f(x)| \leqq L_n(h_x;\ x)-h_x(x)+1/n \leqq$$

$$\leqq x^{-1}L_n((t-x)^2,\ x)+1/n = 2(x+1/n)/nx+1/n \leqq 5/n,$$

provided $x \geqq 1/n$.

If $0<x\leqq 1/n$, then we argue as follows. Since $f(t)=f(x)+\int_x^t f'(\tau)\,d\tau$, we have

$$|L_n'(f;\ x)| = \Big|\sum_{k=0}^{\infty} n\big(F_{n,k+1}(f)-F_{n,k+2}(f)\big)p_{n,k}(x)\Big| =$$

$$= \Big|\sum_{k=0}^{\infty} n \int_0^{\infty} ne^{-nt}((nt)^k/k!)(1-nt/(k+1)) \int_x^t f'(\tau)\,d\tau\,dt\Big| =$$

$$= \Big|\sum_{k=0}^{\infty} n \int_0^{\infty} p'_{n,k+1}(t) \int_x^t f'(\tau)\,d\tau\,dt\Big| = \Big|\sum_{k=0}^{\infty} n \int_0^{\infty} p_{n,k+1}(t)f'(t)\,dt\Big| \leqq \|f'\| \leqq 1$$

and so, according to what we have proved above

$$|L_n(f;\ x)-f(x)| \leqq 2|x-1/n|+|L_n(f;\ 1/n)-f(1/n)| \leqq 7/n \quad (0<x\leqq 1/n)$$

and the sufficiency part of the theorem has been established.

(2) Necessity. Let $W_1(t)=1/t$, $W_2(t)\equiv 1$. Using Taylor's formula (2.4) and the strong localization proved in Theorem 2 (clearly the proof also works for L_n) one can easily prove that if $f\in C_B$ is twice continuously differentiable on $[0,\infty)$ then

$$\lim_{n\to\infty} n(L_n(f;\ x)-f(x)) = \lim_{n\to\infty}\Big(nL_n(f'(x)(t-x);\ x)+nL_n\Big(\int_x^t (t-\tau)f''(\tau)\,d\tau;\ x\Big) =$$

$$= f'(x)+xf''(x) = (xf'(x))' = [(1/W_2)(f'/W_1)'](x)$$

and so [4, Theorem 5.7] gives (use also the localization) that

$$L_n(f;\ x)-f(x) = o(1/n)$$

implies that f has the form $f(x)=c\log x+d$ and so $(f\in C_B)$ f is constant. This proves the first half of the theorem.

Now suppose $L_n(f,x)-f(x)=O(1/n)$ uniformly on $[0,\infty)$. We set

$$g(x) = \overline{\lim_{n\to\infty}}\, n(L_n(f;\ x)-f(x)).$$

If $[a,b]\subseteq(0,\infty)$ is arbitrary, then $g\in L^1(a,b)$. Using once more the localization result for L_n we get from [4, Theorem 5.8] that on (a,b) f has the form

$$f(x) = d+c\log x+\int_a^x (1/t)\int_a^t g(s)\,ds\,dt,$$

i.e. f' is absolutely continuous on (a, b), and since (a, b) is arbitrary, it follows that f' is locally absolutely continuous on $(0, \infty)$. By [4, Lemma 5.9] and our localization result

$$\overline{\lim_{n\to\infty}}\, n\big(L_n(f;\ x)-f(x)\big) = \big(tf'(t)\big)'\big|_{t=x}$$

at every x, where the right hand side makes sense, and so $L_n(f; x)-f(x)=O(1/n)$ yields $\big(x(f'(x))'=O(1)$ and this completes the proof of the theorem.

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A generalization of a theorem of G. Freud on the differentiability of functions

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0. Introduction

Let f be a function defined on R^1 and let m be a positive integer. Then f has a Taylor polynomial of order m at $x=a$ if and only if there is a number C_m such that

$$\Delta_h^m f(x) = C_m h^m + \big((x-a)^m + h^m\big)\varepsilon(x, h), \tag{0.1}$$

where $\varepsilon(x, h) \to 0$, as $x \to a$ and $h \to 0$.

This result was announced by G. FREUD in [3]. It is our purpose to prove an L^p-version of (0.1) for functions in R^n, $1 \leqq p \leqq \infty$, and differentiability of order $l > 0$. This involves finding the L^p-form of (0.1). (See Section 1 for the exact definitions.) The methods of proof use approximation theorems of Jackson type due to H. WHITNEY [10] and JU. A. BRUDNYI [1]. As an application we consider the problem to characterize differentiability in terms of L^p-differentiability together with certain additional conditions. Such problems have been studied by the author in [4]—[6]. See also B.-M. STOCKE [8].

Section 1 contains our notation and the definitions. Our results are then stated in Section 2 and proved in Sections 4 and 5. The lemmas needed in the proofs are given in Section 3.

1. Notation and definitions

1.1. We use standard notation for points $x = (x_1, ..., x_n)$ and real or complex valued functions $f(x)$ in R^n. For $E \subset R^n$ we denote Lebesgue outer measure and Lebesgue measure by $|E|^*$ and $|E|$ respectively. Integration with respect to Lebesgue measure is written $\int_E f(x)\,dx$ and $L^p(E)$ denotes the usual Lebesgue classes, $1 \leqq p \leqq$

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$\leqq\infty$. All functions considered are measurable. We define $\|f\|_{L^p(E)}=\left(\int_E |f(x)|^p dx\right)^{1/p}$, $1\leqq p\leqq\infty$, with the usual modification when $p=\infty$. When $E=R^n$ we just write $\|f\|_p$.

We let $I=I(a,s)$ denote a cube in R^n with its sides parallel to the axes, its centre in a and having diameter s. The ball with centre at a and radius r is denoted by $B(a,r)$. The density of a measurable set E at x_0 is defined by $\lim_{r\to 0}|E\cap B(x_0,r)|\cdot$ $\cdot|B(x_0,r)|^{-1}$, if the limit exists. We let $c(\alpha,\beta,\ldots)$ denote constants, which may be different at each occurrence, depending on $\alpha,\beta,\ldots$. Constants only depending on n are denoted by c. Polynomials P are written $P(x)=\sum C_\alpha x^\alpha$, where the summation is over multi indices α.

1.2. Let $l>0$. A function f is said to be (ordinary) differentiable of order l at $x=a$ in R^n if there is a (Taylor) polynomial $P(x)$ of degree at most l such that $R(x,a)=f(x)-P(x)=o(|x-a|^l)$, as $x\to a$. We say that f is L^p-differentiable at $x=a$ of order l if

$$\left(|B(a,r)|^{-1}\int_{B(a,r)}|R(x,a)|^p\,dx\right)^{1/p}=o(r^l),\quad r\to 0,$$

for a suitable polynomial P. Here we make the usual modification when $p=\infty$. The polynomials P are unique in all cases.

The differences $\Delta_h^m f(x)$ are defined inductively by $\Delta_h^0 f(x)=f(x)$ and $\Delta_h^{m+1}f(x)=$ $=\Delta_h^m f(x+h)-\Delta_h^m f(x)$. It is easily verified that if $P(x)=\sum_{|\alpha|\leqq m}C_\alpha x^\alpha$, then $\Delta_h^m P(x)=$ $=m!\sum_{|\alpha|=m}C_\alpha\cdot h^\alpha$, for all x. For more properties of $\Delta_h^m f$, see [5] and [9, p. 102].

1.3. In this section we define a smoothness property for functions called C_s^j, using j-th order differences. We considered a slightly different property, also denoted by C_s^j, in [5]. For a comparison of these properties, see the second remark following Definition 1.1.

Definition 1.1. Let j be a positive integer and let $0\leqq s\leqq j+1$. Let f be a function defined in a neighbourhood of $x=a$ in R^n. We say that f has property C_s^j at $x=a$ if there exist numbers C_α, $|\alpha|=j$, such that for every $\varepsilon>0$ there are t and δ, $0<t<\min(\varepsilon,1)$ and $\delta>0$, such that

$$\sup_{|h|\leqq t|x-a|}\left|\Delta_h^j f(x)-j!\sum_{|\alpha|=j}C_\alpha h^\alpha\right|\leqq\varepsilon|x-a|^s,\tag{1.1}$$

for all x, $0<|x-a|<\delta$. We take $C_\alpha=0$, $|\alpha|=j$, if $s\leqq j$.

Replacing (1.1) by

$$\left(|B(0,t|x-a|)|^{-1}\int_{B(0,t|x-a|)}\left|\Delta_h^j f(x)-j!\sum_{|\alpha|=j}C_\alpha h^\alpha\right|^p dh\right)^{1/p}\leqq\varepsilon|x-a|^s,\tag{1.2}$$

$1\leqq p\leqq\infty$, gives an equivalent definition of property C_s^j. For the proof, see [5, Lemma 5.2].

The numbers C_α in (1.1) and (1.2) are unique, when $s>j$. In the case when $s\leqq j$, (1.1) and (1.2) are independent of the choice of C_α in the sense that they hold with all $C_\alpha=0$ if and only if they hold for some arbitrary C_α, $|\alpha|=j$.

Remark. We get an equivalent definition of property C_s^j if we in (1.1) or (1.2) replace $\Delta_h^j f$ by the binary differences $B_h^j f$, see [5, p. 51]. In proving this it is no loss of generality to assume that $C_\alpha=0$, $|\alpha|=j$, in (1.1) and (1.2) and hence the proof of [5, Lemma 5.4] applies. We leave the details to the reader.

Remark. The present definition of property C_s^j differs from the one in [5, p. 53] only when $s>j$. For example in the case of [5, Theorem 3.2], the two definitions agree. It can be proved that there is an alternative formulation of Theorem 3.2 in [5] based on (a)—(c) in [5, pp. 53—54] and using property C_s^j. Finally we note that property C_s^1 in the same as property B_s in [4, p. 9], when $0<s\leqq 1$.

2. Main results

2.1. Our first theorem is a characterization of L^p-differentiability which generalizes G. Freud's result in [3].

Theorem 2.1. *Let $f(x)$ be a function defined in a neighbourhood of $x=a$ in R^n. Let $l>0$, $1\leqq p\leqq\infty$ and let $m\leqq l<m+1$, where m is a non-negative integer. Then f is L^p-differentiable at $x=a$ if and only if there are numbers C_α, $|\alpha|=m$, such that*

$$\sup_{|h|\leqq t}\Big(|B(a,t)|^{-1}\int_{B(a,t)}\big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\big|^p\,dx\Big)^{1/p}=o(t^l),\quad as\quad t\to 0. \tag{2.1}$$

We make the usual modification in (2.1) when $p=\infty$.

As we mentioned in the introduction, Theorem 2.1 also holds for the case of differentiability in the ordinary sense with (2.1) replaced by

$$\sup_{|h|\leqq s}\ \sup_{|x-a|\leqq s}\big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\big|=o(s^l), \tag{2.2}$$

as $s\to 0$. Essentially the same proof applies. We omit the details.

Remark. In the case $0<l<1$, we have $m=0$ and $\Delta_h^0 f(x)=f(x)$. Then (2.1) is just the definition of L^p-differentiability. In the case when $l\geqq 1$ and (2.1) holds, there exist numbers C_α, $|\alpha|\leqq m-1$, such that the L^p-differential of f at $x=a$ is given by $P(x)=\sum_{|\alpha|\leqq m}C_\alpha(x-a)^\alpha$.

2.2. Next we use ideas from [4]—[6] and combine Theorem 2.1 with (2.2) to prove results on the connection between differentiability and L^p-differentiability.

Theorem 2.2. *Let $l>0$ and $1\leqq p\leqq\infty$. Let m be an integer such that $m\leqq l<m+1$, when $l\geqq 1$ and $m=1$, when $0<l<1$. Then f is differentiable at $x=a$ of order l if and only if f has an L^p-differential of order l at $x=a$ and f has property C_l^m at $x=a$.*

The following variant of Theorem 2.2 can be proved with the same methods.

Theorem 2.3. *Let $0<l<m$, where m is an integer, and let $1\leqq p\leqq\infty$. Then a function f is differentiable at $x=a$ of order l if and only if f is L^p-differentiable at $x=a$ of order l and f has property C_l^m at $x=a$.*

Theorems 2.2 and 2.3 should be compared to the corresponding results in [4] and [5]. Theorem 2.3 generalizes Theorem 3.2 in [5] and Theorem 2.2 is a generalization of the alternative formulation of Theorem 3.2 in [5] which was mentioned in the second remark following Definition 1.1. Examples show that property C_l^m alone does not imply differentiability of order l in Theorems 2.2 and 2.3.

Our results in [4], [5] and the present paper can be summarized as follows. We want to characterize differentiability of a function $f(x)$ at $x=a$ by L^p-differentiability together with certain additional conditions. These additional conditions can roughly be described as follows (differentiability of order l, $1\leqq m\leqq l<m+1$, where m is an integer):

(i) ([4]) f_m has property C_l^1 at $x=a$, where $f_m(x)=f(x)-\sum_{1\leqq|\alpha|\leqq m} C_\alpha(x-a)^\alpha$,

(ii) ([5]) f has properties C_s^j at $x=a$, for $1\leqq j\leqq m+1$, and suitable s,

(iii) (present paper) f has property C_l^m (or C_l^{m+1}) at $x=a$.

The case (i) was applied to Bessel potentials of L^p-functions in [4]. The advantage of (iii) compared to (ii) is that it has just one single condition.

2.3. It was proved in [4, Lemma 5.2] that property C_l^1, $0<l\leqq 1$, at $x=a$ implies that $|f(x)-b|\leqq M\cdot|x-a|^l$, for suitable numbers b and M, when x is close to a. Our next theorem generalizes this result. Roughly speaking, we prove that property C_l^m is of Lipschitz type.

Theorem 2.4. *Let m be a positive integer and $s>m-1$, and let f be a function. Assume that there are $\varepsilon>0$, $\delta>0$ and $0<t<1$ such that $0<|x-a|\leqq\delta$ implies*

$$\sup_{|h|\leqq t|x-a|} |\Delta_h^m f(x)| \leqq \varepsilon|x-a|^s. \tag{2.3}$$

Then there is a unique polynomial $P(x)$ of degree at most $(m-1)$ such that

$$|f(x)-P(x)| \leqq M|x-a|^s \tag{2.4}$$

for $0<|x-a|\leqq 4\delta/5$, where M is a suitable number depending on n, m, s and t.

Corollary. *Let f have property C_l^m at $x=a$, where l and m are as in Theorem 2.2. Then there is a unique polynomial P of degree less than l such that*

$$|f(x)-P(x)| \leqq M|x-a|^l$$

for $0<|x-a|<\delta$, *where* $\delta>0$ *and* M *are suitable constants.*

The corollary follows easily from Definition 1.1 and Theorem 2.4.

3. Some lemmas

It is well known that if a function $f(x)$ satisfies $\limsup_{h\to 0} |\Delta_h^m f(x)|<\infty$, for every x in a measurable set E, then f is bounded in a neighbourhood of a.e. $x\in E$, cf. [7, p. 249]. We need the following L^p-form of that result.

Lemma 3.1. *Let m be a positive integer and $1\leqq p\leqq\infty$. Assume that for every x_0 in a measurable set $E\subset R^n$, there is $r_0=r_0(x_0)>0$ such that*

$$\int_{|h|\leqq r_0} dh \int_{|x-x_0|\leqq r_0} |\Delta_h^m f(x)|^p\, dx<\infty,$$

if $1\leqq p<\infty$, *and*

$$\operatorname*{ess\,sup}_{|x-x_0|\leqq r_0}\ \operatorname*{ess\,sup}_{|h|\leqq r_0} |\Delta_h^m f(x)|<\infty,$$

if $p=\infty$. *Then for a.e.* $x_0\in E$ *there is* $r=r(x_0)>0$ *such that* f *belongs to* $L^p(B)$, *where* $B=B(x_0, r)$.

Proof of Lemma 3.1. Let $1\leqq p<\infty$. It is no loss of generality to assume that f is finite in E. Define for $i=1, 2, \ldots$

$$E_i=\Big\{x_0\in E;\ \int_{|h|\leqq 1/i} dh \int_{|x-x_0|\leqq 1/i} |\Delta_h^m f(x)|^p\, dx\leqq i\Big\},$$

then $E=\bigcup_1^\infty E_i$. In contrast to the case in [7], the sets E_i are here measurable. There exists a closed subset P of E, with $|E\setminus P|$ arbitrarily small, such that the restriction of f to P is continuous on P. Define $B_i=\{x;\ E_i$ has density one at $x\}$, $C=\{x;\ P$ has density one at $x\}$ and $A_i=B_i\cap C$, $i=1, 2, \ldots$. Let $A=\bigcup_1^\infty A_i$, then

$$E\setminus A\subset\Big(\bigcup_1^\infty (E_i\setminus B_i)\Big)\cup(E\setminus P)\cup(P\setminus C).$$

We get $|E\setminus A|\leqq|E\setminus P|$, since $E_i\setminus B_i$ and $P\setminus C$ have measure zero. It is our purpose to prove that f is locally in L^p near every point in A. Let i be fixed and let $x_0\in A_i$.

We now have the following identity

$$(-1)^m(f(x)-f(x_0)) = \sum_{s=0}^{m-1}(-1)^s\binom{m}{s}(f(x_0)-f(z+sh))+\Delta_h^m f(z),$$

where $z+mh=x$. Let x be fixed, $0<|x-x_0|<r$. We use [5, Lemma 4.3] to split the last term in the above identity as follows

$$(-1)^m(f(x)-f(x_0)) = \sum_{s=0}^{m-1}(-1)^s\binom{m}{s}(f(x_0)-f(z+sh))+$$
$$+\sum_{i=1}^{m-1}(-1)^i\binom{m}{i}\Delta^m_{(i/m)(k-h)}f(z+ih)+(-1)^m\Delta^m_{k-h}f(x)-$$
$$-\sum_{j=1}^{m}(-1)^j\binom{m}{j}\Delta^m_{h+(j/m)(k-h)}f(z).$$

Let $|z-x_0|\leqq M|x-x_0|$, where M is to be chosen below, then $|h|\leqq(M+1)r$. Taking the L^p-norm over $|k|\leqq r$ we get

$$|f(x)-f(x_0)| \leqq \sum_{s=0}^{m-1}\binom{m}{s}|f(x_0)-f(z+sh)|+ \tag{3.1}$$
$$+c(n, M)\sum_{j=0}^{m-1}\Big(|B(0, r)|^{-1}\int_{|w|\leqq(M+2)r}|\Delta^m_w f(z+jh)|^p\,dw\Big)^{1/p}+$$
$$+c(n, m, M)\Big(|B(0, r)|^{-1}\int_{|w|\leqq(M+2)r}|\Delta^m_w f(x)|^p\,dw\Big)^{1/p} = I_1(z)+I_2(z)+I_3(x).$$

There are $r>0$ and $M=M(r)$ such that the density at x_0 of the set

$$S = \{z;\ (j/m)x+(1-j/m)z\in P,\ \text{ for } j = 1, 2, \dots, m-1 \text{ and } |z-x_0| \leqq M|x-x_0|\}$$

is arbitrarily close to one, for all $0<|x-x_0|<r$, cf. [7, p. 267]. Further,

$$\int_{B(x_0,r)} I_2(z)^p\,dz \leqq c(n, M)r^{-n}i^p$$

if $0<r\leqq c(n, m)\cdot(1/i)$. Hence, if r is small enough, we can for every x, $0<|x-x_0|<r$, find $z\in S$ such that $I_1(z)$ is arbitrarily small and $I_2(z)\leqq c(n, M, r, p)$. Integrating the p-th power of (3.1) w.r.t. x over the set $B(x_0, r)$ yields

$$\int_{B(x_0,r)}|f(x)-f(x_0)|^p\,dx \leqq c(n, M, r, p)$$

for some $r>0$.

We have proved that f is locally in L^p at every point of A. The conclusion of the lemma follows since $|E\setminus A|$ can be made arbitrarily small. The case $p=\infty$ is treated analogously and we omit the details.

Lemma 3.2. *Let* $1 \leqq p \leqq \infty$ *and assume that* (2.1) *holds. Then* f *is locally in* L^p *at* a.

Proof of Lemma 3.2. Let $1 \leqq p < \infty$. By (2.1) there are positive numbers δ and M such that $|h| \leqq \delta$ and $|x_0 - a| \leqq \delta$ imply

$$\int_{B(x_0, \delta)} |\Delta_h^m f(x)|^p \, dx \leqq M^p.$$

Then by Lemma 3.1 there are x_0, with $|x_0 - a| < \delta/2$, and positive numbers r and N such that $\int_{B(x_0, r)} |f(x)|^p dx = N^p < \infty$. There is nothing to prove unless $r \leqq \delta$. Consider the identity

$$\Delta_h^m f(x) = \sum_{i=0}^{m-1} (-1)^{m-i} \binom{m}{i} f(x+ih) + f(x+mh). \tag{3.2}$$

Let $|h| = r/m$. We integrate the p-th power of (3.2) w.r.t. x over $B(x_0, s)$, where $s = r - (m-1) \cdot |h| = r/m$. This gives

$$\int_{B(x_0+mh, s)} |f(y)|^p \, dy \leqq (M + 2^m N)^p.$$

Since a certain portion of $B(x_0 + mh, s)$ lies outside $B(x_0, r)$, a simple geometrical argument shows that $\int_{B(x_0, r_1)} |f(y)|^p dy$ is finite, where $r_1 = r + |h|/2$. If $r_1 > \delta$ we are done. Otherwise, repeating this procedure a finite number of times, we find $r_0 > \delta$ such that $\int_{B(a, \delta/2)} |f(x)|^p dx \leqq \int_{B(x_0, r_0)} |f(x)|^p dx < \infty$. The case $p = \infty$ is handled analogously. This completes the proof of Lemma 3.2.

Our next lemma is a slightly generalized form of a lemma due to De Giorgi, see [2, p. 140]. We omit the proof.

Lemma 3.3. *Let* $I_1 = I(a, u)$ *and* $I_2 = I(b, v)$ *be two cubes such that* $I_1 \subset I_2$. *Let* j *be a non-negative integer and* $1 \leqq p \leqq \infty$. *Then*

$$\sup_{I_2} |D^\alpha P(x)| \leqq c(n, j)(v/u)^j v^{-|\alpha|} \left(|I_1|^{-1} \int_{I_1} |P(x)|^p dx\right)^{1/p}$$

for all polynomials P *of degree* $\leqq j$. *The lemma also holds for balls instead of cubes.*

We make the usual modification in the right hand side of the inequality when $p = \infty$.

4. Proof of Theorems 2.1 and 2.2

4.1. Proof of Theorem 2.1. The proof of the necessity follows easily by integrating the identity

$$\Delta_h^m f(x)-m!\sum_{|\alpha|=m} C_\alpha h^\alpha = \Delta_h^m (f-P)(x),$$

where $P(x)=\sum_{|\alpha|\leqq m} C_\alpha (x-a)^\alpha$. The details are left to the reader.

Suppose that (2.1) holds. We may assume that $C_\alpha=0$ for $|\alpha|=m$, and that $f(x)=0$ outside some neighbourhood of $x=a$. Then by Lemma 3.2 we may also assume that f is in L^p.

Let $I_s=I(a,s)$ be a cube. Define $I_{s,h}=\{x;\ x+kh\in I_s,\ \text{for } k=0,1,\ldots,m\}$, then $I_{s,h}\subset I_s$. Now by [1, Theorem 1'] there is, for every $s>0$, a polynomial P_s of degree at most $(m-1)$ such that

$$\|f-P_s\|_{L^p(I_s)} \leqq c(n,m)\sup_{|h|\leqq s}\|\Delta_h^m f\|_{L^p(I_{s,h})}.$$

Then by our assumption (2.1) we get

$$\|f-P_s\|_{L^p(I_s)}\leqq s^l v(s), \tag{4.1}$$

where $v(s)$ is a non-decreasing function tending to zero as $s\to 0$.

Now let P_k be the polynomial P_s, when $s=2^{-k}$, $k=1,2,\ldots$. We intend to prove that the sequence $\{P_k\}_1^\infty$ converges uniformly on compact sets to a polynomial P with the desired properties. More exactly, we prove that there is a polynomial P of degree at most $(m-1)$ such that

$$|D^\alpha P_k(a)-D^\alpha P(a)| = o(2^{-k(l-|\alpha|)}),\quad \text{as } k\to\infty, \tag{4.2}$$

for $|\alpha|\leqq m-1$. It then follows from Taylor's formula that for $s=2^{-k}$ we have $\sup_{x\in I_s}|P_k(x)-P(x)|=o(2^{-kl})$, as $k\to\infty$. Combining this with (4.1) we get the conclusion of the theorem with the polynomial P defined above.

It remains to prove (4.2). Now (4.1) and Lemma 3.3 give that for $|\alpha|\leqq m-1$,

$$|D^\alpha P_{k+1}(a)-D^\alpha P_k(a)| \leqq c(n,m)\Big(|I_{k+1}|^{-1}\int_{I_{k+1}}|P_{k+1}(x)-P_k(x)|^p\,dx\Big)^{1/p}\leqq$$

$$\leqq c(n,m)2^{k(|\alpha|-l)}v(2^{-k}).$$

Then for $i>k$ we get by summation

$$|D^\alpha P_i(a)-D^\alpha P_k(a)|\leqq \sum_k^{i-1}|D^\alpha P_{j+1}(a)-D^\alpha P_j(a)|\leqq c(n,m,l)2^{k(|\alpha|-l)}v(2^{-k}).$$

Hence the sequences $\{D^\alpha P_k(a)\}_1^\infty$, $|\alpha|\leqq m-1$, converge and there is a polynomial P such that

$$|D^\alpha P_k(a)-D^\alpha P(a)| = o(2^{k(|\alpha|-l)}),$$

as $k\to\infty$. This proves (4.2) and thereby completes the proof of Theorem 2.1.

4.2. Proof of Theorem 2.2. We omit the necessity part of the proof since it is straightforward, cf. the corresponding results in [4] and [5]. Assume that f is L^p-differentiable of order l and that f has property C_l^m at $x=a$, for l and m as in the theorem. We first assume that $1\leqq m\leqq l<m+1$. Let $P(x)=\sum_{|\alpha|\leqq m} C_\alpha(x-a)^\alpha$ be the L^p-differential of f at $x=a$. Then the constants C_α, $|\alpha|=m$, in $P(x)$ are the same as the constants C_α in (2.1) by the remark following Theorem 2.1. We claim that it is no loss of generality to assume that we have the same constants C_α, $|\alpha|=m$, in (1.1) and (2.1).

When $l=m$ our claim follows from the fact that in this case (1.1) is independent of the choice of constants C_α. Now let $m<l<m+1$ and denote these constants in (1.1) and (2.1) by C_α and C'_α respectively. Then

$$\Big|m!\sum_{|\alpha|=m}(C_\alpha-C'_\alpha)h^\alpha\Big| \leqq \Big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\Big|+\Big|\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C'_\alpha h^\alpha\Big|.$$

Let $0<\varepsilon<1$ be arbitrary and choose t and δ as in Definition 1.1. Let $0<s<$ $<\min(t,\delta/2)$ and $|h|\leqq st$. We integrate the above inequality to the p-th power w.r.t. x over the set $E_s=\{x;\ s\leqq|x-a|\leqq 2s\}$. Then using (1.1) and (2.1) we get

$$\sup_{|h|\leqq st}\Big|m!\sum_{|\alpha|=m}(C_\alpha-C'_\alpha)h^\alpha\Big| \leqq o(s^l)+2^l\varepsilon s^l, \quad \text{as} \quad s\to 0.$$

It follows easily that $C_\alpha=C'_\alpha$, $|\alpha|=m$, by letting s tend to zero, since $l>m$. Thereby our claim is proved.

Now consider the identity

$$f(x)-P(x) = (-1)^m\Big(\Delta_h^m f(x)-m!\sum_{|\alpha|=m}C_\alpha h^\alpha\Big)-$$
$$-\sum_{k=1}^m\binom{m}{k}(-1)^k(f(x+hk)-P(x+kh)).$$

Let $\varepsilon>0$ be arbitrary. Choose t and δ as in Definition 1.1 and let $0<|x-a|<\delta$. Then integrating this identity to the p-th power w.r.t. h over the set $|h|\leqq t|x-a|$ and using (1.1) and the definition of the L^p-derivative yields

$$|f(x)-P(x)| \leqq \varepsilon|x-a|^l+o(|x-a|^l), \quad \text{as} \quad x\to a.$$

This proves the theorem when $l\geqq 1$. The case $0<l<1$ is much simpler and its proof is left to the reader.

5. Proof of Theorem 2.4

The proof is a combination of the methods of proof in Theorem 2.2 and [4, Lemma 5.1]. Therefore we will not go into so many details.

Let x_0 be such that $|x_0=a|=4\delta/5$ and let L denote the line segment between a and x_0. Define a sequence $\{x_i\}_1^\infty$ of points on L such that $|x_i-a|=r_i=r_0(1-t/4)^i$, $i=1, 2, \dots$, where $r_0=|x_0-a|$. Then $|x_i-x_{i+1}|=(t/4)|x_i-a|$. Define $B_i= =B(x_i, (t/4)|x_i-a|)$, $i=0, 1, \dots$. If x and $x+mh$ belong to B_i, then $|h|\leqq t|x-a|$ and hence (2.3) and the definition of B_i give

$$|\Delta_h^m f(x)| \leqq \varepsilon|x-a|^s \leqq \varepsilon \cdot 2^s|x_i-a|^s.$$

Now by Brudnyi [1, p. 158] there is a polynomial P_i of degree at most $m-1$ such that

$$|f(x)-P_i(x)| \leqq c(n, m, s)\varepsilon|x_i-a|^s, \tag{4.3}$$

for all $x\in B_i$, $i=0, 1, \dots$. We claim that the sequence $\{P_i(x)\}_0^\infty$ of polynomials converges uniformly on compact sets to a polynomial $P(x)$ with the desired properties.

We first note that for $x\in B_i\cap B_{i+1}$, $i=0, 1, \dots$, we have $|P_i(x)-P_{i+1}(x)|\leqq \leqq c(n, m, s)\varepsilon|x_i-a|^s$. Define $E_i=B(a, 2|x_i-a|)$, $i=0, 1, \dots$. Now $B_i\cap B_{i+1}$ contains a ball B_i' of radius $(t/8)|x_i-a|$ and $B_i'\subset E_i$, $i=0, 1, \dots$. We apply Lemma 3.3 to the balls B_i' and E_i and we get

$$|D^\alpha P_i(a)-D^\alpha P_{i+1}(a)| \leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$$

for $|\alpha|\leqq m-1$ and $i=0, 1, \dots$.

By the triangle inequality and summation of the last inequalities we get, since $s>m-1$,

$$|D^\alpha P_i(a)-D^\alpha P_j(a)| \leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$$

for $j>i$. It now follows that there is a polynomial P of degree at most $(m-1)$ such that $|D^\alpha P(a)-D^\alpha P_i(a)|\leqq c(n, m, s, t)\varepsilon|x_i-a|^{s-|\alpha|}$ for $|\alpha|\leqq m-1$ and $i=0, 1, \dots$. This proves the first part of our claim. Taylor's formula and (4.3) yield the estimate

$$|f(x)-P(x)| \leqq c(n, m, s, t)\varepsilon|x-a|^s \tag{4.4}$$

for $x\in\bigcup_1^\infty B_i$.

We now repeat the procedure above with other points x_0', $|x_0'-a|=4\delta/5$, such that the corresponding sets $\bigcup_1^\infty B_i'$ cover the set $\{x;\ 0<|x-a|\leqq 4\delta/5\}$. This can be done in N steps, where N only depends on n and t. If x_0 and x_0' are such that the sets $\bigcup_1^\infty B_i$ and $\bigcup_1^\infty B_i'$ overlap, then the polynomials constructed above must be identical, because of (4.4). It follows that (4.4) holds for all x, $0<|x-a|\leqq 4\delta/5$. This settles the remaining part of our claim.

Finally, the uniqueness of the polynomial P follows easily from (2.4), since $s>m-1$. This completes the proof of Theorem 2.4.

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On approximation by Euler means of orthogonal series

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1. Introduction

Let $\{\varphi_n(x)\}$ be an orthonormal system on the interval $[0, 1]$. We consider orthogonal series

(1) $$\sum_{n=0}^{\infty} c_n \varphi_n(x), \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty$$

and their Euler means of order q, $0<q<1$, ((E, q)-means)

(2) $$t_n(x) = \sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} s_k(x) \quad (n = 0, 1, \ldots),$$

where $s_k(x) = \sum_{\nu=0}^{k} c_\nu \varphi_\nu(x)$. Our main interest is directed to the rate of convergence of strong (E, q)-means

(3) $$\tau_n^{(\gamma)}(x) = \Big\{\sum_{k=0}^{n} \binom{n}{k} q^k (1-q)^{n-k} |s_k(x) - f(x)|^\gamma\Big\}^{1/\gamma} \quad (n = 0, 1, \ldots)$$

on $[0, 1]$ with $0<q<1$ and $\gamma>0$; $f(x)$ is connected with the given series (1) by the Riesz—Fischer theorem.

To this end we assume

(4) $$\sum_{n=0}^{\infty} c_n^2 \lambda^2(n) < \infty,$$

where $\{\lambda(n)\}$ should be a nondecreasing sequence of positive real numbers tending to infinity which satisfies for a suitable δ, $0<\delta<1$, the following condition

(5) $$\lambda(n) \leqq C_\delta \lambda([\delta n]) \quad (n = 1, 2, \ldots).$$

Let the class Λ consist of all such sequences $\{\lambda(n)\}$. For (E, q)-means, V. I. Koljada [3] proved

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Theorem A. (a) *Let* $\{\lambda(n)\}\in\Lambda$ *and* $0<q<1$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)\log^2 n<\infty$, *then*

$$t_n(x)-f(x) = o_x(1/\lambda(n))$$

holds true almost everywhere (a.e.).

(b) *Let* $\{\lambda(n)\}\in\Lambda$. *If the sequence* $\{\mu(n)\}$ *of positive numbers satisfies the condition* $\lambda(n)=o(\mu(n))$, *then there exists an orthogonal series* $\sum_{n=0}^{\infty} c_n\varphi_n(x)$ *with* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)\log^2 n<\infty$ *and*

$$\overline{\lim_{n\to\infty}}\,\mu(n)|t_n(x)-f(x)| = \infty.$$

Here we prove for strong (E, q)-means

Theorem 1. *Let* $\{\lambda(n)\}\in\Lambda$ *and* $0<q<1$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then* $t_n(x)-f(x)=o_x(\lambda^{-1}(n))$ *a.e. implies*

$$\tau_n^{(\gamma)}(x) = o_x(1/\lambda(n)) \quad a.e.$$

for every $\gamma>0$.

We show that the conditions in Theorem 1 are not redundant:

Remark 1. For any method (E, q), $0<q<1$, there exists an orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x)$ with $\sum_{n=0}^{\infty} c_n^2\lambda^2(n)<\infty$, $\{\lambda(n)\}\in\Lambda$, such that

$$\tau_n^{(\gamma)}(x) \neq o_x(1/\lambda(n)) \quad \text{a.e.} \quad (\gamma \geqq 1).$$

Remark 2. For any method (E, q), $0<q<1$, there exists an orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x)$, $\sum_{n=0}^{\infty} c_n^2<\infty$, with $|t_n(x)-f(x)|=o_x(1/\lambda(n))$ a.e. $(\{\lambda(n)\}\in\Lambda)$ and

$$\tau_n^{(\gamma)}(x) \neq o_x(1/\lambda(n)) \quad \text{a.e.} \quad (\gamma > 0).$$

The proof of the remarks will be found in Section 4.

The condition $t_n(x)-f(x)=o_x((\lambda(n))^{-1})$ in our theorem may be substituted by a condition concerning the rate of approximation of certain partial sums, as follows from the next theorem. For this purpose we consider sequences of natural numbers $\{m_i\}$ with a gap condition

$$\alpha\sqrt{m_i} < m_{i+1}-m_i < \beta\sqrt{m_{i+1}} \quad (0<\alpha<\beta<\infty). \tag{6}$$

Theorem 2. *Let the sequences* $\{m_i\}$ *and* $\{m_i^*\}$ *satisfy a gap condition* (6) *and let* $\{\lambda(n)\}\in\Lambda$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then for partial sums and* (E, q)*-means,* $0<q<1$; *of the series* (1)

(a) $s_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. implies* $s_{m_i^*}(x)-f(x)=o_x(1/\lambda(m_i^*))$ *a.e.;*

(b) $t_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. implies* $t_n(x)-f(x)=o_x(1/\lambda(n))$ *a.e.;*

(c) $s_{m_i}(x)-f(x)=o_x(1/\lambda(m_i))$ *a.e. holds if and only if* $t_n(x)-f(x)=o_x(1/\lambda(n))$ *a.e.*

As an immediate consequence of the last theorem we get a result comparing (E, q)-means with different orders, proved by E. MARTIN [5] for a subclass of Λ:

Corollary. *Let* $\{\lambda(n)\}\in\Lambda$. *If* $\sum_{n=2}^{\infty} c_n^2\lambda^2(n)<\infty$, *then for* (E, q)*-means* $t_n^{(q)}(x)$ *and* (E, p)-means $t_n^{(p)}(x)$, $0<p, q<1$, *of the series* (1) *it holds:*

$$t_n^{(q)}(x)-f(x)=o_x(1/\lambda(n)) \quad \textit{a.e. implies} \quad t_n^{(p)}(x)-f(x)=o_x(1/\lambda(n)) \quad \textit{a.e.}$$

2. Lemmas

For the proof of our theorems we require some auxiliary results. In the following we assume a method (E, q) with a fixed order q, $0<q<1$, and we put $\bar{n}=\min\{k\in\mathrm{N}\colon n\leqq qk\}$. Obviously $[q\bar{n}]=n$ holds. Now it is possible to define the sequences $\{n_i\}$ and $\{\bar{n}_i\}$ by the following relations

$$n_0=1, \quad n_{i+1}=n_i+\left[\sqrt{\bar{n}_i}\right] \quad (i=0, 1, \ldots). \tag{7}$$

We put for brevity

$$e_{n\nu}=\binom{n}{\nu} q^\nu(1-q)^{n-\nu} \quad (\nu\leqq n)$$

and consider the differences

$$t_n(x)-s_{[qn]}(x)=\sum_{k=0}^{n} d_{nk}c_k\varphi_k(x)$$

with

$$d_{nk}=\begin{cases} -\sum\limits_{\nu=0}^{k-1} e_{n\nu} & (1\leqq k\leqq qn),\\ \sum\limits_{\nu=k}^{n} e_{n\nu} & (qn<k\leqq n),\\ 0 & (k=0 \quad \text{or} \quad k>n). \end{cases}$$

The following estimates can be found in L. KANTOROWITSCH [1] (Lemma 1), resp. H. SCHWINN [7] (p. 20).

Lemma 1. *Let* $0<q'<q<1$. *Then we have*

(a) $e_{nk}\leqq d_{n(k+1)}\leqq A(s)/n^s \quad (0<k\leqq q'n;\ s>0)$;

(b) $\sum\limits_{i:[q'\bar{n}_i]\leqq k\leqq\bar{n}_i} d_{\bar{n}_i k}^2\leqq B \quad (k=1, 2, \ldots)$.

With regard to some structural properties of Λ, V. I. KOLJADA [3] (Lemma 1) proved

Lemma 2. *Let* $\{\lambda(n)\}\in\Lambda$.

(a) *There exists a constant* $p>0$ *with* $\lambda(n)=O(n^p)$.

(b) *For every* δ', $0<\delta'<1$, *it yields* $\lambda(n)\leqq C_{\delta'}\lambda([\delta' n])$.

Lemma 3. *Let the coefficients of the series* (1) *satisfy condition* (4) *with* $\{\lambda(n)\}\in\Lambda$, *and with respect to* (E, q), $0<q<1$, *let the sequences* $\{n_i\}$, $\{\bar{n}_i\}$ *be defined as in* (7). *Then*

$$t_{\bar{n}_i}(x)-s_{n_i}(x)=o_x\big(1/\lambda(\bar{n}_i)\big)\quad a.e.$$

Proof. This lemma is evident if we show

$$\sum:=\sum_{i=1}^{\infty}\int_0^1\lambda^2(\bar{n}_i)\big(t_{\bar{n}_i}(x)-s_{n_i}(x)\big)^2\,dx=\sum_{i=1}^{\infty}\lambda^2(\bar{n}_i)\sum_{k=1}^{\bar{n}_i}d^2_{\bar{n}_i k}c_k^2<\infty.$$

With the aid of Lemma 2 (a) we have $\lambda(n)=O(n^p)$. Choosing $s=p+1$ in Lemma 1 (a), we get $(0<q'<q<1)$

$$\sum\nolimits_1:=\sum_{i=1}^{\infty}\lambda^2(\bar{n}_i)\sum_{k=1}^{[q'\bar{n}_i]}d^2_{\bar{n}_i k}c_k^2=O(1)\sum_{i=1}^{\infty}\frac{1}{\bar{n}_i^2}\sum_{k=1}^{\infty}c_k^2<\infty.$$

Finally, with Lemma 1 (b) and Lemma 2 (b), we get

$$\sum\nolimits_2=\sum_{i=1}^{\infty}\lambda^2(\bar{n}_i)\sum_{k=[q'\bar{n}_i]+1}^{\bar{n}_i}d^2_{\bar{n}_i k}c_k^2\leqq C^2_{q'}\sum_{i=1}^{\infty}\sum_{k=[q'\bar{n}_i]+1}^{\bar{n}_i}d^2_{\bar{n}_i k}c_k^2\lambda^2(k)\leqq C^2_{q'}B\sum_{k=1}^{\infty}c_k^2\lambda^2(k)<\infty$$

which proves together with $\sum_1<\infty$ the convergence of $\sum$.

Using the sequence $\{n_i\}$ defined in (7) we construct the following sequence of functions

$$\sigma_n^*(x)=\big(1/(n+n_{i+1}-2n_i+1)\big)\sum_{k=n_i}^{n}\big(s_k(x)-s_{n_i}(x)\big)\quad(n_i\leqq n<n_{i+1};\ i=0,1,\ldots).$$

The next lemma which is an analogue of a result of G. Sunouchi [9] can be found in [8]:

Lemma 4. *For any* $\gamma>0$,

$$\int_0^1\Big\{\big(1/2(n_{i+1}-n_i)\big)\sum_{n=n_i+1}^{n_{i+1}-1}|s_n(x)-s_{n_i}(x)-\sigma_n^*(x)|^\gamma\Big\}^{2/\gamma}dx\leqq$$

$$\leqq A(\gamma)\sum_{n=n_i+1}^{n_{i+1}-1}c_n^2\quad(i=0,1,\ldots).$$

Lemma 5. *Condition* (4) *with* $\{\lambda(n)\}\in\Lambda$ *implies*

$$\sigma_n^*(x)=o_x\big(1/\lambda(n)\big)\quad a.e.:$$

Proof. We consider
$$\Delta_i(x) := \max_{n_i<n<n_{i+1}} |\sigma_n^*(x)|^2$$
and get on account of $\sigma_{n_i}^*(x)\equiv 0$ when applying the Cauchy inequality
$$\Delta_i(x) \leqq \Big\{\sum_{n=n_i+1}^{n_{i+1}-1} |\sigma_n^*(x)-\sigma_{n-1}^*(x)|\Big\}^2 \leqq (n_{i+1}-n_i)\sum_{n=n_i+1}^{n_{i+1}-1}(\sigma_n^*(x)-\sigma_{n-1}^*(x))^2.$$
Using the relations $\lambda(n_{i+1})\leqq C^*\lambda(n_i)$ (cf. (5), Lemma 2 (b)) and $n_{i+1}-n_i<n+$ $+n_{i+1}-2n_i\leqq 2(n_{i+1}-n_i)$ if $n_i<n<n_{i+1}$, we get
$$\int_0^1 \lambda^2(n_{i+1})\Delta_i(x)\,dx =$$
$$= O(1)\lambda^2(n_i)(n_{i+1}-n_i)\sum_{n=n_i+1}^{n_{i+1}-1}(1/(n+n_{i+1}-2n_i)^4)\sum_{k=n_i+1}^{n}(k+n_{i+1}-2n_i)^2c_k^2 =$$
$$= O(1)\sum_{k=n_i+1}^{n_{i+1}-1} c_k^2\lambda^2(k).$$
Thus $\int_0^1 \sum_{i=0}^{\infty}\lambda^2(n_{i+1})\Delta_i(x)\,dx<\infty$ and B. Levi's theorem leads us to
$$\lambda(n)|\sigma_n^*(x)| \leqq \lambda(n_{i+1})\max_{n_i\leqq n<n_{i+1}}|\sigma_n^*(x)| = o_x(1) \quad \text{a.e.}$$
$(n_i\leqq n<n_{i+1})$ which proves our lemma.

Dependent on the order q of (E, q) and with the aid of sequence $\{\bar{n}_i\}$ (cf. (7)) we introduce a sequence $\{\bar{\lambda}(n)\}$ by the definition
$$\bar{\lambda}(n) = \lambda(\bar{n}_i) \quad (n_i \leqq n < n_{i+1};\ i = 0, 1, \ldots).$$
Then the method (E, q) transforms $\{(\lambda(n))^{-1}\}$, resp. $\{(\bar{\lambda}(n))^{-1}\}$ with the following properties:

Lemma 6. *Let* $\{\lambda(n)\}\in\Lambda$ *and* $\gamma>0$. *Then*

(a) $\sum_{k=0}^{n} e_{nk}(1/\lambda^\gamma(k)) = O(1/\lambda^\gamma(n))$,

(b) $\sum_{k=0}^{n} e_{nk}(1/\bar{\lambda}^\gamma(k)) = O(1/\bar{\lambda}^\gamma(n))$,

(c) $e_{n0} = o(1/\lambda^\gamma(n))$.

Proof. (a) On account of Lemma 2 (a) $\lambda(n)=O(n^p)$; choosing $s=p\gamma$ in Lemma 1 (a) we get with (5) and with $0<q'<q$
$$\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}(1/\lambda^\gamma(k)) \leqq \lambda^\gamma(n)\Big\{\lambda^{-\gamma}(1)\,d_{n,[q'n]+1}+\lambda^{-\gamma}([q'n])\sum_{k=[q'n]+1}^{n} e_{nk}\Big\} = O(1).$$

(b) By the relation $\lambda(n)\leqq\bar{\lambda}(n)=O(\lambda(n))$ this case is equivalent to (a). (c) follows similarly.

3. Proof of the theorems

Proof of Theorem 1. With the aid of sequence $\{n_i\}$ (cf. (7)) depending on the method $(E, q)=(e_{nk})$ we consider with $j(k)=i$ if $n_i \leqq k < n_{i+1}$ the following estimation

$$\{\lambda(n)\tau_n^{(\gamma)}(x)\}^\gamma \leqq C(\gamma)\{\lambda^\gamma(n)e_{n0}|s_0(x)-f(x)|^\gamma+$$

$$+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|s_k(x)-s_{n_{j(k)}}(x)-\sigma_k^*(x)|^\gamma+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|\sigma_k^*(x)|^\gamma+$$

$$+\lambda^\gamma(n)\sum_{k=1}^{n} e_{nk}|s_{n_{j(k)}}(x)-f(x)|^\gamma\} =: C(\gamma)\{\tau_n^{(\mathrm{I})}(x)+\tau_n^{(\mathrm{II})}(x)+\tau_n^{(\mathrm{III})}(x)+\tau_n^{(\mathrm{IV})}(x)\}.$$

With the aid of Lemma 6 (c), Lemma 5 and Lemma 6 (a) we get a.e.

$$\tau_n^{(\mathrm{I})}(x)=o_x(1); \quad \tau_n^{(\mathrm{III})}(x)=o_x(1). \tag{8}$$

With regard to the assumptions of the theorem and Lemma 3 at first $|s_{n_i}(x)-f(x)| \leqq \leqq |s_{n_i}(x)-t_{\bar n_i}(x)|+|t_{\bar n_i}(x)-f(x)|=o_x((\lambda(\bar n_i))^{-1})$ is true and with Lemma 6 (b)

$$\tau_n^{(\mathrm{IV})}(x)=o_x(1) \quad \text{a.e..} \tag{9}$$

To prove $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ a.e., in the following for the sake of brevity, we put

$$\delta_k(x)=s_k(x)-s_{n_{j(k)}}(x)-\sigma_k^*(x).$$

We notice at first that it suffices to consider exponents $\gamma \geqq 2$; if $\gamma<2$ it follows according to Hölder's inequality that

$$\Big\{\sum_{k=1}^{n} e_{nk}|\delta_k(x)|^\gamma\Big\}^{1/\gamma} \leqq \Big\{\sum_{k=1}^{n} e_{nk}\Big\}^{(1-\gamma/2)(1/\gamma)}\Big\{\sum_{k=1}^{n} e_{nk}|\delta_k(x)|^2\Big\}^{1/2}. \tag{10}$$

Since $\sum_{k=0}^{n} e_{nk}=1$, $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ with $\gamma=2$ implies this relation for $0<\gamma<2$, too. In the next step we divide $\tau_n^{(\mathrm{II})}(x)$ and consider now with $\gamma \geqq 2$, $0<q'<q$,

$$\tau_n'(x) := \lambda^\gamma(n)\sum_{k=1}^{[q'n]} e_{nk}|\delta_k(x)|^\gamma.$$

Lemma 1 (a) and Lemma 2 (a) lead us to

$$\tau_n'(x)=O(n^{p\gamma})(A(s)/n^{s-1})\sum_{i=0}^{\overline{j(n)}}(1/(n_{i+1}-n_i))\sum_{k=n_i+1}^{n_{i+1}-1}|\delta_k(x)|^\gamma$$

and with Lemma 4 using $\{\sum a_n\}^{1/p} \leqq \sum a_n^{1/p}$ $(a_n \geqq 0; p \geqq 1)$ we get with $s=(p+1)\gamma+1$

$$\int_0^1 \{\tau_n'(x)\}^{2/\gamma}\,dx = O(1/n^2) \sum_{i=0}^{j(n)} \int_0^1 \{(1/2(n_{i+1}-n_i)) \sum_{k=n_i+1}^{n_{i+1}-1} |\delta_k(x)|^\gamma\}^{2/\gamma}\,dx =$$

$$= O(1/n^2) \sum_{k=0}^{\infty} c_k^2.$$

This shows that $\int_0^1 \sum_{n=1}^{\infty} \{\tau_n'(x)\}^{2/\gamma}\,dx = O(1) \sum_{k=0}^{\infty} c_k^2 < \infty$, i.e.

(11) $$\tau_n'(x) = o_x(1) \quad \text{a.e..}$$

In the last step of the proof it remains to consider

$$\tau_n''(x) = \lambda^\gamma(n) \sum_{k=[q'n]+1}^{n} e_{nk} |\delta_k(x)|^\gamma.$$

With the aid of (5) and estimation $e_{nk}=O(n^{-1/2})$ (cf. A. RÉNYI [6], p. 127) it holds

$$\tau_n''(x) = O(1) \sum_{k=[q'n]+1}^{n} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma.$$

Using Lemma 4 and the facts that $\{n_i\}$ satisfies a gap condition (6) and $n_{i+1}=O(n_i)$, we obtain by $\gamma \geqq 2$

$$\int_0^1 \{\sum_{k=2}^{\infty} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma\}^{2/\gamma}\,dx \leqq \int_0^1 \sum_{i=0}^{\infty} \{\sum_{k=n_i+1}^{n_{i+1}-1} (\lambda^\gamma(k)/\sqrt{k})|\delta_k(x)|^\gamma\}^{2/\gamma}\,dx =$$

$$= O(1) \sum_{i=0}^{\infty} \lambda^2(n_i) \int_0^1 \{(1/2(n_{i+1}-n_i)) \sum_{k=n_i+1}^{n_{i+1}-1} |\delta_k(x)|^\gamma\}^{2/\gamma}\,dx =$$

$$= O(1) \sum_{i=0}^{\infty} \lambda^2(n_i) \sum_{k=n_i+1}^{n_{i+1}} c_k^2 = O(1) \sum_{k=2}^{\infty} c_k^2 \lambda^2(k) < \infty.$$

In the same way as I. J. MADDOX [4] did we conclude that $\tau_n''(x)=o_x(1)$ a.e. which finally shows together with (11) that $\tau_n^{(\mathrm{II})}(x)=o_x(1)$ a.e.. Considering in addition (8) and (9), the proof of Theorem 1 is complete.

Proof of Theorem 2. (a) It is easy to see that the number of the members in $\{m_i^*\}$ between two adjacent m_k and m_{k+1} is bounded, if both sequences satisfy (6). Defining $k(i)$ by $m_{k(i)} \leqq m_i^* < m_{k(i)+1}$, we get with Lemma 3

$$\sum_{i=0}^{\infty} \int_0^1 \lambda^2(m_i^*)(s_{m_i^*}(x) - s_{m_{k(i)}}(x))^2\,dx =$$

$$= O(1) \sum_{i=0}^{\infty} \lambda^2(m_i^*) \sum_{k=m_i^*+1}^{m_{i+1}^*} c_k^2 = O(1) \sum_{k=1}^{\infty} c_k^2 \lambda^2(k) < \infty,$$

i.e. $s_{m_i^*}(x)-s_{m_{k(i)}}(x)=o_x((\lambda(m_i^*))^{-1})$ a.e., which together with $s_{m_{k(i)}}(x)-f(x)=$ $=o_x((\lambda(m_i^*))^{-1})$ a.e. proves the assertion.

(b) We have to prove

$$\mu_i(x) := \max_{m_i<n<m_{i+1}} |t_n(x)-t_{m_i}(x)| = o_x(1/\lambda(m_{i+1})) \quad \text{a.e.}.$$

Similarly to [7] (p. 25) we get with the aid of the Cauchy inequality and taking into account condition (6)

$$\mu_i(x) \leqq \Big\{\sum_{\nu=m_i+1}^{m_{i+1}} \nu(t_\nu(x)-t_{\nu-1}(x))^2\Big\}^{1/2}\Big\{\sum_{\nu=m_i+1}^{m_{i+1}} 1/\nu\Big\}^{1/2} =$$
$$= O(1)\Big\{(m_{i+1}/\sqrt{m_i}) \sum_{\nu=m_i+1}^{m_{i+1}} (t_\nu(x)-t_{\nu-1}(x))^2\Big\}^{1/2}.$$

By virtue of the identity $t_\nu(x)-t_{\nu-1}(x)=\sum_{k=1}^{\nu}(k/\nu)e_{\nu k}c_k\varphi_k(x)$ (cf. K. Knopp, G. G. Lorentz [2]) and $m_{i+1}=O(m_i)$

$$\int_0^1 \lambda^2(m_{i+1})\mu_i^2(x)\,dx = O(\lambda^2(m_i))\sqrt{m_i}\sum_{\nu=m_i+1}^{m_{i+1}}\sum_{k=1}^{\nu}((k/\nu)e_{n\nu})^2c_k^2. \tag{12}$$

With an arbitrarily chosen q', $0<q'<q$, we divide the inner sums and consider the terms (cf. Lemma 2 (a), Lemma 1 (a), $s=p+3/2$)

$$\begin{aligned}\lambda^2(m_i)\sqrt{m_i}\sum_{\nu=m_i+1}^{m_{i+1}-1}\sum_{k=1}^{[q'm_i]}((k/\nu)e_{\nu k})^2c_k^2 &=\\ = O(1)m_i^{2p}m_im_i^{-2s}\sum_{k=1}^{\infty}c_k^2 = O(1)\frac{1}{m_i^2}\sum_{k=1}^{\infty}c_k^2.\end{aligned} \tag{13}$$

Next, using the estimation $(k/\nu)e_{\nu k}=\binom{\nu-1}{k-1}q^k(1-q)^{\nu-k}=O(\nu^{-1/2})$ (cf. A. Rényi [6], p. 127) and the conditions $m_{i+1}=O(m_i)$ and (5), we get

$$\lambda^2(m_i)\sqrt{m_i}\sum_{\nu=m_i+1}^{m_{i+1}-1}\sum_{k=[q'm_i]+1}^{\nu}((k/\nu)e_{\nu k})^2c_k^2 \leqq$$
$$\leqq C_{q'}^2\sum_{k=1}^{m_{i+1}-1}c_k^2\lambda^2(k)\sum_{\nu=\max(m_i,k)}^{m_{i+1}-1}\binom{\nu-1}{k-1}q^k(1-q)^{\nu-k}$$

and this together with (13) yields (cf. (12))

$$\int_0^1\sum_{i=1}^{\infty}\lambda^2(m_{i+1})\mu_i^2(x)\,dx = O(1)\sum_{i=1}^{\infty}(1/m_i^2)\sum_{k=1}^{\infty}c_k^2+$$
$$+O(1)\sum_{k=1}^{\infty}c_k^2\lambda^2(k)q^k\sum_{v=k}^{\infty}\binom{\nu-1}{k-1}(1-q)^{\nu-k} = O(1)\sum_{k=1}^{\infty}c_k^2+O(1)\sum_{k=1}^{\infty}c_k^2\lambda^2(k)<\infty$$

which proves $\mu_i(x)=o_x((\lambda(m_{i+1}))^{-1})$ a.e.. Together with the assumption $|t_{m_i}(x)-f(x)|=$ $=o_x((\lambda(m_i))^{-1})$ a.e. the statement of the theorem is evident.

(c) This is a consequence of Lemma 3 and of the assertions (a) and (b).

4. Proof of the remarks

Remark 1 follows easily from Theorem A and the relation (analogous to (10))

$$|t_n(x)-f(x)| \leqq \tau_n^{(1)}(x) \leqq \tau_n^{(\gamma)}(x) \quad (\gamma > 1).$$

Proof of Remark 2. (a) We construct at first for a given (E, q) the following sequence $\{n_k^*\}$: With fixed ε, $0<q-\varepsilon<q$, we take at first n_0 such that $n_0^*(q-\varepsilon)>1$. Assuming that $n_0^*, \ldots, n_{k-1}^*$ are determined we choose n_k^* as the smallest number which satisfies $(q-\varepsilon)n_k^*>n_{k-1}^*+\sqrt{n_{k-1}^*}$.

(b) Putting $\nu_k=[\sqrt{n_k^*}/4]$ we define our orthogonal system $\{\psi_n(x)\}$ with the aid of the Rademacher functions $r_n(x)=\operatorname{sign}(\sin(2^n\pi x))$, $0\leqq x\leqq 1$, $n=0, 1, \ldots$. To this end we consider the sets

$$I_k^{(0)}=[0, 1-1/2^{n_k^*}), \quad I_k^{(1)}=[1-1/2^{n_k^*}, 1-1/2^{n_k^*+1}],$$
$$I_k^{(2)}=(1-1/2^{n_k^*+1}, 1-1/2^{n_k^*+2}), \quad I_k^{(3)}=[1-1/2^{n_k^*+2}, 1]$$

$(k=0, 1, \ldots)$. Let us further denote, for an arbitrary interval $I=(a, b)$ (or $I=[a, b]$),

$$f(x;\ I)=f((x-a)/(b-a)) \quad (x\in I).$$

Then we define $\{\psi_n(x)\}$ in the following way:

$$\psi_n(x)=r_n(x), \quad 0\leqq n\leqq n_0^*, \quad \text{resp.} \quad n_k^*+4\nu_k<n\leqq n_{k+1}^*, \quad k=0, 1, \ldots;$$

and if $n_k^*<n\leqq n_k^*+4\nu_k$ we distinguish four cases: if $n=n_k^*+4j+1$ $(0\leqq j<\nu_k)$:

$$\psi_n(x)=\begin{cases}(2^{n_k-2}/(2^{n_k}-1))^{1/2}r_n(x) & (x\in I_k^{(0)}),\\ (2^{n_k-1})^{1/2}r_{j+1}(x;\ I_k^{(1)}) & (x\in I_k^{(1)}),\\ (2^{n_k})^{1/2}r_{j+1}(x;\ I_k^{(2)}) & (x\in I_k^{(2)}),\\ (2^{n_k})^{1/2}r_{j+1}(x;\ I_k^{(3)}) & (x\in I_k^{(3)});\end{cases} \tag{14a}$$

if $n=n_k^*+4j+2$ $(0\leqq j<\nu_k)$:

$$\psi_n(x)=\begin{cases}-\psi_{n-1}(x) & (x\in I_k^{(0)} \text{ resp. } x\in I_k^{(1)}),\\ \psi_{n-1}(x) & (x\in I_k^{(2)} \text{ resp. } x\in I_k^{(3)});\end{cases} \tag{14b}$$

if $n=n_k^*+4j+3$ $(0\leqq j<\nu_k)$:

$$\psi_n(x)=\begin{cases}-\psi_{n-2}(x) & (x\in I_k^{(0)} \text{ resp. } x\in I_k^{(2)}),\\ \psi_{n-2}(x) & (x\in I_k^{(1)} \text{ resp. } x\in I_k^{(3)});\end{cases} \tag{14c}$$

if $n = n_k^* + 4j \quad (1 \leqq j \leqq \nu_k)$:

(14d)
$$\psi_n(x) = \begin{cases} \psi_{n-3}(x) & (x \in I_k^{(0)} \text{ resp. } x \in I_k^{(3)}), \\ -\psi_{n-3}(x) & (x \in I_k^{(1)} \text{ resp. } x \in I_k^{(2)}). \end{cases}$$

We note that for $n_k^* < n, m \leqq n_k^* + 4\nu_k$, $l = 0, \ldots, 3$

$$\int_{I_k^{(l)}} \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1/4 & n_k^* + 4j < m,\ n \leqq n_k^* + 4(j+1) \\ 0 & \text{otherwise} \end{cases}$$

and for $0 < n \leqq n_0^*$ resp. $n_k^* + 4\nu_k < n \leqq n_{k+1}^*$, $0 \leqq m < \infty$,

$$\int_0^1 \psi_n(x)\psi_m(x)\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$$

It is then easy to see that $\{\psi_n(x)\}$ is an orthonormal system on $[0, 1]$.

Finally we define the coefficients c_n with an arbitrarily chosen $\alpha > 1/4$:

$$c_n = \begin{cases} 0 & 0 \leqq n \leqq n_0^* \text{ resp. } n_k^* + 4\nu_k < n \leqq n_{k+1}^*, \quad k = 0, 1, \ldots, \\ (n_k^*)^{-\alpha} & n_k^* < n \leqq n_k^* + 4\nu_k, \quad k = 0, 1, \ldots. \end{cases}$$

Because of the relation $n_{k+1}^*/n_k^* \geqq 1/(q-\varepsilon) > 1$,

$$\sum_{n=0}^{\infty} c_n^2 = \sum_{k=0}^{\infty} (n_k^*)^{-2\alpha} \sum_{n=n_k^*+1}^{n_k^*+4\nu_k} 1 = O(1) \sum_{k=0}^{\infty} (n_k^*)^{-2\alpha+1/2} < \infty,$$

(c) To prove the statement of Remark 2 we consider the partial sums $s_m(x)$ of the series $\sum_{n=0}^{\infty} c_n \psi_n(x)$. If $m > n_{k_0}^* + 4\nu_{k_0}$ we have on $I_{k_0}^{(0)}$ (cf. (14a)—(14d))

(15a)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) = f(x),$$

$$n_k^* + 4\nu_k \leqq m \leqq n_{k+1}^* \quad \text{resp.} \quad m = n_k^* + 2j \quad (1 \leqq j \leqq 2\nu_k), \quad x \in I_{k_0}^{(0)},$$

resp.

(15b)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) + (n_k^*)^{-\alpha}\big(2^{n_k^*-2}/(2^{n_k^*}-1)\big)^{1/2} r_m(x),$$

$$m = n_k^* + 4j + 1 \quad (0 \leqq j < \nu_k;\ k > k_0), \quad x \in I_{k_0}^{(0)},$$

resp.

(15c)
$$s_m(x) = s_{n_{k_0}^* + 4\nu_{k_0}}(x) - (n_k^*)^{-\alpha}\big(2^{n_k^*-2}/(2^{n_k^*}-1)\big)^{1/2} r_{m-2}(x),$$

$$m = n_k^* + 4j + 3 \quad (0 \leqq j < \nu_k;\ k > k_0), \quad x \in I_{k_0}^{(0)}.$$

Consequently $\{s_m(x)\}$ converges on $[0, 1]$ and $f(x) = s_{n_k^* + 4\nu_k}(x)$ if $x \in I_k^{(0)}$ $(k = 0, 1, \ldots)$. Let us now consider $t_n(x)$ on $I_{k_0}^{(0)}$; we assume $n_k^* < n \leqq n_{k+1}^*$ $(k > k_0)$ and get with

(15a)—(15c)

$$t_n(x)-f(x) = \sum_{m=0}^{n} e_{nm}(s_m(x)-f(x)) =$$

$$= \sum_{m=0}^{n_{k_0}^*+1} e_{nm}(s_m(x)-f(x)) + \sum_{m=n_{k_0}^*+1+1}^{n_{k-1}^*+4\nu_{k-1}} e_{nm}(s_m(x)-s_{n_{k_0}^*+4\nu_{k_0}}(x)) +$$

$$+ \sum_{j:0\leqq j<(1/2)(n-n_k^*)} e_{n,n_k^*+2j+1}(s_m(x)-s_{n_{k_0}^*+4\nu_{k_0}}(x)) = t_n^{(1)}(x)+t_n^{(2)}(x)+t_n^{(3)}(x).$$

It follows immediately as a consequence of Lemma 1 (with s sufficiently large), regarding $n_{k-1}^*+4\nu_{k-1}<(q-\varepsilon)\cdot n_k^*$

(16) $$t_n^{(1)}(x) = o_x(n^{-\alpha});\ t_n^{(2)}(x) = o_x(n^{-\alpha}) \quad (x\in I_{k_0}^{(0)}).$$

Putting now $\mu(n)=\min\{(n-n_k^*)/4, \nu_k\}$ and $\delta(n)=1$ if $n=n_k^*+4j+1$ $(0\leqq j<\nu_k)$ resp. $\delta(n)=0$ for all other n, $n_k^*<n\leqq n_{k+1}^*$, we find with (15b), (15c)

$$t_n^{(3)}(x) = (n_k^*)^{-\alpha}(2^{n_k^*-2}/(2^{n_k^*}-1))^{1/2}\Big\{\sum_{j=0}^{\mu(n)-1} r_{n_k^*+4j+1}(x)(e_{n,n_k^*+4j+1}-e_{n,n_k^*+4j+3})+$$

$$+\delta(n)r_n(x)e_{nn}\Big\} \quad (x\in I_{k_0}^{(0)})$$

which leads with respect to the formerly used estimation $e_{nk}=O(n^{-1/2})$ and to the relations $e_{nk}<e_{n,k+1}$ $(k\leqq q(n+1)-1)$ resp. $e_{nk}>e_{n,k+1}$ $(k>q(n+1)-1)$ to

$$t_n^{(3)}(x) = O(n^{-\alpha})\Big\{\sum_{j=0}^{\nu_k} |e_{n,n_k^*+4j+1}-e_{n,n_k^*+4j+3}|+1/\sqrt{n}\Big\} =$$

$$= O(n^{-\alpha-1/2}) = o(n^{-\alpha}) \quad (x\in I_{k_0}^{(0)}).$$

Together with (16) we finally get

(17) $$t_n(x) = o_x(1/n^\alpha) \quad (x\in[0,1]).$$

(d) On the other hand let us consider for $\tilde{n}_k$ with $[q\cdot\tilde{n}_k]=n_k^*$ the means

$$\tau_{\tilde{n}_k}^{(\gamma)}(x) = \Big\{\sum_{m=0}^{\tilde{n}_k} e_{\tilde{n}_k,m}|s_m(x)-f(x)|^\gamma\Big\}^{1/\gamma} \geqq \Big\{\sum_{m=n_k^*}^{n_k^*+4\nu_k} e_{\tilde{n}_k,m}|s_m(x)-f(x)|^\gamma\Big\}^{1/\gamma}.$$

Taking into account equations (15a)—(15c) we find on an arbitrarily chosen $I_{k_0}^{(0)}$ for $k>k_0$

$$\tau_{\tilde{n}_k}(x) \geqq \tilde{C}\Big\{(n_k^*)^{-\gamma\alpha}\sum_{j=0}^{2\nu_k-2} e_{\tilde{n}_k,n_k^*+2j+1}\Big\}^{1/\gamma} \quad \text{a.e.} \quad (x\in I_{k_0}^{(0)}).$$

Regarding the estimation $e_{nk}\geqq\tilde{C}(q)n^{-1/2}$ if $qn\leqq k\leqq qn+\sqrt{n}$ (cf. A. RÉNYI [6], p. 31), the last relation yields because of $\nu_k\geqq\sqrt{n_k^*}/4$: $\tau_{\tilde{n}_k}^{(k)}(x)\geqq\tilde{C}^*(\tilde{n}_k)^{-\alpha}$ a.e.

($x \in I_{k_0}^{(0)}$), i.e.

(18) $$\tau_n^{(\gamma)}(x) \neq o_x(1/n^\alpha) \quad \text{a.e.} \quad (x \in [0, 1]).$$

Now we may define $\lambda(n) = n^\alpha$. The statement of Remark 2 is proved by (17) and (18).

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Integrability of Rees—Stanojević sums

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1. A sequence $\{a_n\}$ of positive numbers is called quasi-monotone if $n^{-\beta}a_n\downarrow 0$ for some β, or equivalently, if $a_{n+1} \leqq a_n(1+\alpha/n)$.

RAM [3] defined that a sequence $\{a_k\}$ of numbers satisfies condition (S*) if $a_k\to 0$ as $k\to\infty$ and there exists a sequence $\{A_k\}$ such that $\{A_k\}$ is quasi-monotone,

$$\sum_{k=0}^{\infty} A_k < \infty \tag{1}$$

and

$$|\Delta a_k| \leqq A_k \quad \text{for all} \quad k. \tag{2}$$

Condition (S*) is weaker than the condition (S) of Sidon introduced in [4].

REES and STANOJEVIĆ [2] (see also GARRETT and STANOJEVIĆ [1]) introduced the modified cosine sums

$$g_n(x) = (1/2)\sum_{k=0}^{n} \Delta a_k + \sum_{k=1}^{n}\sum_{j=k}^{n} \Delta a_j \cos kx \tag{1.1}$$

and obtained a necessary and sufficient condition for the integrability of the limit of (1.1).

RAM [3] proved the following theorem in which he showed that condition (S*) is sufficient for the integrability of the limit of (1.1).

Theorem A. *Let the sequence $\{a_k\}$ satisfy condition* (S*). *Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \Big[(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx\Big]$$

exists for $x\in(0,\pi]$, and $g(x)\in L(0,\pi)$.

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We say that a sequence $\{a_n\}$ of numbers satisfies condition (S**) if $\{a_n\}$ is a null sequence and

(3) $$n\Delta a_n = O(1) \quad (n \to \infty).$$

We claim that our condition (S**) includes a more general class of sequences $\{a_n\}$ than that of Ram's condition (S*).

Example. The sequence

$$a_n = \frac{(-1)^{n+1}}{n \log (n+1)} \quad (n = 1, 2, \ldots)$$

does not satisfy the conditions (S*) of Ram as $|\Delta a_n| \geqq (n \log (n+1)^{-1})$ and so $\sum |\Delta a_n| = \infty$. This in fact contradicts conditions (1) and (2) of (S*). On the other hand this sequence satisfies the condition (3) of (S**).

The object of this paper is to show that condition (S**) is sufficient for the integrability of the limit of (1.1).

Theorem. *Let the sequence* $\{a_n\}$ *satisfy condition* (S**). *Then*

$$g(x) = \lim_{n\to\infty} \sum_{k=1}^{n} [(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx]$$

exists for $x \in (0, \pi]$, *and* $g(x) \in L(0, \pi)$.

2. We require the following lemma for the proof of our theorem.

Lemma. *Let* $\{a_n\}$ *be a null sequence and* $n\Delta a_n = O(1)$, $n \to \infty$. *Then*

$$\sum (n+1)\Delta^2 a_n < \infty.$$

Proof. Applying Abel's transformation, we find

$$\sum_{m=0}^{n} \Delta a_m = \sum_{m=0}^{n} 1 \cdot \Delta a_m = \sum_{m=0}^{n-1} (m+1)\Delta^2 a_m + (n+1)\Delta a_n,$$

and since $(n+1)\Delta a_n \to 0$ but $\sum_{m=0}^{n} \Delta a_m = a_0 - a_n \to a_0$ as $a_n \to 0$, $n \to \infty$, then

$$\sum_{m=0}^{n-1} (m+1)\Delta^2 a_m \to a_0,$$

i.e. the series $\sum_{m=0}^{\infty} (m+1)\Delta^2 a_m$ converges.

3. Proof of the Theorem. We have

$$g_n(x) = \sum_{k=1}^{n} [(1/2)\Delta a_k + \sum_{j=k}^{n} \Delta a_j \cos kx] =$$

$$= \sum_{k=1}^{n} (1/2)\Delta a_k + \sum_{k=1}^{n} \Delta a_k \cos kx - a_{n+1} D_n(x) + (1/2) a_{n+1}.$$

Using Abel's transformation, we obtain

$$
\begin{aligned}
(4.1)\qquad g_n(x) &= \sum_{k=1}^{n}(1/2)\Delta a_k + \sum_{k=1}^{n-1}\Delta a_k(D_k(x)+1/2) + a_n(D_n(x)+1/2) - \\
&\quad - a_{n+1}D_n(x) - a_1 + (1/2)a_{n+1} = \sum_{k=1}^{n-1}\Delta a_k D_k(x) + a_n D_n(x) - a_{n+1}D_n(x).
\end{aligned}
$$

Applying again Abel's transformation, we have

$$
(4.2)\qquad g_n(x) = \sum_{k=1}^{n-2}(k+1)\Delta^2 a_k F_k(x) + n\Delta a_{n-1}F_{n-1}(x) + a_n D_n(x) - a_{n+1}D_n(x),
$$

where $D_n(x)$ and $F_n(x)$ denotes the Dirichlet and Fejér kernels respectively.

If $x\not\equiv 0 \pmod{2\pi}$, then since $a_n\to 0$, the last two terms of the right hand side of (4.2) tends to zero as $n\to\infty$. Moreover, at $x\not\equiv 0 \pmod{2\pi}$ $F_n(x)$ always remains finite as $n\to\infty$ and since $n\Delta a_n\to 0$ therefore $n\Delta a_{n-1}F_{n-1}(x)\to 0$ as $n\to\infty$.

Since $F_k(x)=o\big(1/(k+1)x^2\big)$ if $x\not\equiv 0$ and $\sum(k+1)\Delta^2 a_k$ is convergent then the series $\sum_{k=1}^{\infty}(k+1)\Delta^2 a_k\, F_k(x)$ converges. Hence for $x\not\equiv 0 \pmod{2\pi}$

$$
g(x) = \lim_{n\to\infty} g_n(x) = \sum_{k=1}^{\infty}(k+1)\Delta^2 a_k F_k(x).
$$

The integrability of $g(x)$ follows from the lemma; indeed, we have

$$
\int_0^{\pi} g(x)\,dx = \sum_{k=1}^{\infty}(k+1)\Delta^2 a_k \int_0^{\pi} F_k(x)\,dx = (\pi/2)\sum_{k=1}^{\infty}(k+1)\Delta^2 a_k < \infty;
$$

since $\int_0^{\pi} F_n(x)\,dx=\pi/2$.

Corollary. *Let $\{a_n\}$ be a null sequence and $n\Delta a_n=o(1)$, $n\to\infty$. Then*

$$
(1/x)\sum_{k=1}^{\infty}\Delta a_k \sin(k+1/2)x = h(x)/x
$$

converges for $x\in(0,\pi]$, and $h(x)/x\in L(0,\pi)$.

Proof. From (4.1) we have

$$
g(x) = \sum_{k=1}^{\infty}\Delta a_k D_k(x) = \left(\sum_{k=1}^{\infty}\Delta a_k \sin(k+1/2)x\right)\Big/2\sin(x/2) = h(x)/2\sin(x/2).
$$

According to the theorem, $g(x)$ exists for $x\not\equiv 0$, and $g(x)\in L[0,\pi]$ if $n\Delta a_n=o(1)$, which implies our result.

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Характеристические функции распределений притягивающихся к устойчивому закону с показателем $\alpha=1$

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Функция распределения (ф. р.) $F(x)$ притягивается к (ф. р.) $G(x)$, если для соответствующих характеристических функций (х. ф.) $f(t)$ и $g(t)$ существуют последовательности постоянных $\{a_n\}, \{b_n\}$, $b_n>0$, таких, что при любом t, $-\infty<t<\infty$.

(1) $$\lim_{n\to\infty} \exp\{-ita_n/b_n\} f^n(t/b_n) = g(t).$$

Множество предельных х. ф. в (1) совпадает с четырехпараметрическим семейством устойчивых х. ф. $g(t, \alpha, \beta, \gamma, c)$. В интересующем нас случае $\alpha=1$ имеет место представление

(2) $$\ln g(t) = i\gamma t - c|t|(1+(2/\pi)\beta i \ln|t| \operatorname{sgn} t),$$

где γ — действительная постоянная, $c>0$, $-1\leqq\beta\leqq 1$.

Обозначим $D(\alpha, \beta)$ — множество ф. р., притягивающихся к устойчивой ф. р. $G(x, \alpha, \beta, \gamma, c)$. К настоящему времени подробно изучены (см. [2], [6], [4]) свойства $F(x)$ и $f(t)$ связанные с условием $F(x)\in D(\alpha, \beta)$. Например, теорема 2.6.5 из [2] содержит необходимые и достаточные условия равенства (1) в терминах х. ф. $f(t)$. Однако в работе [4] отмечено, что в случае $F(x)\in D(1, \beta)$ утверждение этой теоремы неверно. В свою очередь приведенное в [4] условие (см. конец раздела 2 в [4]) для случая $F(x)\in D(1, 0)$ также является неточным.

Целью настоящей заметки является исчерпывающее рассмотрение данного вопроса.

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Теорема. 1. *Если* $F(x)\in D(1,\beta)$, *то при* $t\to 0$

$$\ln f(t) = -(\pi/2)\left(1-F(1/|t|)+F(-1/|t|)\right)+i\left[t\int_0^{1/|t|}\left(1-F(u)-F(-u)\right)du - \right. \tag{3}$$
$$\left. -CE(\beta)\left(1-F(1/|t|)-F(-1/|t|)\right)\right]+o\left(1-F(1/|t|)+F(-1/|t|)\right),$$

где C — *постоянная Эйлера,* $E(\beta)=0$ *при* $\beta=0$, $E(\beta)=1$ *при* $\beta\neq 0$.

2. $F(x)\in D(1,\beta)$ *тогда и только тогда, когда при* $t\to 0$

$$\ln f(t) = -|t|h(1/|t|)+it\left[\int_0^{1/|t|}\left((2\beta/\pi)(h(u)/u)+q(u)\right)du+o\left(h(1/|t|)\right)\right], \tag{4}$$

где $h(u)/u$ *и* $q(u)$ *интегрируемые на любом конечном интервале из* $[0,\infty)$ *функции такие, что при* $u\to\infty$ $h(u)$ — *медленно меняющаяся функция, а* $q(u)= =o\left(h(u)/u\right)$.

Следствие. *Если* $F(x)\in D(1,\beta)$, *то при* $t\to 0$

$$|f(t)|\sim\exp\left\{-(\pi/2)\left(1-F(1/|t|)+F(-1/|t|)\right)\right\}\sim\exp\left\{-|t|h(1/|t|)\right\}, \tag{5}$$

(6)

$$|\ln f(t)|\sim\left|t\int_0^{1/|t|}\left(1-F(u)-F(-u)\right)du\right|\sim\left|t\int_0^{1/|t|}\left((2\beta/\pi)(h(u)/u)+q(u)\right)du\right|,\quad \beta\neq 0,$$

$$|\ln f(t)|\sim\left\{t^2h^2(1/|t|)+\left[t\int_0^{1/|t|}q(u)\,du\right]^2\right\}^{1/2},\quad \beta=0, \tag{7}$$

где $h(u)$ *и* $q(u)$ *те же, что и в* (4).

Оценка (5) уточняет соответствующее утверждение, полученное в [2] на стр. 110 в случае $\alpha=1$. Оценки (6) и (7) не следуют, соответственно, из теоремы 2.6.5 монографии [2] и из условия приведенного в работе [4], поскольку в [2] на месте интеграла в представлении (4) находится функция $(2\beta/\pi)h(1/|t|)\ln|t|$, а в работе [4] в представлении (4) отсутствует функция $q(u)$. Указанные обстоятельства позволяют построить

Пример 1. Пусть $F(x)=1/3$ при $|x|\leqq 3$, а при $x>3$

$$F(-x)=1/x,\quad 1-F(x)=(1+1/\ln x)/x,$$

тогда (см. условие (12) и равенство (13), приводимые ниже) $F(x)\in D(1,0)$, и из представления (3) следует, что при $t\to 0$

$$\ln f(t) = -\pi|t|-it\left(\ln\ln 3-\ln\left|\ln|t|\right|\right)+o(t),$$

в то время как соответствующие утверждения из [2] и [4] дают в этом случае неверную оценку

$$\ln f(t) = -\pi|t| + i\gamma t + o(t), \quad \gamma = \text{const}.$$

Заметим, что в общем случае $o(h(1/|t|))$ в правой части (4) нельзя внести под знак интеграла. (Эту проблему указали автору А. В. Нагаев и Л. А. Анорина.) Это показывает

Пример 2. Пусть случайным величинам η, ζ, ξ соответствуют ф. р. $F_\eta(x)$, $F_\zeta(x)$, $F_\xi(x)$ и х. ф.

$$f_\eta(t) = a \int_3^\infty x^{-2} \ln x \exp\{itx\}\, dx, \quad a = \Big(\int_3^\infty x^{-2} \ln x\, dx\Big)^{-1}, \tag{8}$$

$$f_\zeta(t) = b \sum_{k=1}^\infty 2^{-k^2} \exp\{it2^{k^2}\}, \quad b = \Big(\sum_{k=1}^\infty 2^{-k^2}\Big)^{-1}, \tag{9}$$

$$f_\xi(t) = \big(f_\eta(t) + f_\zeta(t)\big)/2. \tag{10}$$

Далее будет показано, что $F_\xi(x) \in D(1, 1)$, а функции

$$\varphi(u) = \operatorname{Im} u \ln f_\xi(1/u), \quad \psi(u) = \operatorname{Re} u \ln f_\xi(1/u), \tag{11}$$

где $u=1/t$, $t \neq 0$, не являются абсолютно непрерывными ни в одном интервале. Следовательно, $\varphi(u)$ нельзя представить в виде интеграла с переменным верхним пределом $u=1/t$. Отметим также, что $f_\zeta(t)$ и $f_\xi(t)$ являются новыми примерами нигде не дифференцируемых х. ф.

Доказательство теоремы. Известно (см. [2] стр. 93), что $F(x) \in D(1, \beta)$ тогда и только тогда, когда при $x \to \infty$

$$F(-x) = x^{-1} l(x)\big(c_1 + o(1)\big), \quad 1 - F(x) = x^{-1} l(x)\big(c_2 + o(1)\big), \tag{12}$$

$$(c_2 - c_1)/(c_1 + c_2) = \beta, \tag{13}$$

где $c_1 \geqq 0$, $c_2 \geqq 0$, $c_1 + c_2 > 0$, $l(x)$ — медленно меняющаяся функция.

1. Обозначим

$$m^-(x) = x\big(1 - F(x) - F(-x)\big), \quad m^+(x) = x\big(1 - F(x) + F(-x)\big). \tag{14}$$

Ввиду условия (12) имеем

$$m^-(x) = l(x)\big(c_2 - c_1 + o(1)\big), \quad m^+(x) = l(x)\big(c_1 + c_2 + o(1)\big). \tag{15}$$

Легко проверить, что

$$f(t) - 1 = it \int_0^\infty \frac{\cos tu}{u} m^-(u)\, du - |t| \int_0^\infty \frac{\sin|tu|}{u} m^+(u)\, du. \tag{16}$$

Слегка изменив рассуждения леммы 2.5.1 из [2], получим ввиду (15) при $t\to 0$

$$\int_0^\infty \frac{\sin|tu|}{u} m^+(u)\,du = \frac{\pi}{2} m^+\left(\frac{1}{|t|}\right)(1+o(1)), \tag{17}$$

$$\int_0^{1/|t|} \frac{1-\cos tu}{u} m^-(u)\,du = E(\beta) m^-\left(\frac{1}{|t|}\right)\int_0^1 \frac{1-\cos u}{u}\,du + o\left(l\left(\frac{1}{|t|}\right)\right), \tag{18}$$

$$\int_{1/|t|}^\infty \frac{\cos tu}{u} m^-(u)\,du = E(\beta) m^-\left(\frac{1}{|t|}\right)\int_1^\infty \frac{\cos u}{u}\,du + o\left(l\left(\frac{1}{|t|}\right)\right), \tag{19}$$

где $E(\beta)$ определено в (3). Из (18) и (19) следует, что

$$\int_0^\infty \frac{\cos tu}{u} m^-(u)\,du = \int_0^{1/|t|} \frac{m^-(u)}{u}\,du - CE(\beta) m^-\left(\frac{1}{|t|}\right) + o\left(l\left(\frac{1}{|t|}\right)\right), \tag{20}$$

так как (см. [3], стр. 627)

$$\int_0^1 \frac{1-\cos u}{u}\,du - \int_1^\infty \frac{\cos u}{u}\,du = C,$$

где C — постоянная Эйлера. Ввиду представления (16), соотношений (15), (17) и (20)

$$f(t)-1 = it\left(\int_0^{1/|t|} \frac{m^-(u)}{u}\,du - CE(\beta) m^-\left(\frac{1}{|t|}\right)\right) - \frac{\pi}{2}|t| m^+\left(\frac{1}{|t|}\right)(1+o(1)). \tag{21}$$

Далее, легко проверить, что

$$\left|\mathrm{Im}(f(t)-1)\right| < |t|\int_0^\infty \frac{\cos tu}{u} m^+(u)\,du = |t| L\left(\frac{1}{|t|}\right), \tag{22}$$

где $L(u)$ — медленно меняющаяся функция. Соотношения (14), (15), (21), (22) вместе с оценкой

$$\ln f(t) = f(t) - 1 + O\left(|f(t)-1|^2\right)$$

доказывают утверждение пункта 1.

2. Лемма 1. *Медленно меняющаяся функция $h(x)$ допускает представление*

$$h(x) = \int_0^x (q_1(u)/u)\,du + q_2(x), \tag{23}$$

где $q_1(x)/x$ — интегрируемая на любом конечном интервале из $[0, \infty)$ функция и при $x\to\infty$ $q_i(x) = o(h(x))$, $i=1, 2$.

Доказательство леммы 1. Известно (см. [1], гл. II), что

$$h(x) = h_1(x)+q_3(x), \tag{24}$$

где $h_1(x)\in C_\infty$ и при $x\to\infty$ $q_3(x)=o(h(x))$. Используя (24) и представление Карамата (см. [6], стр. 342)

$$h(x) = a(x)\exp\left\{\int_1^x (\varepsilon(u)/u)\,du\right\}, \tag{25}$$

где $\varepsilon(x)\to 0$ и $a(x)\to c$, $0<c<\infty$, при $x\to\infty$, легко проверить, что

$$h_1'(x) = o(h_1(x)/x), \quad x\to\infty. \tag{26}$$

Из (24) и (26) следует (23), где можно положить

$$q_1(x) = x(h_1'(x)+h_1(0)\exp\{-x\}) = o(h(x)),$$

$$q_2(x) = q_3(x)+h_1(0)\exp\{-x\} = o(h(x)).$$

Лемма 1 доказана. Необходимость представления (4) вытекает теперь из (3), (14), (15) и (23).

Достаточность представления (4) будет доказана, если найдутся последовательности $\{a_n\}$, $\{b_n\}$ такие, что

$$\exp\{-ita_n/b_n\}f^n(t/b_n)\to g(t). \tag{27}$$

Положим при достаточно больших n

$$b_n = \{\inf\{t\colon \operatorname{Re}\ln f(t) = c/n\}\}^{-1}, \quad a_n = nb_n\operatorname{Im}\ln f(1/b_n)-\gamma b_n. \tag{28}$$

Используя (4) и свойства медленно меняющихся функций, нетрудно проверить, что при $n\to\infty$

$$b_n = c^{-1}nh(b_n)(1+o(1)), \tag{29}$$

$$a_n = n\int_0^{b_n} Q(u)\,du-\gamma b_n+o(b_n), \quad Q(u) = (2\beta/\pi)(h(u)/u)+q(u). \tag{30}$$

Из (29), (30) следует, что при любом фиксированном $t\neq 0$ и $n\to\infty$

$$nh(b_n/|t|)/b_n = (nh(b_n)/b_n)(h(b_n/|t|)/h(b_n)) = c+o(1), \tag{31}$$

$$\begin{aligned}(n/b_n)\left(\int_0^{b_n/|t|} Q(u)\,du-\int_0^{b_n} Q(u)\,du\right) &= (n/b_n)\int_{b_n}^{b_n/|t|} Q(u)\,du = \\ = \int_1^{1/|t|} \frac{h(ub_n)}{h(b_n)}\frac{1}{u}\left(\frac{2\beta c}{\pi}+o(1)\right)du &= -\frac{2\beta c}{\pi}\ln|t|+o(1).\end{aligned} \tag{32}$$

Представление (4) и соотношения (29)—(32) позволяют записать, что при $n\to\infty$ и любом фиксированном $t\neq 0$

$$n \ln f(t/b_n)-ita_n/b_n = it\gamma+itnb_n^{-1}\Big(\int\limits_0^{b_n/|t|} Q(u)\,du-\int\limits_0^{b_n} Q(u)\,du\Big)-$$

$$-|t|\,nh(b_n/|t|)/b_n+o(1) = i\gamma t-c|t|\big(1-i(2\beta/\pi)\ln|t|\operatorname{sgn} t\big)+o(1),$$

следовательно, выполняется (27). Доказательство пункта 2 и самой теоремы завершено.

Приступим к доказательству оценок (5)—(7). Соотношения (5), (7) вытекают непосредственно из (3), (4). Оценка (6) получается из (3), (4) и известных свойств медленно меняющихся функций (см. [6], стр. 341)

(33) $$\lim_{x\to\infty} h(x)\Big/\int\limits_0^x \big(h(u)/u\big)\,du = 0,$$

(34) $$\lim_{x\to\infty} h(x)\Big/\int\limits_x^\infty \big(h(u)/u\big)\,du = 0,$$

последнее свойство верно, если $\int\limits_0^\infty \big(h(u)/u\big)\,du$ существует.

Пример 2. Из равенств (8)—(10) следует, что случайная величина η имеет плотность

(35) $$p_\eta(x) = \begin{cases} ax^{-2}\ln x, & x\geqq 3\\ 0, & x<3,\end{cases}$$

случайная величина ζ имеет дискретное распределение

(36) $$p_k = P\{\zeta = 2^{k^2}\} = b2^{-k^2}, \quad k=1,2,\ldots,$$

а ф. р. случайной величины ξ определяется равенством

(37) $$F_\xi(x) = \big(F_\eta(x)+F_\zeta(x)\big)/2.$$

Используя (35)—(37), легко проверить, что при $x\to\infty$

(38) $$1-F_\xi(x) = \big(1-F_\eta(x)\big)\big(1/2+o(1)\big) = x^{-1}\ln x\big(a/2+o(1)\big).$$

Поскольку $F_\xi(x)=0$ при $x<0$, то ввиду (38) для $F_\xi(x)$ выполняются условия (12), (13), где $\beta=1$. Поэтому $F_\xi(x)\in D(1,1)$.

Из следующих утверждений (доказательство их приводится ниже):

(а) $\operatorname{Re} f_\eta(t)$, $\operatorname{Im} f_\eta(t)$, $f_\eta(t)$ — дифференцируемые при любом $t\neq 0$ функции,

(б) $\operatorname{Re} f_\zeta(t)$, $\operatorname{Im} f_\zeta(t)$ — почти всюду недифференцируемые функции,

(в) $f_\zeta(t)$ — всюду недифференцируемая функция, (г) $f_\xi(t)$ — не имеет производной в точке $t=0$, и равенства (10) следует, что $f_\xi(t)$ всюду недифференцируемая функция, а $\operatorname{Re} f_\xi(t)$ и $\operatorname{Im} f_\xi(t)$ почти всюду недифференцируемые функции. Следовательно, функции определенные в (11) тоже почти всюду недифференцируемы и поэтому не являются (см. [5], стр. 229) абсолютно непрерывными.

Теперь докажем утверждения (а)—(г).

(а) Достаточно установить дифференцируемость функций

$$\operatorname{Re} f_\eta(t) = a\int_3^\infty \cos tx\,x^{-2}\ln x\,dx, \quad \operatorname{Im} f_\eta(t) = a\int_3^\infty \sin tx\,x^{-2}\ln x\,dx, \quad t \neq 0.$$

Дифференцируя под знаком интеграла, мы получим формальные (пока) равенства

$$(\operatorname{Re} f_\eta(t))' = -a\int_3^\infty \sin tx\,x^{-1}\ln x\,dx,$$

(39)

$$(\operatorname{Im} f_\eta(t))' = a\int_3^\infty \cos tx\,x^{-1}\ln x\,dx, \quad t \neq 0.$$

Интегралы в (39) не только существуют, но и сходятся равномерно по t, $t\in[c, d]$, в любом промежутке $[c, d]$ не содержащем нуля. Применяя известную теорему анализа о дифференцировании под знаком интеграла, мы видим, что равенства (39) и утверждение (а) справедливы.

(б) Из (36) получается оценка

(40) $$\sum_{k=m}^\infty p_k = b2^{-m^2}+\theta_1 2^{-(m+1)^2}, \quad b < \theta_1 < 2b.$$

Обозначим

(41) $$x_k = 2^{k^2}.$$

Легко проверить, что

$$f_\zeta(t+h)-f_\zeta(t) = 2i\sum_{k=1}^\infty \sin(hx_k/2)\exp\{ix_k(t+h/2)\}p_k =$$

(42)

$$= 2i\Big(\sum_{k=1}^m + \sum_{k=m+1}^\infty\Big) = 2i\Big(\sum\nolimits_1+\sum\nolimits_2\Big).$$

Положим

(43) $$h = 2^{1-m^2-m}.$$

Тогда при $k\leqq m$ и $m\to\infty$, ввиду (36), (41)—(43) имеем

(44) $$p_k\sin(hx_k/2) = hx_k/2+O(h^3x_k^3p_k) = h\big(b/2+O(2^{-2m})\big),$$

(45) $$\exp\{ix_k(t+h/2)\} = \exp\{ix_kt\}\big(1+O(2^{-m})\big),$$

(46) $$\sum\nolimits_1 = (h/2)\Big[b\sum_{k=1}^m \exp\{ix_kt\}+O(m2^{-m})\Big].$$

Если же $k \geqq m+1$, то ввиду (36), (40)—(43)

$$(47)\qquad \left|\sum\nolimits_2\right| \leqq b2^{1-(m+1)^2} = bh2^{-m-1}.$$

Представление (42) и соотношения (46), (47) позволяют записать, что при h, выбранном в соответствии с (43), при $m\to\infty$ справедлива оценка

$$(48)\qquad f_\zeta(t+h)-f_\zeta(t) = ih\left(b\sum_{k=1}^{m}\exp\{ix_k t\}+o(1)\right).$$

Из (48) вытекает, что

$$(49)\qquad \operatorname{Re}f_\zeta(t+h)-\operatorname{Re}f_\zeta(t) = -h\left(b\sum_{k=1}^{m}\sin[\pi(2^{k^2}t/\pi)]+o(1)\right),$$

$$(50)\qquad \operatorname{Im}f_\zeta(t+h)-\operatorname{Im}f_\zeta(t) = h\left(b\sum_{k=1}^{m}\cos[\pi(2^{k^2}t/\pi)]+o(1)\right).$$

Используя представление дробной доли вещественного числа в двоичной системе счисления в виде бесконечной последовательности нулей и единиц, легко проверить, что справедлива

Лемма 2. *Мера Лебега множества вещественных чисел u, $-\infty<u<\infty$, таких, что при достаточно большом N, $N=N(u)$, для всех $n>N$ выполняется неравенство*

$$1/4 < \{2^{n^2}u\} < 3/4, \quad \{z\} \text{ — дробная доля } z,$$

равна нулю.

Лемма 2 равносильна утверждению: почти все вещественные числа u, $-\infty<u<\infty$, обладают тем свойством, что существует бесконечно много натуральных чисел n_k, для которых

$$(51)\qquad \left(\{2^{n_k^2}u\}-1/4\right)\left(\{2^{n_k^2}u\}-3/4\right) \geqq 0.$$

В свою очередь (51) при $u=t/\pi$ равносильно неравенству

$$|\cos[\pi(2^{n_k^2}t/\pi)]| \geqq 1/\sqrt{2},$$

которое означает, что почти для всех t, $-\infty<t<\infty$, сумма из правой части (50) расходится при $m\to\infty$. Поэтому не существует предел

$$\lim_{|h\to 0}\left(\operatorname{Im}f_\zeta(t+h)-\operatorname{Im}f_\zeta(t)\right)/h$$

и функция $\operatorname{Im}f_\zeta(t)$ не дифференцируема почти при всех t. Аналогично доказывается, что $\operatorname{Re}f_\zeta(t)$ также почти всюду не дифференцируемая функция.

(в) Для того, чтобы доказать, что $f_\zeta(t)$ нигде не дифференцируемая функция, достаточно заметить, что сумма в (48) расходится, поскольку все слагаемые по модулю равны единице при любом t.

(г) Из (38) следует, что при $x \to \infty$

$$x\left(1 - F_\xi(x)\right) \to \infty.$$

Это противоречит (см. [6], стр. 635) критерию дифференцируемости х. ф. в нуле, поэтому $f'_\xi(0)$ не существует.

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The approximate point spectrum of a pure quasinormal operator

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In this paper all Hilbert spaces are over the complex scalars. If $\mathscr{H}$ is a Hilbert space, let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. (In this paper the term operator shall mean an element of $\mathscr{L}(\mathscr{H})$.) If T is an operator, let $\sigma(T)$ denote the spectrum of T, let $\sigma_{ap}(T)$ denote the approximate point spectrum of T, let $\tilde{T}$ denote the image of T in the Calkin algebra $\mathscr{L}(\mathscr{H})/\mathscr{C}$ under the natural projection, where $\mathscr{C}$ denotes the ideal of all compact operators in $\mathscr{L}(\mathscr{H})$, and let $\sigma_e(T)$ denote the essential spectrum of T, i.e., $\sigma_e(T)=\sigma(\tilde{T})$. C. R. PUTNAM proved in [12] that the planar Lebesgue measure of the spectrum of a pure hyponormal operator is positive. There exist, however, pure hyponormal operators which have essential spectra of measure zero. (The unilateral shift is an example.) Let T be a pure hyponormal operator. It follows from Putnam's inequality [12] that $\pi\|\tilde{T}^*\tilde{T}-\tilde{T}\tilde{T}^*\|\leqq \leqq m_2(\sigma_e(T))$, where m_2 denotes planar Lebesgue measure. So if $m_2(\sigma_e(T))=0$, then the self-commutator T^*T-TT^* is compact. The converse is not true, even in the subnormal case. (See Example 2.4.) Yet the following question which was posed by the present author in [15] remains open: If T is a pure subnormal operator that has a finite rank self-commutator, then is $m_2(\sigma_e(T))=0$? The results of this paper were motivated by the above question. In Section 1 we show that the above question is equivalent to a similar question about the approximate point spectrum and to a question posed by J. Conway about the measure of the spectrum of the minimal normal extension of a pure subnormal operator. In Section 2 we compute the approximate point spectrum of a pure quasinormal operator and then present a formula for the planar Lebesgue measure of it. In Section 3 we present a class of pure subnormal operators for which the answer to the above question is affirmative.

We present here some terminology and notation. Let T be an operator. Recall that T is *hyponormal* if $T^*T-TT^*\geqq 0$, T is *subnormal* if T has a normal extension,

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and T is quasinormal if T commutes with T^*T. It is known that each quasinormal operator is subnormal and each subnormal operator is hyponormal. Each operator T is unitarily equivalent to $T_1 \oplus T_2$, where T_1 is normal and T_2 is pure, i.e., if $\mathcal{M}$ is a reducing subspace for T_2 and $T_2|\mathcal{M}$ is normal, then $\mathcal{M}=(0)$. The operator T_1 is the *normal part* of T and T_2 is the *pure part* of T. (Note that either of the operators T_1 or T_2 may be the zero operator on the zero Hilbert space.) Observe that if T is a hyponormal operator, then any eigenspace of T reduces T. Thus the point spectrum of a pure hyponormal operator is empty. We shall use this fact freely. Finally, let $\mathcal{K}(T)$ denote the kernel of T and $\mathcal{R}(T)$ the range of T.

1. Pure subnormal operators with finite rank self-commutators

We begin this section by observing that if T is a pure subnormal operator on a Hilbert space $\mathcal{H}$ and N is its minimal normal extension on a Hilbert space $\mathcal{K}$, where $\mathcal{H} \subseteq \mathcal{K}$, then N is unitarily equivalent to the operator

$$\begin{bmatrix} T & X \\ 0 & S^* \end{bmatrix} \tag{1}$$

on $\mathcal{H} \oplus \mathcal{H}_1$ for some Hilbert space $\mathcal{H}_1$. But since $\mathcal{K}$ is the closed linear span of $\{(N^*)^n x \colon x \in \mathcal{H},\ n \text{ a nonnegative integer}\}$ [6], we have $\dim(\mathcal{K}) = \dim(\mathcal{H})$. Olin has observed in [10] that N^* is the minimal normal extension of S if and only if T is pure (and it follows that N is the minimal normal extension of T if and only if S is pure). Thus, since T is pure, an argument similar to the one above shows that $\dim(\mathcal{K}) = \dim(\mathcal{H}_1)$. Hence N is unitarily equivalent to the operator in (1) on $\mathcal{H} \oplus \mathcal{H}$.

We now state the following questions.

Questions. Suppose that T is a pure subnormal operator that has a finite rank self-commutator and suppose that N is its minimal normal extension.

A. Then is $m_2(\sigma_e(T))=0$?
B. Then is $m_2(\sigma(N))=0$?
C. Then is $m_2(\sigma_{ap}(T))=0$?

As mentioned earlier Question A was posed by the present author in [15] and Question B was posed by J. Conway in [6]. We shall show that the three questions are equivalent. In order to see that Questions A and B are equivalent, let T and N be as above and observe that N is unitarily equivalent to the operator in (1) on $\mathcal{H} \oplus \mathcal{H}$. The operator S is also a pure subnormal operator (and is called the *dual* of T). Since N is normal, a matrix calculation shows that $T^*T - TT^* = XX^*$

and $S^*S-SS^*=X^*X$. Since T has a finite rank self-commutator, S also has a finite rank self-commutator and X has finite rank. Hence $\sigma_e(N)=\sigma_e(T)\cup\sigma_e(S^*)$. Since N is normal, $\sigma(N)\setminus\sigma_e(N)$ is countable; thus $m_2(\sigma(N))=m_2(\sigma_e(N))$. It follows that $m_2(\sigma(N))=0$ if and only if $m_2(\sigma_e(T))=0$ and $m_2(\sigma_e(S))=0$. Now suppose that the answer to Question A is affirmative. Then, since both T and S are pure subnormal operators having finite rank self-commutators, $m_2(\sigma_e(T))=$ $=m_2(\sigma_e(S))=0$. Thus $m_2(\sigma(N))=0$. So the answer to Question B is affirmative also. On the other hand it is clear from the above discussions that if the answer to Question B is affirmative, then the answer to Question A is affirmative also. So Questions A and B are equivalent.

The following theorem and corollary will show that Question C is equivalent to Questions A and B. Recall that an operator T is *semi-Fredholm* if either $\mathscr{K}(T)$ or $\mathscr{K}(T^*)$ is finite dimensional and $\mathscr{R}(T)$ is closed and is *Fredholm* if both $\mathscr{K}(T)$ and $\mathscr{K}(T^*)$ are finite dimensional and $\mathscr{R}(T)$ is closed. Recall also that $\sigma_e(T)=$ $=\{\lambda\in\mathbf{C}\colon T-\lambda$ is not Fredholm$\}$ and $\sigma_{ap}(T)=\{\lambda\in\mathbf{C}\colon T-\lambda$ is not bounded below$\}$. If T is a pure hyponormal operator, then, since the point spectrum of T is empty, $\sigma_{ap}(T)=\{\lambda\in\mathbf{C}\colon \mathscr{R}(T-\lambda)$ is not closed$\}$ and thus $\sigma_{ap}(T)\subseteq\sigma_e(T)$. If T is semi-Fredholm, let $i(T)=\dim(\mathscr{K}(T))-\dim(\mathscr{K}(T^*))$ denote the index of T.

Theorem 1.1. *Suppose that T is a hyponormal operator and that T^*T-TT^* has rank n, where n is a nonnegative integer. Then if λ is a complex number and $T-\lambda$ is semi-Fredholm, then $0\geqq i(T-\lambda)\geqq -n$.*

The following corollary follows from Theorem 1.1 and from the above characterizations of the essential spectrum and the approximate point spectrum of a pure hyponormal operator.

Corollary 1.2. *If T is a pure hyponormal operator and T^*T-TT^* has finite rank, then $\sigma_{ap}(T)=\sigma_e(T)$.*

The following lemma is needed in the proof of Theorem 1.1. Its proof is an easy exercise.

Lemma 1.3. *If $\mathscr{H}$ is a Hilbert space, $\mathscr{M}$ an arbitrary subspace of $\mathscr{H}$, $\mathscr{N}$ a finite dimensional subspace of $\mathscr{H}$, and $\mathscr{H}=\mathscr{M}+\mathscr{N}$, then* $\dim(\mathscr{M}^\perp)\leqq\dim(\mathscr{N})$.

Proof of Theorem 1.1. We first consider the case that T is a pure hyponormal operator on a Hilbert space $\mathscr{H}$. Let $P=\sqrt{T^*T-TT^*}$. Then for each complex number λ, $(T-\lambda)^*(T-\lambda)=(T-\lambda)(T-\lambda)^*+P^2$. By Theorem 2.2 of [8] we have $\mathscr{R}((T-\lambda)^*)\subseteq\mathscr{R}((T-\lambda))+\mathscr{R}(P)$. Since $\mathscr{R}(P)$ is finite dimensional, $\mathscr{R}((T-\lambda))^-+$ $+\mathscr{R}(P)$ is closed and contains $\mathscr{R}((T-\lambda)^*)^-$. But $\mathscr{R}((T-\lambda)^*)^-=\mathscr{H}$ since $\mathscr{K}(T-\lambda)=$ $=(0)$. Thus $\mathscr{H}=\mathscr{R}((T-\lambda))^-+\mathscr{R}(P)$. Lemma 1.3 implies that $\dim(\mathscr{K}((T-\lambda)^*))=$ $=\dim(\mathscr{R}((T-\lambda))^\perp)\leqq\dim(\mathscr{R}(P))=n$. Thus if $T-\lambda$ is semi-Fredholm, then $0\geqq i(T-\lambda)\geqq -n$. The general case follows readily.

2. The approximate point spectrum of a pure quasinormal operator

If $\mathscr{H}$ is a Hilbert space, let $\hat{\mathscr{H}}=\sum_{k=1}^{\infty}\oplus\mathscr{H}_k$, where for each positive integer k, $\mathscr{H}_k=\mathscr{H}$. If $T\in\mathscr{L}(\mathscr{H})$, define an operator $\hat{T}$ in $\mathscr{L}(\hat{\mathscr{H}})$ by $\hat{T}=\sum_{k=1}^{\infty}\oplus T_k$, where for each positive integer k, $T_k=T$. Let $V_{\mathscr{H}}$ denote the unilateral shift on $\hat{\mathscr{H}}$, i.e., $V_{\mathscr{H}}(x_1, x_2, \ldots)=(0, x_1, x_2, \ldots)$ for each $(x_1, x_2, \ldots)$ in $\hat{\mathscr{H}}$. Arlen Brown proved in [3] that each pure quasinormal operator is unitarily equivalent to $V_{\mathscr{H}}\hat{P}$, for some Hilbert space $\mathscr{H}$ and for some positive definite operator P in $\mathscr{L}(\mathscr{H})$, i.e., P is positive and $\mathscr{K}(P)=(0)$. The present author showed in [14] that

$$\sigma(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|P\|\},$$
$$\sigma_e(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}\cup\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|\tilde{P}\|\}$$

if $\mathscr{H}$ is infinite dimensional, and

$$\sigma_e(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$$

if $\mathscr{H}$ is finite dimensional. Here we compute the approximate point spectrum of $V_{\mathscr{H}}\hat{P}$.

Theorem 2.1. *If P is a positive definite operator on a nonzero Hilbert space $\mathscr{H}$, then $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$.*

Proof. Let $\Gamma=\{\lambda\in\mathbf{C}\colon |\lambda|\in\sigma(P)\}$ and let E be the spectral measure of P. Suppose that $\lambda\in\mathbf{C}\setminus\Gamma$. If $|\lambda|>\|P\|=\|V_{\mathscr{H}}\hat{P}\|$, then $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. So assume that $|\lambda|<\|P\|$. There exists a positive number ε such that $(|\lambda|-\varepsilon, |\lambda|+\varepsilon)\cap\sigma(P)=\emptyset$. Let $\mathscr{M}=\mathscr{R}(E([|\lambda|+\varepsilon, \|P\|]))$, let $R=P|\mathscr{M}$, and let $Q=P|\mathscr{M}^{\perp}$. Then $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $V_{\mathscr{M}}\hat{R}\oplus V_{\mathscr{M}^{\perp}}\hat{Q}$. Since $\|V_{\mathscr{M}^{\perp}}\hat{Q}\|=\|Q\|\leqq|\lambda|-\varepsilon$, $\lambda\notin\sigma(V_{\mathscr{M}^{\perp}}\hat{Q})$. Also since $\sigma(R)\subseteqq[|\lambda|+\varepsilon, \|P\|]$, we have $\|\hat{R}x\|\geqq(|\lambda|+\varepsilon)\|x\|$ for each x in $\hat{\mathscr{M}}$. Thus $\|(V_{\mathscr{M}}\hat{R}-\lambda)x\|\geqq\|V_{\mathscr{M}}\hat{R}x\|-\|\lambda x\|=\|\hat{R}x\|-\|\lambda x\|\geqq\varepsilon\|x\|$ for each x in $\hat{\mathscr{M}}$. It follows that $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{M}}\hat{R})$, and, therefore, $\lambda\notin\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. We have shown that $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})\subseteqq\Gamma$.

Now suppose that $\mu\in\Gamma$. For each positive integer n, let $\mathscr{M}_n=\mathscr{R}(E([|\mu|-1/n, |\mu|+1/n]))$, let $R_n=P|_{\mathscr{M}_n}$, and let $Q_n=P|_{\mathscr{M}_n^{\perp}}$. Note that $V_{\mathscr{H}}\hat{P}$ is unitarily equivalent to $V_{\mathscr{M}_n}\hat{R}_n\oplus V_{\mathscr{M}_n^{\perp}}\hat{Q}_n$ and that $\sigma(V_{\mathscr{M}_n}\hat{R}_n)=\{\lambda\in\mathbf{C}\colon |\lambda|\leqq\|R_n\|\}$. Since for any operator T, $\partial\sigma(T)\subseteqq\sigma_{\mathrm{ap}}(T)$, we have $\{\lambda\in\mathbf{C}\colon |\lambda|=\|R_n\|\}\subseteqq\sigma_{\mathrm{ap}}(V_{\mathscr{M}}\hat{R}_n)\subseteqq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. But $|\mu|\leqq\|R_n\|\leqq|\mu|+1/n$, $n=1, 2, \ldots$; thus $\|R_n\|\to|\mu|$. Hence, since $\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$ is closed, $\{\lambda\in\mathbf{C}\colon |\lambda|=|\mu|\}\subseteqq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$. This argument shows that $\Gamma\subseteqq\sigma_{\mathrm{ap}}(V_{\mathscr{H}}\hat{P})$, and the proof is complete.

We next discuss the relation between quasinormal operators and some other important classes of operators. In the following let $\mathscr{H}$ be a separable, infinite dimensional Hilbert space, let P be a positive definite operator on $\mathscr{H}$, and let (BQT) denote the class of biquasitriangular operators in $\mathscr{L}(\mathscr{H})$. The present author showed in [15] that a hyponormal operator T belongs to (BQT) if and only if $\sigma_{ap}(T)=\sigma(T)$. This fact shall be used freely in the following discussions. The following corollary is easy to verify.

Corollary 2.2. *The operator* $V_{\mathscr{K}}\hat{P}\in(\mathrm{BQT})$ *if and only if* $\sigma(P)=[0, \|P\|]$.

Let $(\mathrm{Ni})^-$ denote the norm-closure of the class of nilpotent operators in $\mathscr{L}(\mathscr{H})$. (See [11] for a discussion of the classes (BQT) and $(\mathrm{Ni})^-$.) Apostol, Foiaş, and Voiculescu gave the following characterization of $(\mathrm{Ni})^-$ in [1]: $(\mathrm{Ni})^-= =\{T\in(\mathrm{BQT})$: both $\sigma(T)$ and $\sigma_e(T)$ are connected and $0\in\sigma_e(T)\}$. Hence we have the following corollary.

Corollary 2.3. *The operator* $V_{\mathscr{K}}\hat{P}\in(\mathrm{BQT})$ *if and only if* $V_{\mathscr{K}}\hat{P}\in(\mathrm{Ni})^-$.

Let (EN) denote the class of essentially normal operators, i.e., $T\in(\mathrm{EN})$ if and only if T^*T-TT^* is compact, and let $(N+K)=\{N+K\in\mathscr{L}(\mathscr{H})$: N is normal and K is compact$\}$. It is known that $(N+K)=(\mathrm{BQT})\cap(\mathrm{EN})$ [4], [11]. Suppose that $V_{\mathscr{K}}\hat{P}\in(\mathrm{EN})$. Then P is compact and Corollary 2.2 implies that $V_{\mathscr{K}}\hat{P}\notin(\mathrm{BQT})$. Hence there are no pure quasinormal operators in the class $(N+K)$.

In [11] C. Pearcy observed that each operator in $\mathscr{L}(\mathscr{H})$ has a nontrivial invariant subspace if and only if each operator in $(\mathrm{Ni})^-$ does. He then wrote $(\mathrm{Ni})^-$ as the disjoint union of four subsets and he conjectured that if there exists an operator in $\mathscr{L}(\mathscr{H})$ that does not have a nontrivial invariant subspace, then that operator belongs to the "mysterious" fourth subset which consists of those operators in $(\mathrm{Ni})^-$ that are neither essentially normal nor quasinilpotent. The above shows that pure quasinormal operators in (BQT) are examples of operators in this fourth subset of $(\mathrm{Ni})^-$. But, of course, pure quasinormal operators do have nontrivial invariant (and hyperinvariant) subspaces.

Even though there are no pure quasinormal operators in $(N+K)$, there are general nonnormal quasinormal operators in this class. For example, let N be a normal operator such that $\sigma(N)=\sigma_e(N)=\overline{\mathbf{D}}$, where $\mathbf{D}$ denotes the open unit disk, and let V denote the unilateral shift (of multiplicity one). Then $N\oplus V$ is clearly quasinormal and essentially normal, and, since $\sigma_{ap}(N\oplus V)=\sigma(N\oplus V)=\overline{\mathbf{D}}$, $N\oplus V\in(\mathrm{BQT})$. Thus $N\oplus V\in(N+K)$.

The following example shows that there are also pure subnormal operators in $(N+K)$.

Example 2.4. Let S denote the Bergman shift, i.e., the Bergman operator for $\mathbf{D}$, and let N be its minimal normal extension. (See [6] for a discussion of Bergman operators.) The operator N is unitarily equivalent to the operator

$$\begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}.$$

on $\mathcal{H}\oplus\mathcal{H}$, where $\mathcal{H}$ is the Hilbert space on which S acts. We shall show that the dual T of S belongs to $(N+K)$. The operator T is a pure subnormal operator. It is known that $\sigma(N)=\sigma_e(N)=\sigma(S)=\overline{\mathbf{D}}$ and $\sigma_e(S)=\partial\mathbf{D}$ [6], and it is easy to verify that $\sigma(T)\subseteq\overline{\mathbf{D}}$. Since S is a weighted unilateral shift, S has a compact self-commutator. Thus X is compact and $T\in(\mathrm{EN})$ because $S^*S-SS^*=XX^*$ and $T^*T-TT^*=X^*X$. Observe that N is a compact perturbation of $S\oplus T^*$. Suppose that $\lambda\in\mathbf{D}$. Then $N-\lambda$ is not semi-Fredholm; thus $T^*-\lambda$ is not semi-Fredholm since $S-\lambda$ is Fredholm. Hence $\bar{\lambda}\in\sigma_{\mathrm{ap}}(T)$ since the point spectrum of T is empty. It follows that $\sigma_{\mathrm{ap}}(T)=\sigma(T)=\overline{\mathbf{D}}$; thus $T\in(\mathrm{BQT})$. Consequentially, $T\in(N+K)$.

Clancey and Morrell gave an example of a pure hyponormal operator T that is not subnormal and that has a rank one self-commutator such that $\sigma_e(T)=\sigma(T)=\overline{\mathbf{D}}$ [5]. By Corollary 1.2 $\sigma_{\mathrm{ap}}(T)=\overline{\mathbf{D}}$. Thus $T\in(\mathrm{BQT})\cap(\mathrm{EN})=(N+K)$. On the other hand if T is a pure quasinormal operator that has a finite rank self-commutator, then $T\notin(N+K)$. These facts motivate the following question.

Question 2.5. Does $(N+K)$ or, equivalently, does (BQT) contain any pure subnormal operators that have finite rank self-commutators?

Now let $\mathcal{H}$ be a nonzero Hilbert space of arbitrary dimension, let P be a positive definite operator in $\mathcal{L}(\mathcal{H})$, and let m_1 denote Lebesgue measure on the real line. It is easy to verify that

$$m_2(\sigma(V_{\mathcal{H}}\hat{P}))=\int_{[0,a]}2\pi r\,dm_1(r)\quad\text{and}\quad m_2(\sigma_e(V_{\mathcal{H}}\hat{P}))=\int_{[0,b]}2\pi r\,dm_1(r),$$

where $a=\|P\|$ and $b=\|\tilde{P}\|$ if $\mathcal{H}$ is infinite dimensional and $b=0$ otherwise. (To get the second equation we used the fact that $\{c\in\sigma(P)\colon c>\|\tilde{P}\|\}$ is countable.) We shall now develop a similar formula for $m_2(\sigma_{\mathrm{ap}}(V_{\mathcal{H}}\hat{P}))$. We shall need the following notation and lemma.

Let $\mathcal{B}$ denote the family of Borel subsets of $[0,+\infty)$. If $E\in\mathcal{B}$, let $\Lambda(E)=\{\lambda\in\mathbf{C}\colon|\lambda|\in E\}$, and let $\mathcal{D}=\{\Lambda(E)\colon E\in\mathcal{B}\}$. It is clear that $\mathcal{D}$ is a σ-algebra consisting of Borel subsets of $\mathbf{C}$ and that $\Lambda\colon\mathcal{B}\to\mathcal{D}$ is a one-to-one mapping of $\mathcal{B}$ onto $\mathcal{D}$ that preserves all of the Boolean operations.

Lemma 2.6. *Suppose that* $E\in\mathcal{B}$. *Then* $m_2(\Lambda(E))=\int_E 2\pi r\,dm_1(r)$.

Proof. For in $\mathcal{B}$, let $\mu(\Lambda(E))=\int_E 2\pi r\,dm_1(r)$. It is clear that μ is a measure on $\mathcal{D}$ and that if $E=(a,b]$, then $\mu(\Lambda(E))=m_2(\Lambda(E))$. An application of the theorem of Caratheodory shows that $\mu(\Lambda(E))=m_2(\Lambda(E))$ for all E in $\mathcal{B}$.

Theorem 2.1 and Lemma 2.6 imply the following theorems.

Theorem 2.7. $m_2(\sigma_{\mathrm{ap}}(V_{\mathcal{H}}\hat{P}))=\int_{\sigma(P)}2\pi r\,dm_1(r)$.

Theorem 2.8. $m_2(\sigma_{ap}(V_{\mathscr{H}}\hat{P}))=0$ *if and only if* $m_1(\sigma(P))=0$.

For comparison, we state the following theorem.

Theorem 2.9. $m_2(\sigma_e(V_{\mathscr{H}}\hat{P}))=0$ *if and only if P is compact.*

The spectrum of a pure quasinormal operator is connected and its essential spectrum has at most countably many connected components. In the following example, we present a pure quasinormal operator whose approximate point spectrum has uncountably many connected components each of which is a circle. We then use Theorem 2.7 to compute the measure of its approximate point spectrum.

Example 2.9. Let C denote the Cantor set and let $g\colon [0,1]\to[0,1]$ be the Cantor ternary function (cf. [13]). Recall that g is defined as follows: Let $r=\sum_{n=1}^{\infty} a_n/3^n$ be the ternary expansion of a number in $[0,1]$. Let $N=+\infty$ if $a_n\neq 1$ for each positive integer n, and otherwise let N be the smallest positive integer such that $a_N=1$. Let $b_n=a_n/2$ for $n<N$ and let $b_N=1$. Then $g(r)=\sum_{n=1}^{N} b_n/2^n$. Recall also that g is a continuous, monotonic increasing function of $[0,1]$ onto itself that is constant on the intervals in the complement in $[0,1]$ of C. Let $t>0$. Define a function $f\colon [0,1]\to[0,1+t]$ by $f(r)=g(r)+tr$. The function f is a monotone homeomorphism of $[0,1]$ onto $[0,1+t]$. Let $F=f(C)$, let P be a positive definite operator on a Hilbert space $\mathscr{H}$ such that $\sigma(P)=F$, and let $T=V_{\mathscr{H}}\hat{P}$. Since F is uncountable and totally disconnected, it follows from Theorem 2.1 that $\sigma_{ap}(T)$ has uncountably many connected components.

We now compute $m_2(\sigma_{ap}(T))$ by evaluating $\int_F 2\pi r\,dm_1(r)$. (It is easy to see that $m_2(\sigma_e(T))=m_2(\sigma(T))=\pi\|P\|^2=\pi(1+t)^2$.) Let S_n^k, $k=1,2,\dots,2^{n-1}$, be the disjoint subintervals of $[0,1]\setminus C$ that have measure equal to $1/3^n$, and let $T_n^k=$ $=f(S_n^k)$, $k=1,2,\dots,2^{n-1}$, $n=1,2,\dots$. Let $S=[0,1]\setminus C$ and $T=[0,1+t]\setminus F$. Then $\bigcup_{n,k} S_n^k=S$ and $\bigcup_{n,k} T_n^k=T$. We will first evaluate $\int_T 2\pi r\,dm_1(r)$. Fix n and k. Now S_n^k and T_n^k are open intervals and $g=(2k-1)/2^n$ on S_n^k. Thus f is differentiable on S_n^k and, therefore, by the change of variable theorem,

$$\int_{T_n^k} 2\pi r\,dm_1(r) = \int_{S_n^k} 2\pi f(r)f'(r)\,dm_1(r) = 2\pi t(2k-1)/6^n+t^2\int_{S_n^k} 2\pi r\,dm_1(r).$$

Thus

$$\int_T 2\pi r\,dm_1(r) = \sum_{n=1}^{\infty}\sum_{k=1}^{2^{n-1}} \Big(2\pi t(2k-1)/6^n+t^2\int_{S_n^k} 2\pi r\,dm_1(r)\Big) =$$

$$= 2\pi t\Big(\sum_{n=1}^{\infty}(1/6^n)\sum_{k=1}^{2^{n-1}}(2k-1)\Big)+t^2\int_S 2\pi r\,dm_1(r).$$

Using the facts that

$$\sum_{k=1}^{2^{n-1}} (2k-1) = (2^{n-1})^2 \quad \text{and} \quad \int_S 2\pi r\, dm_1(r) = \int_{[0,1]} 2\pi r\, dm_1(r) = \pi$$

$$\text{(since } m_1(C) = 0\text{)},$$

we get $\int_T 2\pi r\, dm_1(r) = \pi t + \pi t^2$. Hence

$$\int_F 2\pi r\, dm_1(r) = \int_{[0,1+t]} 2\pi r\, dm_1(r) - \int_T 2\pi r\, dm_1(r) = \pi(1+t);$$

thus $m_2(\sigma_{ap}(T)) = \pi(1+t)$.

3. Quasinormals plus commuting normals

In this section we present a class of pure subnormal operators that contains the class of pure quasinormal operators and show that for this class of operators the answer to the equivalent questions posed in Section 1 is affirmative.

Let S be a subnormal operator on a Hilbert space $\mathscr{H}$. Halmos has shown that S has a normal extension of the form

$$\begin{bmatrix} S & X \\ 0 & R^* \end{bmatrix} \tag{1}$$

on $\mathscr{H} \oplus \mathscr{H}$ (cf. [2], [9]). In fact, as mentioned earlier, if S is pure, then the minimal normal extension of S is unitarily equivalent to an operator of the form (1) on $\mathscr{H} \oplus \mathscr{H}$. If S is pure and is unitarily equivalent to its dual R, then we say that S is a *self-dual* subnormal operator. (See [7] for a discussion of the dual of a pure subnormal operator.) If S is self-dual, then the minimal normal extension of S is unitarily equivalent to the operator

$$\begin{bmatrix} S & Z \\ 0 & S^* \end{bmatrix} \tag{2}$$

on $\mathscr{H} \oplus \mathscr{H}$. As we mentioned earlier, Olin has observed in [10] that the operator in (1) is the minimal normal extension of S if and only if R is pure. Also note that any matrix of form (1) is normal if and only if $S^*S - SS^* = XX^*$, $R^*R - RR^* = X^*X$, and $S^*X = XR$.

Theorem 3.1. *Suppose that S is a subnormal operator on a Hilbert space $\mathscr{H}$ that has a normal extension of the form* (1) *and suppose that N is a normal operator in $\mathscr{L}(\mathscr{H})$ that commutes with S, R, and X. Then $T = S + N$ is also subnormal. Moreover, T is pure if and only if S is pure.*

Proof. An application of Fuglede's theorem shows that both N and N^* commute with S, R, X, S^*, R^*, and X^*. Let $Q=R+N^*$. Then $T^*T-TT^*=S^*S-SS^*=XX^*$, $Q^*Q-QQ^*=R^*R-RR^*=X^*X$, and $T^*X=XQ$. Therefore, the operator

$$\begin{bmatrix} T & X \\ 0 & Q^* \end{bmatrix}$$

on $\mathcal{H}\oplus\mathcal{H}$ is a normal extension of T, i.e., T is subnormal.

Now suppose that $S=S_1\oplus S_2$ on $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$, where S_1 is a normal operator on the Hilbert space $\mathcal{H}_1$ and S_2 is a pure operator on the Hilbert space $\mathcal{H}_2$. Then $N=[N_{ij}]$, relative to the decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{H}_2$. Since $NS=SN$ and $N^*S=SN^*$, a matrix calculation shows that $N_{21}S_1=S_2N_{21}$ and $N_{12}^*S_1=S_2N_{12}^*$. Theorem 1.2 of [15] shows that $N_{12}=N_{21}=0$. Thus $N=N_{11}\oplus N_{22}$ and $T=(S_1+N_{11})\oplus\oplus(S_2+N_{22})$, where S_1+N_{11} is normal since S_1 and N_{11} are commuting normal operators. This argument shows that if T is pure, then S is also pure. Since $S=T-N$, a similar argument shows that if S is pure, then T is also pure.

Corollary 3.2. *Suppose that S is a subnormal operator on a Hilbert space $\mathcal{H}$ that has a normal extension of the form* (2) *and suppose that N is a normal operator in $\mathcal{L}(\mathcal{H})$ that commutes with S and Z. Then $T=S+N$ is also subnormal.*

We remark that if S is a subnormal operator with a normal extension of form (2) and if U is a unitary operator that commutes with S (for example, if $U=\alpha 1_{\mathcal{H}}$, where α is a complex number such that $|\alpha|=1$), then the operator in (2) is unitarily equivalent to the operator

$$\begin{bmatrix} S & ZU \\ 0 & S^* \end{bmatrix}$$

on $\mathcal{H}\oplus\mathcal{H}$. Thus even a normal extension of the form (2) is not unique.

The following theorem is well-known. (Recall that if S is a quasinormal operator, then $S^*S-SS^*\geqq 0$.)

Theorem 3.3. *If S is a quasinormal operator, then the matrix in* (2) *is a normal extension of S if $Z=\sqrt{S^*S-SS^*}$. In particular, if S is a pure quasinormal operator, then S is self-dual.*

Proof. It is clear that $S^*S-SS^*=Z^2$. To show that $S^*Z=ZS$, observe that $Z^2S=(S^*S-SS^*)S=0$. Hence $ZS=0$ since Z is self-adjoint, and thus $S^*Z=(ZS)^*=0$. Therefore, the operator in (2) is normal.

Corollary 3.4. *If S is a quasinormal operator on a Hilbert space $\mathcal{H}$ and N is a normal operator in $\mathcal{L}(\mathcal{H})$ that commutes with S, then $T=S+N$ is subnormal.*

Proof. By Fuglede's theorem N also commutes with S^* and, therefore, with $Z=\sqrt{S^*S-SS^*}$. Hence T is subnormal by Corollary 3.2.

Let $\mathscr{K}$ be a Hilbert space, and let $\mathscr{S}=\{S+N\in\mathscr{L}(\mathscr{K})$: S is a pure quasinormal, N is normal, and $NS=SN\}$. The set $\mathscr{S}$ is a subset of the set of all pure subnormal operators in $\mathscr{L}(\mathscr{K})$ and contains the pure quasinormals. We show that the operators in $\mathscr{S}$ have a fairly simple structure.

Theorem 3.5. *If $T\in\mathscr{S}$, then there exist a Hilbert space $\mathscr{H}$ and commuting operators P and N in $\mathscr{L}(\mathscr{H})$, where P is positive definite and N is normal, such that T is unitarily equivalent to the operator $V_{\mathscr{H}}\hat{P}+\hat{N}$ on $\hat{\mathscr{H}}$.*

Proof. We have $T=S+N_1$, where S is a pure quasinormal operator, N_1 is normal, and $N_1S=SN_1$. There exist a Hilbert space $\mathscr{H}$ and a positive definite operator P in $\mathscr{L}(\mathscr{H})$ such that S is unitarily equivalent to $V_{\mathscr{H}}\hat{P}$. So T is unitarily equivalent to $V_{\mathscr{H}}\hat{P}+N_0$, where N_0 is normal and commutes with $V_{\mathscr{H}}\hat{P}$. Then $N_0=[N_{ij}]$ on $\hat{\mathscr{H}}$. Since N_0 commutes with $V_{\mathscr{H}}\hat{P}$, a matrix calculation shows that $N_{1,j+1}P=0$ and $N_{i+1,j+1}P=PN_{ij}$, $i,j=1,2,\dots$. An induction argument shows that $N_{ij}=0$ for $i<j$. Since Fuglede's theorem implies that N_0^* commutes with $V_{\mathscr{H}}\hat{P}$, we have by a similar argument that $N_{ij}=0$ for $i>j$. Thus $[N_{ij}]$ is diagonal, N_{ii} is normal, and $N_{i+1,i+1}P=PN_{ii}$, $i=1,2,\dots$. Using the Putnam—Fuglede's theorem, we can see that $PN_{i+1,i+1}=N_{ii}P$ also. Hence P^2, and thus P, commutes with N_{ii}. It follows that $(N_{i+1,i+1}-N_{ii})P=0$; thus $N_{i+1,i+1}=N_{ii}$, $i=1,2,\dots$. Therefore, $N_0=\hat{N}_{11}$, and the proof is complete.

It follows from Theorem 3.5 that if S is a pure quasinormal and N is a nonzero normal operator that commutes with S, then $S+N$ is not quasinormal. Thus $\mathscr{S}$ contains operators that are not quasinormal.

We can say more about the structure of those operators in $\mathscr{S}$ that have compact self-commutators. Let V denote the unilateral shift of multiplicity one.

Theorem 3.6. *Suppose that $T\in\mathscr{S}$ and T^*T-TT^* is compact. Then there exist an index set A, a set of positive numbers $\{c_\alpha\}_{\alpha\in A}$, and a set of complex numbers $\{\lambda_\alpha\}_{\alpha\in A}$ such that T is unitarily equivalent to $\sum_{\alpha\in A}\oplus(\lambda_\alpha+c_\alpha V)$. If the rank of T^*T-TT^* is n, then $A=\{1,2,\dots,n\}$; otherwise $A=\{1,2,\dots\}$.*

Proof. By Theorem 3.5 T is unitarily equivalent to $V_{\mathscr{H}}\hat{P}+\hat{N}$, where P is positive definite in $\mathscr{L}(\mathscr{H})$, N is normal in $\mathscr{L}(\mathscr{H})$, and $PN=NP$. Since T^*T-TT^* is compact, P is compact. Suppose that c is an eigenvalue of P. Then $\mathscr{K}(P-c)$ is finite dimensional and reduces N. Hence $\mathscr{K}(P-c)$ has an orthonormal basis consisting of eigenvectors of N. It follows that $\mathscr{H}$ has an orthonormal basis $\{e_\alpha\}_{\alpha\in A}$ consisting of vectors that are eigenvectors of both P and N. For $\alpha\in A$, let c_α and

λ_α be the eigenvalues of P and N, respectively, associated with e_α, and let $\mathcal{M}_\alpha$ be the one-dimensional subspace of $\mathcal{H}$ spanned by e_α. Then $\mathcal{H}=\sum_{\alpha\in A}\oplus\mathcal{M}_\alpha$ and $\hat{\mathcal{H}}$ is Hilbert space isomorphic to $\sum_{\alpha\in A}\oplus\hat{\mathcal{M}}_\alpha$. Hence $V_{\mathcal{H}}\hat{P}$ is unitarily equivalent to $\sum_{\alpha\in A}\oplus c_\alpha V_{\mathcal{M}_\alpha}$ and $\hat{N}$ is unitarily equivalent to $\sum_{\alpha\in A}\oplus\lambda_\alpha 1_{\hat{\mathcal{M}}_\alpha}$; thus T is unitarily equivalent to $\sum_{\alpha\in A}\oplus(\lambda_\alpha+c_\alpha V_{\mathcal{M}_\alpha})$. The proof is complete since for each α in A, $V_{\mathcal{M}_\alpha}$ is unitarily equivalent to V.

Corollary 3.7. *If $T\in\mathcal{S}$ and T^*T-TT^* has finite rank, then $m_2(\sigma_{ap}(T))=$ $=m_2(\sigma_e(T))=0$.*

Proof. By Theorem 3.6 T is unitarily equivalent to $\sum_{k=1}^{n}\oplus(\lambda_k+c_k V)$. The proof is complete since $m_2(\sigma_{ap}(V))=m_2(\sigma_e(V))=0$.

Corollary 3.7 shows that the answer to the three equivalent questions posed in Section 1 is affirmative for the class of operators $\mathcal{S}$. In regard to Question 2.5, note that Corollary 3.7 also implies that if $T\in\mathcal{S}$ and T has a finite rank self-commutator, then $T\notin(N+K)$, since $\sigma_{ap}(T)\neq\sigma(T)$. Recall that if T is a pure quasinormal operator, then T^*T-TT^* is compact if and only if $m_2(\sigma_e(T))=0$ (see Theorem 2.9), and if T^*T-TT^* is compact, then $m_2(\sigma_{ap}(T))=0$ (see Theorem 2.8). The next example shows that this is not the case for the class of operators $\mathcal{S}$.

Example 3.8. Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis for a Hilbert space $\mathcal{H}$, let $\{\lambda_k\}_{k=1}^\infty$ be an enumeration of all the "rational" complex numbers in $\mathbf{D}$, let $\{c_k\}_{k=1}^\infty$ be a sequence of positive numbers such that $\sum_{k=1}^\infty c_k^2<1/2$, let $G_k=\{\lambda\in\mathbf{D}: |\lambda-\lambda_k|<c_k\}$, $k=1,2,\dots$, and let $G=\bigcup_{k=1}^\infty G_k$. Note that G is an open subset of $\mathbf{C}$ and that $m_2(G)\leqq\sum_{k=1}^\infty\pi c_k^2<\pi/2$. Since $\bar{G}=\bar{\mathbf{D}}$, $m_2(\partial G)>\pi/2$. Define a positive definite operator P and a normal operator N in $\mathcal{L}(\mathcal{H})$ by $Pe_k=c_k e_k$ and $Ne_k=\lambda_k e_k$. Let $T=V_{\mathcal{H}}\hat{P}+$ $+\hat{N}$. Observe that $T\in\mathcal{S}$ and that T^*T-TT^* is compact (since P is compact). Observe also that T is unitarily equivalent to $\sum_{k=1}^\infty\oplus(\lambda_k+c_k V)$ and that $\partial G_k=$ $=\sigma_{ap}(\lambda_k+c_k V)\subseteqq\sigma_{ap}(T)$. Thus $\left(\bigcup_{k=1}^\infty\partial G_k\right)^-\subseteqq\sigma_{ap}(T)$. It follows that $m_2(\sigma_{ap}(T))>0$ since $\partial G\subseteqq\left(\bigcup_{k=1}^\infty\partial G_k\right)^-$. We also have $m_2(\sigma_e(T))>0$ since $\sigma_{ap}(T)\subseteqq\sigma_e(T)$.

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The closure of invertible operators on a Hilbert space

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1. Introduction. Let H be a separable infinite dimensional Hilbert space and let $B(H)$ be the Banach algebra of all bounded linear operators on H. Denote by $\mathbf{G}$ the group of all invertible operators in $B(H)$, then what is the condition for an operator to be in the (norm) closure $\overline{\mathbf{G}}$ or the boundary bdy $\mathbf{G}$ of $\mathbf{G}$? FELDMAN and KADISON [3] considered this problem and characterized elements in the closure of invertible operators in a weakly closed subalgebra of $B(H)$. In the setting of Banach space operators, KELLY and HOGAN [8] gave some sufficient conditions for an operator to lie in the boundary of invertible operators from a view point of conservative operators. TREESE and KELLY [10], also in the same setting, showed a characterization of such operators under the restriction that they have closed ranges. Recall that the distance dist $(A, \mathbf{S})$ of an operator A to a subset $\mathbf{S}\subset B(H)$ is defined as $\inf\{\|A-S\|: S\in\mathbf{S}\}$. Now another approach to our problem is to estimate, by some familiar parameter, the distance for $\mathbf{S}=\mathbf{G}$ or some other set related to $\mathbf{G}$. In terms of essential minimum modulus, the first author [6] showed some distance formulae on $\mathbf{G}$ and certain subsets of operators with index zero. Independently, BOULDIN [2] also tried a similar approach to the problem and presented distance formulae on $\mathbf{G}$ and on the set $\mathbf{F}$ of all Fredholm operators.

In this paper we shall continue the study on the closure $\overline{\mathbf{G}}$ and the boundary bdy $\mathbf{G}$ of $\mathbf{G}$. In Section 2 we clarify operators in bdy $\overline{\mathbf{G}}$ and show that the interior int $\overline{\mathbf{G}}$ of $\overline{\mathbf{G}}$ coincides with the set of Fredholm operators with index zero. In Section 3 we characterize closed range operators in $\overline{\mathbf{G}}$, which refines results in [1] and [10]. In Section 4, as an extension of [2] or [6], we determine the distance dist $(A, \mathbf{S})$ when $\mathbf{S}$ is the subset of Fredholm operators with an index or the boundary bdy $\mathbf{G}$.

Throughout this paper we assume that the Hilbert space H is separable infinite dimensional. The index ind A of an operator A is defined by dim ker $A-$dim ker A^*, where dim ker B is the dimension of the kernel of B and $\infty-\infty$ is understood to

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be zero [9]. The minimum (resp. essential minimum) modulus $m(A)$ (resp. $m_e(A)$) of $A \in B(H)$ is defined as the number

$$\inf\{\lambda:\ \lambda \in \sigma(|A|)\} \quad (\text{resp. } \inf\{\lambda:\ \lambda \in \sigma_e(|A|)\}).$$

Here $\sigma(|A|)$ (resp. $\sigma_e(|A|)$) is the spectrum (resp. essential spectrum) of $|A|:=(A^*A)^{1/2}$. Let $\mathbf{I}_n$ be the set of all operators with index n. Now, as a preliminary we state a result due to Bouldin [2, Theorem 3] (which was essentially shown in [6, Theorem 4]).

Theorem 1.1. *Let* $A \in B(H)$.
(1) *If* $A \in \mathbf{I}_0$ *then* $\operatorname{dist}(A, \mathbf{G})=0$.
(2) *If* $A \notin \mathbf{I}_0$ *then* $\operatorname{dist}(A, \mathbf{G})=\max\{m_e(A), m_e(A^*)\}$.

Concerning the index and the essential minimum modulus we want to state three more basic facts.

Lemma 1.2. *Let* $A, B \in B(H)$ *and let* $\|A-B\|<m_e(A)$. *Then* $\operatorname{ind} A=\operatorname{ind} B$ ([2, p. 513]).

Lemma 1.3. *Let* $\operatorname{ind} A=n$. *Then there is an isometry or coisometry* W *according to* $n \leqq 0$ *or* $n \geqq 0$ *such that* $A=W|A|$ *and* $\operatorname{ind} W=n$ ([9, Proof of Theorem 1.3]).

Lemma 1.4. *If* $\operatorname{ind} A \leqq 0$, *then* $m_e(A) \geqq m_e(A^*)$. *Hence, if* $A \in \mathbf{G}$ *or* $A \in \overline{\mathbf{G}}$ *then* $m_e(A)=m_e(A^*)$.

2. Operators in $\overline{\mathbf{G}}$. Let $\mathbf{F}_n=\mathbf{F}\cap\mathbf{I}_n$ be the set of all Fredholm operators with index n. Then, since $\mathbf{G}\subset\mathbf{F}_0\subset\mathbf{I}_0$ we have, by Theorem 1.1,

$$\overline{\mathbf{G}} = \overline{\mathbf{F}}_0 = \overline{\mathbf{I}}_0. \tag{2.1}$$

First, for the boundary of this set we have:

Theorem 2.1. $\operatorname{bdy} \overline{\mathbf{G}}=\{A \in B(H):\ m_e(A)=m_e(A^*)=0\}$.

Proof. Let $m_e(A)=m_e(A^*)=0$. First we show $A \in \overline{\mathbf{G}}$. If $A \in \mathbf{I}_0$ then $A \in \overline{\mathbf{G}}$, say, by (2.1), and if $A \notin \mathbf{I}_0$ then by Theorem 1.1 (2) $\operatorname{dist}(A, \mathbf{G})=0$, so that again we have $A \in \overline{\mathbf{G}}$. Now, to see $A \in \operatorname{bdy} \overline{\mathbf{G}}$ let $\varepsilon>0$ and suppose, without loss of generality, that $\operatorname{ind} A \leqq 0$. Then $A=W|A|$ for an isometry W with $\operatorname{ind} W \leqq 0$, by Lemma 1.3. Since $m_e(A)=0$, we see, from [4, Theorem 1.1], that $\dim E([0, \varepsilon))$ is infinite, where $E(\cdot)$ is the spectral measure of $|A|$. For brevity, write $E_\varepsilon=E([0, \varepsilon))$ and $E_\varepsilon^\perp=1-E_\varepsilon$ ($E_\varepsilon^\perp$ becomes the orthogonal projection onto the subspace $E([\varepsilon, \infty))H$). Define an operator $V \in B(H)$ as

$$Vx = x \quad \text{for} \quad x \in E_\varepsilon^\perp H, \quad \text{and}$$

$$Vx_n = x_{n+1} \quad \text{for an orthonormal basis} \quad \{x_n\} \quad \text{of} \quad E_\varepsilon H.$$

Furthermore, put

$$B_\varepsilon = \int \max\{\lambda-\varepsilon, 0\}\, dE(\lambda)$$

and $C_\varepsilon = WV(B_\varepsilon+\varepsilon)$. Then, we easily see that

$$VE_\varepsilon^\perp = E_\varepsilon^\perp, \quad E_\varepsilon^\perp B_\varepsilon = B_\varepsilon, \quad \||A|-B_\varepsilon\| \leqq \varepsilon \quad \text{and} \quad m_e(C_\varepsilon) \geqq \varepsilon.$$

Since $\operatorname{ind} W \leqq 0$ (and $\operatorname{ind} V(B_\varepsilon+\varepsilon) = -1$, $W, V(B_\varepsilon+\varepsilon)$ are Fredholm operators), we see $\operatorname{ind} C_\varepsilon \leqq -1$, so that by Theorem 1.1 we have $\operatorname{dist}(C_\varepsilon, \mathbf{G}) \geqq m_e(C_\varepsilon) > 0$ or $C_\varepsilon \notin \overline{\mathbf{G}}$. But

$$\|C_\varepsilon - A\| = \|W(V(B_\varepsilon+\varepsilon)-|A|)\| = \|VB_\varepsilon - |A| - \varepsilon V\| \leqq$$
$$\leqq \|VB_\varepsilon - |A|\| + \varepsilon = \|VE_\varepsilon^\perp B_\varepsilon - |A|\| + \varepsilon = \|B_\varepsilon - |A|\| + \varepsilon \leqq 2\varepsilon.$$

Hence, since ε is arbitrary we see that A is on the boundary $\operatorname{bdy} \overline{\mathbf{G}}$. To see the converse, that is, if $A \in \operatorname{bdy} \overline{\mathbf{G}}$ then $m_e(A) = m_e(A^*) = 0$, suppose otherwise, say, $m_e(A) > 0$. Then by Lemma 1.4 $m_e(A^*) = m_e(A) > 0$, so that A is Fredholm. Besides, since $A \in \operatorname{bdy} \overline{\mathbf{G}} \subset \operatorname{bdy} \mathbf{G}$, we can find an operator $D \in \mathbf{G}$ such that $\|A-D\| < m_e(A)$. Hence $\operatorname{ind} A = \operatorname{ind} D = 0$ (say, by Lemma 1.2), so that $A \in \mathbf{F}_0$. But, since $\mathbf{F}_0$ is an open subset of $\overline{\mathbf{G}}$ we see that A is an interior point of $\overline{\mathbf{G}}$, which is a contradiction.

Remark. Denote by $\mathbf{F}_l$ (resp. $\mathbf{F}_r$) the set of all left (resp. right) semi-Fredholm operators or the set $\{A\colon m_e(A) > 0\}$ (resp. $\{A\colon m_e(A^*) > 0\}$). Then, from the proof of Theorem 2.3 (or a similar argument) we see

$$\overline{\mathbf{G}} \cap \mathbf{F}_l = \mathbf{F}_0 \quad (= \overline{\mathbf{G}} \cap \mathbf{F}_r). \tag{2.2}$$

If we denote by $\mathbf{G}_l$ (resp. $\mathbf{G}_r$) the set of all left (resp. right) invertible operators, then as (2.2) we can also see

$$\overline{\mathbf{G}} \cap \mathbf{G}_l = \mathbf{G} \quad (= \overline{\mathbf{G}} \cap \mathbf{G}_r).$$

Corollary 2.2. (1) $\operatorname{int} \overline{\mathbf{G}} = \mathbf{F}_0$, *and hence* $\mathbf{F}_0$ *is a regularly open subset in* $B(H)$.
(2) $\operatorname{bdy} \overline{\mathbf{G}} = \operatorname{bdy} \mathbf{F}_0$.
(3) $\operatorname{bdy} \mathbf{G} = \operatorname{bdy} \overline{\mathbf{G}} \cup (\mathbf{F}_0 \setminus \mathbf{G})$.

Proof. (1) Since $\mathbf{F}_0 \subset \operatorname{int} \overline{\mathbf{G}}$ is clear, we may only show the opposite inclusion. Let $A \in \operatorname{int} \overline{\mathbf{G}}$. Then by the theorem $m_e(A) > 0$ or $m_e(A^*) > 0$. Hence, in either case we have (say, by (2.2)) $A \in \mathbf{F}_0$.

(2) Clear by the theorem and (2.1).

(3) Note that $\operatorname{bdy} \mathbf{G} \supset \operatorname{bdy} \overline{\mathbf{G}}$, and that $A \in \operatorname{bdy} \mathbf{G} \setminus \operatorname{bdy} \overline{\mathbf{G}}$ if and only if $A \in \mathbf{F}_0 \setminus \mathbf{G}$.

3. Closed range operators in $\overline{\mathbf{G}}$. In this section we show some necessary and sufficient conditions for an operator to lie in $\overline{\mathbf{G}}$ or $\operatorname{bdy} \mathbf{G}$ under the restriction that the operator has closed range. For simplicity, we denote by $A \in (\mathrm{CR})$ if $A \in B(H)$ has closed range. It is well-known [1], [5] that if $A \in (\mathrm{CR})$ then there exists the unique (Moore—Penrose) generalized inverse $A^\dagger \in B(H)$ of A satisfying the following four identities;

$$AA^\dagger A = A,\quad A^\dagger AA^\dagger = A^\dagger,\quad (AA^\dagger)^* = AA^\dagger \quad\text{and}\quad (A^\dagger A)^* = A^\dagger A.$$

The products $AA^\dagger$ and $A^\dagger A$ are the orthogonal projections onto the ranges $AH(=\ker^\perp A^*$, the orthogonal complement of $\ker A^*)$ and $A^*H(=\ker^\perp A)$, respectively. The next fact [7, Proposition 2.3] is useful for our discussion.

Lemma 3.1. *Let $\{A_n\}$ be a sequence of operators with closed range, and suppose that it converges to $A\in$(CR) uniformly, that is, $A_n \to A$. Then the following conditions are equivalent.*

(1) $\sup_n \|A_n^\dagger\| < \infty$.

(2) $A_n A_n^\dagger \to AA^\dagger$.

(3) $A_n^\dagger A_n \to A^\dagger A$.

The equivalence (2) and (3) or (3′) of the following result was essentially shown by Beutler [1, Theorem 1].

Theorem 3.2. *Let $A\in$(CR). Then the following conditions are equivalent.*

(1) $A\in\overline{\mathbf{G}}$.

(2) $A\in\mathbf{I}_0$.

(3) $A=BP$ *for an operator* $B\in\mathbf{G}$ *and an orthogonal projection* P.

(3′) $A=PB$ *for an operator* $B\in\mathbf{G}$ *and an orthogonal projection* P.

Proof. (1)⇒(2) Let $\{A_n\}$ be a sequence in $\mathbf{G}$, and let $A_n \to A$. Put $C_n = A_n A^\dagger$ and $C=AA^\dagger$. Then $C_n, C\in$(CR) and $C_n \to C$. Furthermore, since $\ker^\perp C_n = AH$ we have $C_n^\dagger C_n = AA^\dagger = C = C^\dagger C$ (cf. $C=C^\dagger$). Hence, by Lemma 3.1 we have $C_n C_n^\dagger \to CC^\dagger = AA^\dagger$. Hence, for a sufficiently large n, we have

$$\|C_n C_n^\dagger - C_n^\dagger C_n\| < 1.$$

This implies $\dim\ker C_n^* = \dim\ker C_n$ or $\operatorname{ind} C_n = 0$. Hence $\operatorname{ind} A^\dagger = 0$, i.e., $\operatorname{ind} A = 0$.

(2)⇒(3) If $A\in\mathbf{I}_0$, then $A=U|A|$ with a unitary U. Since $P:=A^\dagger A$ is an orthogonal projection such that $|A|P=|A|$, and since $B:=U\{|A|+(1-A^\dagger A)\}\in\mathbf{G}$, we see that $A=BP$ is the desired decomposition.

(3)⇒(1) Note that $\operatorname{ind} BP = \operatorname{ind} B + \operatorname{ind} P = 0$ for B and P in (3).

(3)⇔(3′) Note that $A\in\mathbf{I}_0 \Leftrightarrow A^*\in\mathbf{I}_0$.

In [10] Treese and Kelly characterized closed range operators in $\operatorname{bdy}\mathbf{G}$ (in the setting of Banach space operators). From Theorem 3.2 we now deduce a similar characterization of such operators, which is to be compared with [10, Theorem].

Corollary 3.3. *Let* $A \in (\mathrm{CR})$. *Then the following conditions are equivalent.*

(1) $A \in \operatorname{bdy} \mathbf{G}$.

(2) $A \in \mathbf{I}_0 \setminus \mathbf{G}$.

(3) $A = BP$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* $P \neq 1$.

(3′) $A = PB$ *for an operator* $B \in \mathbf{G}$ *and an orthogonal projection* $P \neq 1$.

(4) $A \notin \mathbf{G}$ *and there exists a sequence* $\{B_n\}$ *in* $\mathbf{G}$ *such that* $B_n A^\dagger A \to A$.

Proof. From the theorem we easily see that (1), (2), (3) and (3′) are mutually equivalent. If (3) is assumed, then $B_n = B(P+1/n)$ $(n=1, 2, \ldots)$ are invertible and $B_n A^\dagger A \to AA^\dagger A = A$, that is, (4) is obtained. If we assume (4), then since $B_n A^\dagger A \in \mathbf{I}_0$ we easily see $A \in \bar{\mathbf{I}}_0 = \overline{\mathbf{G}}$, which implies $A \in \operatorname{bdy} \mathbf{G}$, i.e., the condition (1).

Remark. In proving the above corollary by a technique in [10], we would have to add to (4) the uniform boundedness of $\{B_n^{-1}\}$. Related to this, we observe that the sequence $\{(B_n A^\dagger A)^\dagger\}$ of generalized inverses is uniformly bounded; since $B_n A^\dagger A \to A$ and $(B_n A^\dagger A)^\dagger (B_n A^\dagger A) = A^\dagger A$, we have, by Lemma 3.1, $\sup_n \|(B_n A^\dagger A)^\dagger\| < \infty$.

4. Distance formulae related to $\mathbf{F}_n$, bdy $\mathbf{G}$ and bdy $\overline{\mathbf{G}}$. Recall that $\bar{\mathbf{F}}_0 = \bar{\mathbf{I}}_0 = \overline{\mathbf{G}}$, and hence that

$$\operatorname{dist}(A, \mathbf{F}_0) = \max\{m_e(A), m_e(A^*)\} \quad \text{for} \quad A \notin \mathbf{I}_0$$

by Theorem 1.1. As an extension of those facts we have:

Theorem 4.1. *Let* $A \in B(H)$.

(1) *If* $A \in \mathbf{I}_n$, *then* $\operatorname{dist}(A, \mathbf{F}_n) = 0$.

(2) *If* $A \notin \mathbf{I}_n$, *then* $\operatorname{dist}(A, \mathbf{F}_n) = \max\{m_e(A), m_e(A^*)\}$.

Proof. (1) If $A \in \mathbf{I}_n$ then $A = W|A|$ with an isometry (or coisometry) $W \in \mathbf{I}_n$. Let $\varepsilon > 0$ and $B = W(|A| + \varepsilon)$. Then $B \in \mathbf{F}_n$ and $\|A - B\| < \varepsilon$. Hence, $\operatorname{dist}(A, \mathbf{F}_n) < \varepsilon$, which implies the assertion (1).

(2) Let S be a unilateral simple shift on H, and let $B = S^n A$ or $B = AS^{*(-n)}$ according to $n \geqq 0$ or $n \leqq 0$. Then we see $\operatorname{ind} B \neq 0$ because of $\operatorname{ind} S = -1$, and

$$m_e(B) = m_e(A), \quad m_e(B^*) = m_e(A^*). \tag{4.1}$$

Furthermore, we see

$$\bar{\mathbf{F}}_n = (S^{*(n)}\mathbf{G})^- \quad \text{or} \quad \bar{\mathbf{F}}_n = (\mathbf{G}S^{(-n)})^-$$

according to $n \geqq 0$ or $n \leqq 0$. Hence, if $n \geqq 0$, then

$$\operatorname{dist}(A, \mathbf{F}_n) = \operatorname{dist}(A, S^{*(n)}\mathbf{G}) = \operatorname{dist}(B, S^n S^{*(n)}\mathbf{G}) = \operatorname{dist}(B, \mathbf{G})$$

(cf. $(S^n S^{*(n)}\mathbf{G})^- = \overline{\mathbf{G}}$). Hence, by Theorem 1.1 and (4.1) we have the desired identity in (2). For $n \leqq 0$, similarly we can obtain the identity.

Concerning the distance from an operator to the boundary bdy $\mathbf{G}$ or bdy $\overline{\mathbf{G}}$, we have:

Theorem 4.2. *Let* $A \in B(H)$. *Then*

$$\text{(1)}\quad \operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) = \begin{cases} \max\{m_e(A), m_e(A^*)\} & \textit{if } A \notin \overline{\mathbf{G}}, \\ m(A)(= m(A^*)) & \textit{if } A \in \overline{\mathbf{G}}. \end{cases}$$

$$\text{(2)}\quad \operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) = \max\{m_e(A), m_e(A^*)\}.$$

Proof. (1) If $A \notin \overline{\mathbf{G}}$, then clearly

$$\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) = \operatorname{dist}(A, \mathbf{G}) = \max\{m_e(A), m_e(A^*)\}.$$

If $A \in \overline{\mathbf{G}}$, then we consider the two cases $A \in \mathbf{I}_0$ and $A \notin \mathbf{I}_0$. First, if $A \in \mathbf{I}_0$, then $A = U|A|$ for a unitary U. Let $B = U(|A| - m(A))$. Then $m(B) = 0$ and $B \in \operatorname{bdy}\mathbf{G}$. Hence $\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) \leqq \|A - B\| = m(A)$. To see that only the equality sign holds, suppose

$$\operatorname{dist}(A, \operatorname{bdy}\mathbf{G}) < m(A), \tag{4.2}$$

and hence also $m(A) > 0$. Then $A \in \mathbf{G}_l$ or $A \in \mathbf{G}_l \cap \overline{\mathbf{G}} = \mathbf{G}$, and by (4.2) there exists an operator $C \in \operatorname{bdy}\mathbf{G}$ such that $\|A - C\| < m(A)$. Hence, since $\|A^{-1}\| = m(A)^{-1}$ (cf. [2, Theorem 1]), we have

$$\|1 - A^{-1}C\| = \|A^{-1}(A - C)\| \leqq \|A^{-1}\|\,\|A - C\| < 1,$$

so that we easily see $C \in \mathbf{G}$. This is a contradiction. Next, if $A \notin \mathbf{I}_0$ then by Theorem 1.1 we see that $m_e(A) = m_e(A^*) = \operatorname{dist}(A, \mathbf{G}) = 0$. Hence, since $m(A) \leqq m_e(A) = 0$ and since $A \in \overline{\mathbf{G}} \setminus \mathbf{I}_0 \subset \operatorname{bdy}\mathbf{G}$ we again obtain the desired identity with the common value zero. It is easy to see $m(A) = m(A^*)$ for $A \in \mathbf{G}$ and hence for $A \in \overline{\mathbf{G}}$.

(2) If $A \notin \overline{\mathbf{G}}$, then clearly

$$\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) = \operatorname{dist}(A, \overline{\mathbf{G}}) = \max\{m_e(A), m_e(A^*)\}.$$

If $A \in \overline{\mathbf{G}}$, then as (1) we consider the two cases $A \in \mathbf{I}_0$ and $A \notin \mathbf{I}_0$. If $A \in \mathbf{I}_0$, then $A = U|A|$ for a unitary U. Put $B = U(|A| - m_e(A))$. Then $B \in \operatorname{bdy}\overline{\mathbf{G}}$, because $m_e(B) = m_e(B^*) = 0$ (and by Theorem 2.1). Hence, $\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) \leqq \|A - B\| = m_e(A)$. To show that the equality sign holds, suppose $\operatorname{dist}(A, \operatorname{bdy}\overline{\mathbf{G}}) < m_e(A)$, and hence also $m_e(A) > 0$. Then $A \in \mathbf{F}_l \cap \overline{\mathbf{G}} = \mathbf{F}_0$ (say, by (2.2)). Besides, there exists an operator $C \in \operatorname{bdy}\overline{\mathbf{G}}$ such that $\|A - C\| < m_e(A)$. Hence we see $m_e(C) \geqq m_e(A) - \|A - C\| > 0$, so that $C \in \mathbf{F}_l \cap \overline{\mathbf{G}} = \mathbf{F}_0$. But this is a contradiction by Corollary 2.2 (1). If $A \notin \mathbf{I}_0$, then by Theorem 1.1 we have $m_e(A) = m_e(A^*) = \operatorname{dist}(A, \mathbf{G}) = 0$. This implies $A \in \operatorname{bdy}\overline{\mathbf{G}}$ and the identity in (2) holds again.

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On generalized resolvents of nondensely defined symmetric contractions

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1. In 1977 M. G. KREĬN and I. E. OVČARENKO [6] described all generalized selfadjoint contraction resolvents of a nondensely defined symmetric contraction T in Hilbert space using its minimal or maximal selfadjoint contraction extension as the fixed extension.

In this note recent results of H. Langer and the author on the extension of dual pairs of contractions together with well-known results of B. SZ.-NAGY and A. KORÁNYI [10] are used to give a description of the generalized selfadjoint contraction resolvents of T when an arbitrary selfadjoint contraction extension of T is taken as fixed. The results have immediate application to the extension problem for non-negative closed linear relations in Hilbert space.

2. Assume T is a symmetric contraction in a Hilbert space $\mathfrak{H}$, with nondense domain $\mathfrak{D}(T)$. Then results of GR. ARSENE and A. GHEONDEA [1] (cf. also [3]) on dual pairs of contractions, applied to the pair $\{T, T\}$, imply the existence of a bijection $G \to T_G$ between all canonical (i.e. remaining in $\mathfrak{H}$) contraction extensions T_G of T such that $T_G^* \supset T$ and the set of all contractions $G \in [\mathscr{D}_\Gamma]$. This bijection is given by the matrix representation

$$(1) \qquad T_G = \begin{pmatrix} A & D_A \Gamma \\ \Gamma^* D_A & -\Gamma^* A \Gamma + D_\Gamma G D_\Gamma \end{pmatrix}$$

with respect to the decomposition $\mathfrak{H} = \mathfrak{D}(T) \oplus \mathfrak{D}(T)^\perp$, with some contractions $A \in [\mathfrak{D}(T)]$, $A = A^*$, $\Gamma \in [\mathfrak{D}(T)^\perp, \mathscr{D}_A]$. Here $[\mathfrak{H}_1, \mathfrak{H}_2]$ denotes the set of all bounded linear operators from all of $\mathfrak{H}_1$ into $\mathfrak{H}_2$ and we put $[\mathfrak{H}_1] := [\mathfrak{H}_1, \mathfrak{H}_1]$. If B is a contraction from $\mathfrak{H}_1$ into $\mathfrak{H}_2$ we put $D_B := (I - B^* B)^{1/2}$ and $\mathscr{D}_B := \overline{\mathfrak{R}(D_B)}$. $\mathfrak{R}(C)$ denotes the range of the linear operator C.

The extension T_G is selfadjoint if and only if G is selfadjoint. In this case (1) gives an explicit representation of the extremal canonical selfadjoint contraction

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(c.s.c.) extensions of T. Indeed, since $-I \leqq G \leqq I$ the minimal (maximal) c.s.c.-extension T_μ (T_M resp.) is obtained from (1) by taking $G=-I$ ($G=I$ resp.). In particular, $T_M - T_\mu = 2D_\Gamma^2 P_{\mathfrak{D}(T)^\perp}$ (by $P_{\mathfrak{K}}$ we denote the orthogonal projector of $\mathfrak{H}$ onto a subspace $\mathfrak{K}$ of $\mathfrak{H}$). The completely undetermined case (i.e. $(T_M - T_\mu)x \neq 0$ for all $x \in \mathfrak{D}(T)^\perp \setminus \{0\}$, see [5]) thus holds if and only if D_Γ is a bijection $\mathfrak{D}(T)^\perp \to \mathscr{D}_\Gamma$.

Let $\tilde{\mathfrak{H}} \supset \mathfrak{H}$ be a Hilbert space, $\tilde{T} \in [\tilde{\mathfrak{H}}]$ an arbitrary selfadjoint contraction (s.c.) extension of T, and $\tilde{P}$ the orthogonal projector of $\tilde{\mathfrak{H}}$ onto $\mathfrak{H}$. Denote by $\Omega(-1, 1)$ the set $\operatorname{Ext}((-\infty, -1] \cup [1, \infty))$ in the extended complex plane. The operator function

$$z \to R_z(\tilde{T}) := \tilde{P}(z\tilde{T} - I)^{-1}|_{\mathfrak{H}}, \quad z \in \Omega(-1, 1),$$

with values in $\mathfrak{H}$ is called a generalized s.c.-resolvent of T. If $\tilde{T}$ is canonical ($\tilde{\mathfrak{H}} = \mathfrak{H}$) then the generalized s.c.-resolvent is called a canonical s.c.-resolvent (c.s.c.-resolvent) of T. If no ambiguity arises we use the notation R_z for $R_z(\tilde{T})$ in the sequel.

The c.s.c.-extension T_0 of T which corresponds to $G=0$ in (1) will play a special role. We set $\mathring{R}_z := (zT_0 - I)^{-1}$. The operator function

$$z \to X_0(z) := -zD_\Gamma P_{\mathfrak{D}(T)^\perp}(zT_0 - I)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z \in \Omega(-1, 1),$$

with values in $[\mathscr{D}_\Gamma]$ was introduced in [9] and shown to be contractive for $|z| < 1$. We will show in Proposition 3 that there is a close connection between X_0 and the two Q-functions Q_μ and Q_M of T introduced in [5] by the relations

$$\begin{aligned} Q_\mu(z) &:= (C^{1/2}(T_\mu - zI)^{-1} C^{1/2} + I)|_{\mathfrak{D}(T)^\perp}, \\ Q_M(z) &:= (C^{1/2}(T_M - zI)^{-1} C^{1/2} - I)|_{\mathfrak{D}(T)^\perp}, \quad z \in \operatorname{Ext}[-1, 1], \end{aligned} \tag{2}$$

where $C := T_M - T_\mu (= 2D_\Gamma^2 P_{\mathfrak{D}(T)^\perp})$.

3. Let $\mathfrak{G}$ be a Hilbert space. Denote by $\mathscr{N}(\mathfrak{G})$ the set of all functions G holomorphic in $\Omega(-1, 1)$ with values in $[\mathfrak{G}]$ such that

1) $-I \leqq G(x) \leqq I$ if $-1 < x < 1$,

2) the kernel $K(s, t) := \begin{cases} (s-t)^{-1}(G(s) - G(t)), & s \neq t \\ G'(t), & s = t \end{cases} \quad (-1 < s, t < 1)$

is positive definite.

Denote by $\mathscr{N}_0(\mathfrak{G})$ the subset of $\mathscr{N}(\mathfrak{G})$ consisting of those elements of $\mathscr{N}(\mathfrak{G})$ which are independent of z.

Proposition 1. *Assume G is holomorphic in $\Omega(-1, 1)$ with values in $[\mathfrak{G}]$. Then $G \in \mathscr{N}(\mathfrak{G})$ if and only if*

1') $\|G(z)\| \leqq 1$ *if* $|z| < 1$,

2') *the kernel* $K(z, \zeta) := \begin{cases} (z-\bar{\zeta})^{-1}(G(z) - G(\zeta)^*), & z \neq \bar{\zeta} \\ G'(z), & z = \bar{\zeta} \end{cases} \quad (z, \zeta \in \Omega(-1, 1))$

is positive definite.

Proof. We must only prove that 1′) and 2′) follow from 1) and 2). Define $F(s):=s(I-sG(s))^{-1}$, $-1<s<1$. Since $F(0)=0$, $F'(0)=I$ and since for $-1<s<1$, $-1<t<1$, $s\neq t$

$$(s-t)^{-1}(F(s)-F(t))=(I-sG(s))^{-1}(I+st(s-t)^{-1}(G(s)-G(t))(I-tG(t))^{-1},$$

a well known result of B. Sz.-Nagy and A. Korányi [10, Satz C*] yields the existence of a selfadjoint contraction $\tilde{G}$ in some Hilbert space $\tilde{\mathfrak{G}}\supset\mathfrak{G}$ such that $F(s)=sP_{\mathfrak{G}}(I-s\tilde{G})^{-1}|_{\mathfrak{G}}$, $-1<s<1$. Hence, by analytic continuation

$$R_z(\tilde{G}):=P_{\mathfrak{G}}(z\tilde{G}-I)^{-1}|_{\mathfrak{G}}=(zG(z)-I)^{-1},\quad z\in\Omega(-1,1).$$

It is now straightforward to see (cf. [8], [9]) that $R_z(\tilde{G})^{-1}\in[\mathfrak{G}]$ for $z\in\Omega(-1,1)$ and that $G(z)=z^{-1}(R_z(\tilde{G})^{-1}+I)$ (the right-hand side being extended by continuity to $z=0$) is a contraction for $|z|<1$. Thus 1′) holds. Property 2′) follows from the relation

$$(z-\bar{z})^{-1}(G(z)-G(z)^*)=$$

$$=|z|^{-2}R_{\bar{z}}(\tilde{G})^{-1}P_{\mathfrak{G}}(\bar{z}\tilde{G}-I)^{-1}(I-P_{\mathfrak{G}})(z\tilde{G}-I)^{-1}P_{\mathfrak{G}}R_z(\tilde{G})^{-1}\geqq 0,\quad \operatorname{Im} z\neq 0.$$

Remark. In [5] the class $\mathfrak{K}_{\mathfrak{G}}[-1,1]$ of $[\mathfrak{G}]$-valued operator functions k in $\operatorname{Ext}[-1,1]$ with the following properties was introduced:

(i) k is holomorphic in $\operatorname{Ext}[-1,1]$,

(ii) $(\operatorname{Im} z)^{-1}\operatorname{Im} k(z)\leqq 0$ if $\operatorname{Im} z\neq 0$,

(iii) $0\leqq k(x)\leqq I$ if $x>1$ or $x<-1$.

Proposition 1 implies that there is a close connection between $\mathfrak{K}_{\mathfrak{G}}[-1,1]$ and $\mathcal{N}(\mathfrak{G})$. Namely, $k\in\mathfrak{K}_{\mathfrak{G}}[-1,1]$ if and only if $G\colon z\to 2k(z^{-1})-I$ belongs to $\mathcal{N}(\mathfrak{G})$.

Proposition 2. *Assume* $G\in\mathcal{N}(\mathcal{D}_T)$. *Then* $(I-X_0(z)G(z))^{-1}\in[\mathcal{D}_T]$ *for* $z\in\Omega(-1,1)$.

Proof. For $|z|<1$, $X_0(z)G(z)$ is a contraction; in particular, for $z=0$ it is the zero operator. Thus for $|z|<1$ the assertion follows from the maximum modulus theorem.

Next assume that $z\in C_+\cup C_-$, where C_+ (C_-) denotes the open upper (lower) half-plane of the complex plane. The operator $I-X_0(z)G(z)P_{\mathcal{D}_T}$ has the matrix representation

$$I-X_0(z)G(z)P_{\mathcal{D}_T}=\begin{pmatrix} I-X_0(z)G(z) & 0\\ 0 & I\end{pmatrix}$$

with respect to the decomposition $\mathfrak{H}=\mathcal{D}_T\oplus\mathcal{D}_T^{\perp}$. The assertion is hence equivalent to $1\notin\sigma(X_0(z)G(z)P_{\mathcal{D}_T})$. Recall that if B_1, B_2 are bounded operators on $\mathfrak{H}$, then $\sigma(B_1B_2)\setminus\{0\}=\sigma(B_2B_1)\setminus\{0\}$, so

$$-1\notin\sigma(zD_TP_{\mathfrak{D}(T)^{\perp}}(zT_0-I)^{-1}P_{\mathfrak{D}(T)^{\perp}}D_TG(z)P_{\mathcal{D}_T})$$

if and only if

$$-1 \notin \sigma\big(zP_{\mathfrak{D}(T)^\perp} D_\Gamma G(z) P_{\mathcal{D}_\Gamma} D_\Gamma P_{\mathfrak{D}(T)^\perp}(zT_0-I)^{-1}\big).$$

Note that

$$I+zP_{\mathfrak{D}(T)^\perp} D_\Gamma G(z) P_{\mathcal{D}_\Gamma} D_\Gamma P_{\mathfrak{D}(T)^\perp}(zT_0-I)^{-1} =$$
$$= z\big(T_0+D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp} - z^{-1}I\big)(zT_0-I)^{-1}.$$

Now the assertion follows from the fact that $T_0+D_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp}$ is maximal dissipative for $z\in C_+$ and therefore C_- is contained in its resolvent set.

Assume $\hat{G}\in\mathcal{N}_0(\mathcal{D}_\Gamma)$. According to (1), the corresponding c.s.c.-extension of T is $T_{\hat{G}}=T_0+D_\Gamma \hat{G} D_\Gamma P_{\mathfrak{D}(T)^\perp}$. Introduce the corresponding Q-function $Q_{\hat{G}}$ of T with values in $[\mathfrak{D}(T)^\perp]$ by

$$Q_{\hat{G}}(z) := D_\Gamma P_{\mathfrak{D}(T)^\perp}(T_{\hat{G}}-zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z\in \mathrm{Ext}\,[-1,1]$$

(cf. [7]). Note that $Q_{\hat{G}}$ has the matrix representation

$$Q_{\hat{G}}(z) = \begin{pmatrix} D_\Gamma P_{\mathfrak{D}(T)^\perp}(T_{\hat{G}}-zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathfrak{D}(T)^\perp=\mathcal{D}_\Gamma\oplus\ker(D_\Gamma)$, and that if $G_1, G_2\in\mathcal{N}_0(\mathcal{D}_\Gamma)$ then

$$Q_{G_2}(z) = Q_{G_1}(z)\big(I+(G_1-G_2)Q_{G_1}(z)\big)^{-1}.$$

Obviously (see (2))

$$Q_\mu(z) = \big(2Q_{-I}(z)+I\big)\big|_{\mathfrak{D}(T)^\perp}, \quad Q_M(z) = \big(2Q_I(z)-I\big)\big|_{\mathfrak{D}(T)^\perp}.$$

Proposition 3. *Assume* $\hat{G}\in\mathcal{N}_0(\mathcal{D}_\Gamma)$, $z\in\mathrm{Ext}\,[-1,1]$. *Then*

$$(I-X_0(z^{-1})\hat{G})^{-1}X_0(z^{-1}) = -Q_{\hat{G}}(z)|_{\mathcal{D}_\Gamma}.$$

In particular,

$$X_0(z^{-1}) = -Q_0(z)|_{\mathcal{D}_\Gamma},\ (I+X_0(z^{-1}))^{-1}(I-X_0(z^{-1}))=Q_\mu(z)|_{\mathcal{D}_\Gamma},$$
$$-(I-X_0(z^{-1}))^{-1}(I+X_0(z^{-1})) = Q_M(z)|_{\mathcal{D}_\Gamma}.$$

Remark. In the completely undetermined case, $\ker D_\Gamma=\{0\}$, so $Q_{\hat{G}}(z)|_{\mathcal{D}_\Gamma}= =Q_{\hat{G}}(z)$. This can always be assumed without loss of generality.

Proof. By Proposition 2, $(I-X_0(z^{-1})\hat{G})^{-1}\in[\mathcal{D}_\Gamma]$. A direct calculation gives

$$\begin{pmatrix} (I-X_0(z^{-1})\hat{G})^{-1}X_0(z^{-1}) & 0 \\ 0 & 0 \end{pmatrix} = (I-X_0(z^{-1})\hat{G}P_{\mathcal{D}_\Gamma})^{-1}X_0(z^{-1})P_{\mathcal{D}_\Gamma} =$$
$$= -D_\Gamma P_{\mathfrak{D}(T)^\perp}(T_0+D_\Gamma\hat{G}D_\Gamma P_{\mathfrak{D}(T)^\perp}-zI)^{-1}P_{\mathfrak{D}(T)^\perp}D_\Gamma P_{\mathcal{D}_\Gamma} = \begin{pmatrix} -Q_{\hat{G}}(z)|_{\mathcal{D}_\Gamma} & 0 \\ 0 & 0 \end{pmatrix},$$

where the matrix representations are taken with respect to the decomposition $\mathfrak{H}=\mathcal{D}_\Gamma\oplus\mathcal{D}_\Gamma^\perp$.

Being R-functions, $z \to -X_0(z^{-1})$ and Q_G are increasing on $(-\infty, -1) \cup (1, \infty)$. The first function is bounded below (by $-I$) and above (by I). Therefore, the strong limit as $t \nearrow -1$ $(t \searrow 1)$ exists which is a contraction. Also the strong limits of Q_G exist under additional conditions on $\hat{G}$. We next calculate the limits.

Proposition 4. a) $s-\lim\limits_{t\to\pm\infty} Q_G(t) = s-\lim\limits_{t\to\pm\infty} X(t^{-1}) = 0$;

b) $s-\lim\limits_{t\nearrow -1} X_0(t^{-1}) = -I$, $s-\lim\limits_{t\searrow 1} X_0(t^{-1}) = I$;

c) *Assume the completely undetermined case holds. Then*

$$s-\lim_{t\nearrow -1} Q_G(t) = (I+\hat{G})^{-1} \quad \text{if} \quad -1 \notin \sigma(\hat{G}),$$

$$s-\lim_{t\searrow 1} Q_G(t) = -(I-\hat{G})^{-1} \quad \text{if} \quad 1 \notin \sigma(\hat{G}).$$

Proof. Assertion a) is obvious. Assertion b) can be derived from Proposition 3 and [5, Theorem 2.1], or by the argument below. In order to prove c), assume e.g. $-1 \notin \sigma(\hat{G})$. It is easy to see that

$$\begin{aligned} Q_G(t) &= -D_\Gamma\big(\Gamma^*(A+(I-A)(I+(t+1)(A-tI)^{-1})\Gamma + tI - D_\Gamma \hat{G} D_\Gamma\big)^{-1} D_\Gamma = \\ &= \big(I+\hat{G}-(t+1)D_\Gamma^{-1}\Gamma^*(A-tI)^{-1}\Gamma D_\Gamma^{-1} - (t+1)D_\Gamma^{-2}\big)^{-1}, \quad t < -1. \end{aligned} \tag{3}$$

Denote by E the resolution of the identity of the selfadjoint contraction $A \in [\mathfrak{D}(T)]$. Put $t := -1-\varepsilon$, $\varepsilon > 0$, and $f(\tau;\varepsilon) := \varepsilon^2(\tau+1+\varepsilon)^{-2}$, $-1 \leqq \tau \leqq 1$. Then for each $h \in \mathfrak{D}(T)^\perp$,

$$\|(t+1)(A-tI)^{-1}\Gamma h\|^2 = \int_{-1}^{1} f(\tau;\,\varepsilon)(dE_\tau \Gamma h, \Gamma h);$$

so $\lim\limits_{t\nearrow -1} \|(t+1)(A-tI)^{-1}\Gamma h\|^2 = \|E_{-1}\Gamma h\|^2$. But $E_{-1}\Gamma h = 0$, since E_{-1} is the orthogonal projector of $\mathfrak{D}(T)$ onto $\ker(A+I) = \ker(A+I)^{1/2} \subset \ker(I-A^2)^{1/2} = \mathfrak{D}(T) \ominus \mathscr{D}_A$ and $\Gamma\mathfrak{D}(T)^\perp \subset \mathscr{D}_A$. Thus, (3) implies that $(Q_G(t))^{-1}$ decreases to its strong limit $I+\hat{G} \gg 0$ as $t \nearrow -1$. The assertion follows.

4. We first prove the following theorem on the characterization of the generalized s.c.-resolvents of T.

Theorem 1. *Let T be a symmetric nondensely defined contraction in the Hilbert space $\mathfrak{H}$. Then the formula*

$$R_z = \mathring{R}_z - z\mathring{R}_z D_\Gamma G(z)\big(I - X_0(z)G(z)\big)^{-1} D_\Gamma P_{\mathfrak{D}(T)^\perp} \mathring{R}_z, \quad z \in \Omega(-1,1) \tag{4}$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G \in \mathscr{N}(\mathscr{D}_\Gamma)$. The generalized s.c.-resolvent is canonical if and only if $G \in \mathscr{N}_0(\mathscr{D}_\Gamma)$.

Proof. Let $\tilde{T}$ be a s.c.-extension of T in $\tilde{\mathfrak{H}} \supset \mathfrak{H}$. Denote its resolvent by R_z, $z \in \Omega(-1, 1)$. Then [9, Theorem 1] implies the existence of a uniquely determined holomorphic contraction valued operator function $G\colon z\to G(z)\in[\mathscr{D}_\Gamma]$, $|z|<1$, such that

(5) $$R_z = \mathring{R}_z\big(I+zD_\Gamma G(z)D_\Gamma P_{\mathfrak{D}(T)^\perp}\mathring{R}_z\big)^{-1}, \quad |z|<1.$$

If, in particular, the extension $\tilde{T}$ is canonical, then G is a constant contraction.

It is straightforward to see (cf. [8], [9] and the proof of Proposition 1) that G has an analytic continuation to $\Omega(-1, 1)$, which we also denote by G, and that $G\in\mathscr{N}(\mathscr{D}_\Gamma)$. Then (cf. the proof of Proposition 2)

$$\big(I+zD_\Gamma G(z)D_\Gamma P_{\mathfrak{D}(T)^\perp}\mathring{R}_z\big)^{-1}\in[\mathfrak{H}], \quad z\in\Omega(-1, 1),$$

so by analytic continuation the relation (5) holds in the same region. But (4) and (5) are equivalent ([9]).

Assume, conversely, $G\in\mathscr{N}(\mathscr{D}_\Gamma)$ and put for $z\in\Omega(-1, 1)$

$$S(z) := \big(zT_0-I+zD_\Gamma G(z)D_\Gamma P_{\mathfrak{D}(T)^\perp}\big)^{-1}, \quad F(z) := -zS(z).$$

Let the kernel K be given by

$$K(z, \zeta) := \begin{cases}(z-\bar{\zeta})^{-1}\big(F(z)-F(\zeta)^*\big), & z\neq\bar{\zeta}\\ F'(z), & z=\bar{\zeta}.\end{cases}$$

By Proposition 1 and the formula

(6) $$K(z, \zeta) = S(\zeta)^*\big(I+z\bar{\zeta}(z-\bar{\zeta})^{-1}D_\Gamma(G(z)-G(\zeta)^*)D_\Gamma P_{\mathfrak{D}(T)^\perp}\big)S(z), \quad z\neq\bar{\zeta},$$

this kernel is positive definite. In order to construct a s.c.-extension of T we apply a standard technique. Consider the linear set $\tilde{\mathfrak{L}}$ of all finite formal sums $\tilde{f}=\sum\varepsilon_z f_z$ $\big(f_z\in\mathfrak{H},\ z\in\Omega(-1, 1)\big)$ and define in $\tilde{\mathfrak{L}}$ an inner product $(\cdot,\cdot)$ by

$$\big(\sum\varepsilon_z f_z, \sum\varepsilon_\zeta g_\zeta\big) = \sum\big(K(z, \zeta)f_z, g_\zeta\big),$$

which is nonnegative by (6). $\tilde{\mathfrak{L}}$ can be canonically embedded into a Hilbert space $\tilde{\mathfrak{H}}$ and we identify $\tilde{\mathfrak{L}}$ and its image in $\tilde{\mathfrak{H}}$. Since $(\varepsilon_0 f, \varepsilon_0 f)=(K(0, 0)f, f)=(f, f)$ we can also identify $\mathfrak{H}$ with a subset of $\tilde{\mathfrak{H}}$ by the correspondence $f\to\varepsilon_0 f$. Next we define an operator $\tilde{T}$ on $\tilde{\mathfrak{L}}_0:=\{\tilde{f}=\sum\varepsilon_z f_z\in\tilde{\mathfrak{L}}\colon f_0=0\}$ by

$$\tilde{T}(\varepsilon_z f) := z^{-1}(\varepsilon_z f-\varepsilon_0 f), \quad z\neq 0.$$

It follows from the relation

$$(\varepsilon_z f-\varepsilon_0 f, \varepsilon_z f-\varepsilon_0 f) = \big((|z|^2(z-\bar{z})^{-1}S(z)^*D_\Gamma(G(z)-G(z)^*)D_\Gamma P_{\mathfrak{D}(T)^\perp}S(z)+$$
$$+\big(S(z)^*+I\big)\big(S(z)+I\big)\big)f, f\big)\to 0, \quad z\to 0$$

that the domain $\tilde{\mathfrak{L}}_0$ of $\tilde{T}$ is dense in $\tilde{\mathfrak{H}}$. In order to see that $\tilde{T}$ is symmetric with defect numbers $(0, 0)$, let $z, \zeta\in\Omega(-1, 1)$, $z\zeta\neq 0$. Then for $f, g\in\mathfrak{H}$

$$(\tilde{T}\varepsilon_z f, \varepsilon_\zeta g)-(\varepsilon_z f, \tilde{T}\varepsilon_\zeta g) = \big(((z\bar{\zeta})^{-1}(zS(z)-\bar{\zeta}S(\zeta)^*)-\bar{\zeta}^{-1}S(z)+z^{-1}S(\zeta)^*)f, g\big) = 0,$$

so $\tilde{T}$ is symmetric. The denseness of $\mathfrak{R}(z\tilde{T}-I)$ follows from the relation

$$(z\tilde{T}-I)(\zeta\varepsilon_\zeta g-z\varepsilon_z g)=(z-\zeta)\varepsilon_\zeta g.$$

The closure of $\tilde{T}$, which we again denote by $\tilde{T}$, is thus selfadjoint in $\tilde{\mathfrak{H}}$. Observe that $(z\tilde{T}-I)^{-1}\varepsilon_0 f=-\varepsilon_z f$, $z\in\Omega(-1,1)$, and that for $|z|>1$

$$\|(\tilde{T}-zI)^{-1}\varepsilon_0 f\|^2=|z|^{-2}(F'(z^{-1})f,f).$$

A minor modification of the proof of [10, Satz C] then leads to the conclusion $\|\tilde{T}\|\leqq 1$. Furthermore,

$$((z\tilde{T}-I)^{-1}\varepsilon_0 f,\varepsilon_0 g)=-(\varepsilon_z f,\varepsilon_0 g)=-(K(z,0)f,g)=(S(z)f,g),$$

hence $\tilde{P}(z\tilde{T}-I)^{-1}|_{\mathfrak{H}}=S(z)$.

It remains to show that $\tilde{T}$ is an extension of T — it is then minimal since

$$\text{c.l.s. }\{(z\tilde{T}-I)^{-1}\mathfrak{H}:\ z\in\Omega(-1,1)\}=\text{c.l.s. }\{\varepsilon_z f:\ z\in\Omega(-1,1),\ f\in\mathfrak{H}\}=\tilde{\mathfrak{H}}$$

— and that $\tilde{T}$ is canonical if G is constant.

In order to show the first assertion, assume $f\in\mathfrak{D}(T)$, $z\neq 0$. Then (note the relation $z^{-1}(S(z)+I)=S(z)(T_0+D_\Gamma G(z)D_\Gamma P_{\mathfrak{D}(T)^\perp})$)

$$(\tilde{T}\varepsilon_z f-\varepsilon_0 Tf,\varepsilon_\zeta g)=(z^{-1}(K(z,\zeta)-K(0,\zeta))f-K(0,\zeta)Tf,g)=$$
$$=((z^{-1}(S(z)+I)+\bar{\zeta}(z-\bar{\zeta})^{-1}D_\Gamma(G(z)-G(\zeta)^*)D_\Gamma P_{\mathfrak{D}(T)^\perp}S(z)+T)f,S(\zeta)g)=$$
$$=((S(z)+I)Tf,S(\zeta)g)+((z-\bar{\zeta})^{-1}D_\Gamma(G(z)-G(\zeta)^*)D_\Gamma P_{\mathfrak{D}(T)^\perp}S(z)f,\zeta S(\zeta)g)\to 0,$$
$$z\to 0,$$

that is, for each $\tilde{g}\in\tilde{\mathfrak{H}}$, $(\tilde{T}\varepsilon_z f,\tilde{g})\to(Tf,\tilde{g})$, $z\to 0$. But then

$$(\tilde{T}f-Tf,\tilde{g})=(\tilde{T}f-\tilde{T}\varepsilon_z f,\tilde{g})+(\tilde{T}\varepsilon_z f-Tf,\tilde{g})\to 0,\quad z\to 0,$$

for each $\tilde{g}\in\tilde{\mathfrak{H}}$. Thus $\tilde{T}f=Tf$.

The second assertion follows easily. Indeed, if $G\in\mathcal{N}_0(\mathcal{D}_\Gamma)$, then $K(z,\zeta)=$ $=S(\zeta)^*S(z)$ and

$$(\tilde{f},\tilde{f})=\sum(K(z,\zeta)f_z,f_\zeta)=(\sum S(z)f_z,\sum S(\zeta)f_\zeta).$$

But an element $\tilde{f}=\sum\varepsilon_z f_z$ belongs to $\tilde{\mathfrak{H}}\ominus\mathfrak{H}$ if and only if $\sum S(z)f_z=0$. This concludes the proof of the theorem.

The c.s.c.-extension T_0 and its resolvent play a special role in Theorem 1. However, it can be replaced by any canonical or even noncanonical s.c.-extension of T. In order to see this, fix a s.c.-extension $\hat{T}$ in $\hat{\mathfrak{H}}$ corresponding to the function $\hat{G}\in\mathcal{N}(\mathcal{D}_\Gamma)$ according to Theorem 1 and denote its generalized resolvent by $\hat{R}_z$:

$$\hat{R}_z:=P_{\mathfrak{H}}(z\hat{T}-I)^{-1}|_{\mathfrak{H}},\quad z\in\Omega(-1,1).$$

Introduce a function $\hat{Q}$ with values in $[\mathfrak{D}(T)^{\perp}]$ by

$$\hat{Q}(z) := D_\Gamma P_{\mathfrak{D}(T)^\perp} P_{\mathfrak{H}}(\hat{T}-zI)^{-1} P_{\mathfrak{D}(T)^\perp} D_\Gamma, \quad z\in \mathrm{Ext}\,[-1, 1].$$

Note that $\hat{Q}=Q_{\hat{G}}$ if $\hat{G}\in\mathcal{N}_0(\mathcal{D}_\Gamma)$.

Theorem 2. *Let T be a symmetric nondensely defined contraction in the Hilbert space $\mathfrak{H}$. Then the formula*

$$(7)\quad R_z = \hat{R}_z - z\hat{R}_z D_\Gamma(G(z)-\hat{G}(z))\big(I+(\hat{Q}(z^{-1})|_{\mathcal{D}_\Gamma})(G(z)-\hat{G}(z))\big)^{-1} D_\Gamma P_{\mathfrak{D}(T)^\perp}\hat{R}_z, \quad z\in\Omega(-1, 1)$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G\in\mathcal{N}(\mathcal{D}_\Gamma)$. The generalized s.c.-resolvent is canonical if and only if $G\in\mathcal{N}_0(\mathcal{D}_\Gamma)$.

Proof. Since (4) is equivalent to the relation

$$R_z = \mathring{R}_z(I+zD_\Gamma G(z) D_\Gamma P_{\mathfrak{D}(T)^\perp}\mathring{R}_z)^{-1},$$

from which we obtain by specialization

$$(8)\qquad \hat{R}_z = \mathring{R}_z(I+zD_\Gamma\hat{G}(z)D_\Gamma P_{\mathfrak{D}(T)^\perp}\mathring{R}_z)^{-1},$$

it is straightforward to prove (cf. [9, Theorem 2]) that the formula

$$(9)\quad R_z = \hat{R}_z - z\hat{R}_z D_\Gamma(G(z)-\hat{G}(z))(I-X_0(z)G(z))^{-1}(I-X_0(z)\hat{G}(z))D_\Gamma P_{\mathfrak{D}(T)^\perp}\hat{R}_z, \quad z\in\Omega(-1, 1);$$

yields a bijective correspondence between all generalized s.c.-resolvents of T and all functions $G\in\mathcal{N}(\mathcal{D}_\Gamma)$, and that the generalized s.c.-resolvent is canonical if and only if $G\in\mathcal{N}_0(\mathcal{D}_\Gamma)$. It remains to show that relation (9), which still contains the special extension T_0, appearing in the function X_0, can be rewritten in the form (7).

Indeed, (8) and the definition of the function $\hat{Q}$ imply

$$\hat{Q}(z^{-1}) = D_\Gamma P_{\mathfrak{D}(T)^\perp}(T_0+D_\Gamma\hat{G}(z)D_\Gamma P_{\mathfrak{D}(T)^\perp} - z^{-1}I)^{-1}P_{\mathfrak{D}(T)^\perp} D_\Gamma P_{\mathfrak{D}_\Gamma}, \quad z\in\Omega(-1, 1),$$

and, according to Proposition 2, $(I-X_0(z)\hat{G}(z))^{-1}\in[\mathcal{D}_\Gamma]$, $z\in\Omega(-1, 1)$. Hence, the proof of Proposition 3 carries over to yield the relation

$$(I-X_0(z)\hat{G}(z))^{-1}X_0(z) = -\hat{Q}(z^{-1})|_{\mathfrak{D}_\Gamma}, \quad z\in\Omega(-1, 1),$$

and the statement follows from the identity

$$(I-X_0(z)G(z))^{-1}(I-X_0(z)\hat{G}(z)) =$$
$$= \big(I-(I-X_0(z)\hat{G}(z))^{-1}X_0(z)(G(z)-\hat{G}(z))\big)^{-1}, \quad z\in\Omega(-1, 1).$$

Remark 1. It is easy to see that (9) implies the relation

$$\tilde{P}\tilde{T}f = \hat{P}\hat{T}f + D_\Gamma(G(0)-\hat{G}(0))D_\Gamma P_{\mathfrak{D}(T)^\perp}f, \quad f\in\mathfrak{H}.$$

In particular,

$$\tilde{P}\tilde{T}f = T_\mu f + D_\Gamma(G(0)+I)D_\Gamma P_{\mathfrak{D}(T)^\perp}f, \quad f\in\mathfrak{H}$$

(cf. [5, (5.7)]).

Remark 2. Recently E. R. Cekanovskiĭ [2] stated among other things the following theorem: The symmetric nondensely defined contraction T has a canonical non selfadjoint contraction extension $\tilde{T}$ such that $\tilde{T}^*\supset T$ (that is, $\tilde{T}$ is a canonical non selfadjoint extension of the dual pair $\{T, T\}$ of contractions) if and only if T has more than one c.s.c.-extension. This is a consequence of (1) and Theorem 2. Indeed, T has a unique c.s.c.-extension if and only if $D_\Gamma=0$, which is true if and only if $\{T, T\}$ has a unique canonical contraction extension. Moreover, [9, Theorem 1] and Theorem 2 imply that if T has a unique c.s.c.-extension T', then the only generalized resolvent of the dual pair $\{T, T\}$ is the s.c.-resolvent $(zT'-I)^{-1}$. These statements carry over via the Cayley transformation to dual pairs of dissipative linear relations.

Now it is easy to recover the results of M. G. Kreĭn and I. E. Ovčarenko from Theorem 2. To this end, assume that $\hat{G}\in\mathcal{N}_0(\mathcal{D}_\Gamma)$ and that the completely undetermined case holds. If $z\in\operatorname{Ext}[-1, 1]$, the representation formula (7) can be written in the form

$$\tilde{P}(\tilde{T}-zI)^{-1}|_{\mathfrak{H}} = (\hat{T}-zI)^{-1} -$$

(10)

$$-(\hat{T}-zI)^{-1}D_\Gamma(G(z^{-1})-\hat{G})(I+Q_G(z)(G(z^{-1})-\hat{G}))^{-1}D_\Gamma P_{\mathfrak{D}(T)^\perp}(\hat{T}-zI)^{-1}.$$

Assume e.g. $\hat{G}=-I$, and put $k(z):=(1/2)(G(z^{-1})+I)$, $z\in\operatorname{Ext}[-1, 1]$. Then k belongs to $\mathfrak{R}_{\mathcal{D}_\Gamma}[-1, 1]$ according to the remark after Proposition 1. We thus obtain the formula [5, (5.1)]

$$\tilde{P}(\tilde{T}-zI)^{-1}|_{\mathfrak{H}} =$$

$$= (T_\mu-zI)^{-1}-(T_\mu-zI)^{-1}C^{1/2}k(z)(I+(Q_\mu(z)-I)k(z))^{-1}C^{1/2}(T_\mu-zI)^{-1}$$

by specialization from (7).

Formula (10) can further be rewritten to yield a representation similar to the one in [7]. For each $z\in\Omega(-1, 1)$ we define a linear relation $T(z)$ by $T(z):=(G(z^{-1})-\hat{G})^{-1}$. It has the following properties:

1) If $z\in C_+$, then $T(z)$ is a maximal dissipative closed linear relation in $\mathfrak{H}$.

2) $T(z)$ is a holomorphic function (in the sense of [7]) such that $T(\bar{z})=T(z)^*$. Thus [7, Proposition 1.2] implies the existence of a decomposition $\mathcal{D}_\Gamma=(\mathcal{D}_\Gamma)_0\oplus(\mathcal{D}_\Gamma)_\infty$ independent of z such that $(\mathcal{D}_\Gamma)_0$ and $(\mathcal{D}_\Gamma)_\infty$ are reducing subspaces of $T(z)$ for all $z\in\Omega(-1, 1)$, the operator part $T(z)_0$ of $T(z)$ is a maximal dissipative operator in $(\mathcal{D}_\Gamma)_0$ if $\operatorname{Im} z>0$, and the infinite part $T(z)_\infty$ of $T(z)$ is $T(z)_\infty:=\{\{0, g\}: g\in(\mathcal{D}_\Gamma)_\infty\}$.

Put $\Gamma_z := (\hat{T}-zI)^{-1}D_{\Gamma}$. Then (10) can be written in the form

$$\tilde{P}(\tilde{T}-zI)^{-1}|_{\mathfrak{H}} = (\hat{T}-zI)^{-1} - \Gamma_z T(z)^{-1}(I+Q_G(z)T(z)^{-1})^{-1}\Gamma_{\bar{z}}^* =$$
$$= (\hat{T}-zI)^{-1} - \Gamma_z(Q_G(z)+T(z))^{-1}\Gamma_{\bar{z}}^*$$

(see [7, (1.8)]), where for $x>1$ or $x<-1$

$$-(I+\hat{G}) \leqq T(x)^{-1} = G(x^{-1}) - \hat{G} \leqq I - \hat{G}.$$

In particular, $\hat{G} = -I$ gives $0 \leqq T(x)^{-1} \leqq 2I$ (cf. [7, Theorem 4.3] and [4]).

The results can be applied in a straightforward way to solve the extension problem for a nonnegative closed linear relation S in $\mathfrak{H}$ (cf. [6]) by using the transformation $T := -I + 2(S+I)^{-1}$. We leave the details to the reader.

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Compact weighted composition operators on $L^2(\lambda)$

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1. Introduction. Let $(X, \mathcal{S}, \lambda)$ be a sigma-finite measure space and let $T\colon X \to X$ be a non-singular measurable transformation such that the composition transformation C_T defined as $C_T f = f \circ T$ is bounded linear operator on $L^2(\lambda)$. If $\theta \in L^\infty(\lambda)$, then the multiplication operator M_θ defined as $M_\theta f = \theta \cdot f$ is a bounded linear operator on $L^2(\lambda)$. The product $C_T M_\theta$ is an operator on $L^2(\lambda)$ and we call it a weighted composition operator on $L^2(\lambda)$. This class of operators includes some of the well known operators such as multiplication operators, weighted shifts and composition operators [1].

In this note we are interested in studying compact weighted composition operators on $L^2(\lambda)$.

By $B(H)$ we denote the C^*-algebra of all bounded linear operators on a Hilbert space H. If C_T is a composition operator on $L^2(\lambda)$, then f_0 denotes the Radon—Nikodym derivative of the measure λT^{-1} with respect to the measure λ. For any complex valued function on X, $Z_\theta = \{x\colon \theta(x) = 0\}$ and Z'_θ is the complement of Z_θ.

2. Some general results. It has been proved in [4] that if $C_T \in B(L^2(\lambda))$, then $C_T^* C_T = M_{f_0}$, where f_0 is the Radon—Nikodym derivative of the measure λT^{-1} with respect to the measure λ. Also it has been proved in [6] that $f_0 \circ T \neq 0$ (a.e.). By using these results we prove the following theorem.

Theorem 2.1. *Let $C_T \in B(L^2(\lambda))$. Then C_T has dense range if and only if $C_T C_T^* = M_{f_0 \circ T}$.*

Proof. Suppose C_T has dense range and let $f \in L^2(\lambda)$. Then there exists a sequence $\{f_n\}$ in $L^2(\lambda)$ such that $\{C_T f_n\}$ converges to f. Now

$$C_T C_T^* f = \lim_n (C_T C_T^* C_T f_n) = \lim_n (C_T M_{f_0} f_n) = M_{f_0 \circ T}(\lim_n C_T f_n) = M_{f_0 \circ T} f.$$

Hence $C_T C_T^* = M_{f_0 \circ T}$.

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Conversely, suppose $C_T C_T^* = M_{f_0 \circ T}$. Since $f_0 \circ T \neq 0$ (a.e.), $M_{f_0 \circ T}$ and hence $C_T C_T^*$ is an injection. This implies that $\ker C_T^* = \ker C_T C_T^* = \{0\}$ and hence C_T has dense range.

Corollary 2.2. *Let $C_T \in B(l^2)$. Then $C_T C_T^* = M_{f_0 \circ T}$ if and only if C_T is onto.*

Proof. Since the range of C_T is always closed in l^2, the result follows immediately.

Theorem 2.3. *Let $C_T M_\theta \in B(L^2(\lambda))$. Then $(C_T M_\theta)^* C_T M_\theta = M_{|\theta|^2 f_0}$.*

Proof. If $C_T M_\theta \in B(L^2(\lambda))$, then

$$(C_T M_\theta)^* C_T M_\theta = M_{\bar{\theta}} C_T^* C_T M_\theta = M_{|\theta|^2 f_0}.$$

Corollary 2.4. *Let $\theta \in L^\infty(\lambda)$ be such that $Z_\theta \subset (\operatorname{ran} T)'$, the complement of the range of T. Then $(C_T M_\theta)(C_T M_\theta)^* = M_{(|\theta|^2 f_0) \circ T}$ if and only if $C_T M_\theta$ has dense range.*

Proof. Suppose $(C_T M_\theta)(C_T M_\theta)^* = M_{(|\theta|^2 f_0) \circ T}$. Since $Z_\theta \subset (\operatorname{ran} T)'$, $\theta \circ T \neq 0$ (a.e.) and hence $(|\theta|^2 f_0) \circ T \neq 0$ (a.e.). This implies that $\ker (C_T M_\theta)^* = \{0\}$ and hence $C_T M_\theta$ has dense range.

The converse of this theorem follows from Theorem 2.1.

Theorem 2.5. *Let $C_T, M_\theta \in B(L^2(\lambda))$. Then*

(i) $C_T M_\theta = M_{\theta \circ T} C_T$,

(ii) $M_\theta C_T = 0$ *if and only if* $\theta = 0$ *(a.e.)*,

(iii) $C_T M_\theta = 0$ *if and only if* $\theta \circ T = 0$ *(a.e.), and*

(iv) $C_T M_\theta = M_\theta C_T$ *if and only if* $\theta = \theta \circ T$ *(a.e.)*.

Proof. (i) Let $f \in L^2(\lambda)$. Then $C_T M_\theta f = (\theta \circ T)(f \circ T) = M_{\theta \circ T} C_T f$. Hence $C_T M_\theta = M_{\theta \circ T} C_T$.

(ii) Suppose $M_\theta C_T = 0$. Then $M_\theta C_T f = 0$ for every f in $L^2(\lambda)$. Since $(X, \mathcal{S}, \lambda)$ is a sigma-finite measure space, there exists an $f \in L^2(\lambda)$ such that $f \neq 0$ (a.e.). Hence $\theta \cdot (f \circ T) = 0$ (a.e.) implies that $\theta = 0$ (a.e.).

The converse is obvious.

(iii) Since $C_T M_\theta = M_{\theta \circ T} C_T$, the proof follows from (ii).

(iv) The sufficiency of this result is obvious. To prove the necessary part, suppose $C_T M_\theta = M_\theta C_T$. Then $M_{\theta \circ T} C_T = M_\theta C_T$ and hence $M_{\theta \circ T - \theta} C_T = 0$. Thus the result follows from (ii).

The following examples illustrate that there are C_T and M_θ in $B(l^2)$, such that C_T commutes with M_θ. Here l^2 denotes the Hilbert space of square summable sequences of complex numbers.

Example 2.6. Let $X=N$, the set of natural numbers and λ be the counting measure on it. Define $T\colon X\to X$ by $T(n)=1$, if $n=1,2$ and $T(3n+m)=n+2$, if $m=0,1,2$ and $n\in N$. Then $C_T\in B(l^2)$. Define $\theta\colon X\to\mathbf{C}$ by $\theta(n)=2$, if $n=1,2$ and $\theta(n)=3$, if $n>2$. Then $M_\theta\in B(l^2)$ and C_T commutes with M_θ.

Example 2.7. If $\theta\in L^\infty(\lambda)$ is a constant, then M_θ commutes with every $C_T\in B(L^2(\lambda))$.

Example 2.8. Let $C_T\in B(L^2(\lambda))$ be such that $T(E)=E$ for some $E\in\mathscr{S}$ and $0<\lambda(E)<\infty$. Define $\theta=X_E$, the characteristic function of E. Then C_T commutes with M_θ.

3. Compact weighted composition operators. Let $(X,\mathscr{S},\lambda)$ be a sigma-finite measure space. An element $E\in\mathscr{S}$ is said to be an atom if for every non-null measurable subset F of E, either $\lambda(F)=0$ or $\lambda(F)=\lambda(E)$. A measure space $(X,\mathscr{S},\lambda)$ is said to be atomic if every element of $\mathscr{S}$ contains an atom. A measure space $(X,\mathscr{S},\lambda)$ is said to be non-atomic if it does not contain any atom. It has been proved in [4], that no composition operator on L^2 of a non-atomic measure space is compact. It is interesting to note that the weighted composition operator on $L^2(\lambda)$ is compact if and only if it is the zero operator. This is evident from the following theorem.

Theorem 3.1. *The weighted composition operator C_TM_θ on L^2 of a non-atomic measure space is compact if and only if $\theta=0$ (a.e.) on Z'_{f_0}.*

Proof. Suppose C_TM_θ is compact. Then $C_T^*C_TM_\theta$ and hence $M_{\theta f_0}$ is compact. By a theorem of [5], $\theta f_0=0$ (a.e.). If $\theta\neq 0$ (a.e.) on Z'_{f_0}, then $f_0=0$ (a.e.). This implies that $C_T=0$. But no non-singular measurable transformation induces the zero operator. Hence $\theta=0$ (a.e.) on Z'_{f_0}. The converse is obvious.

Corollary 3.2. *The weighted composition operator C_TM_θ on $L^2(\lambda)$ is compact if and only if it is the zero operator.*

Proof. Suppose C_TM_θ is compact. Then $(C_TM_\theta)^*C_TM_\theta$ and hence $M_{|\theta|^2f_0}$ is the zero operator. Hence C_TM_θ is the zero operator.

Corollary 3.3. *No composition operator on $L^2(\lambda)$ is compact.*

Let $\theta\in L^\infty(\lambda)$. We denote

$$X_\theta^\delta=\{x\in X\colon \theta(x)>\delta\}\quad\text{and}\quad M_\delta^\theta=\{f\in L^2(\lambda)\colon f(x)=0 \text{ on } X-X_\delta^\theta\}.$$

It has been proved in [5] that the multiplication operator M_θ on $L^2(\lambda)$ is compact if and only if M_δ^θ is finite dimensional. We shall characterize compact weighted composition operators on L^2 of an atomic measure space. Since $(X,\mathscr{S},\lambda)$ is a sigma finite measure space, without loss of generality we write X as a countable union of atoms and we denote the ith atom by i.

Theorem 3.4. *Let $C_T M_\theta \in B(L^2(\lambda))$. Then $C_T M_\theta$ is compact if and only if either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero.*

Proof. Suppose $C_T M_\theta$ is compact. Then $M_{\theta f_0}$ is compact and hence $M_\delta^{\theta f_0}$ is finite dimensional. This shows that $X_\delta^{\theta f_0}$ contains finite number of atoms. It follows from this that the sequence $\{\theta f_0(i)\}$ converges to zero. Since θ and f_0 are essentially bounded functions, either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero. This completes the necessary part of the theorem.

Conversely, suppose either $\{f_0(i)\}$ or $\{\theta(i)\}$ converges to zero. Then either C_T or M_θ is compact. Hence $C_T M_\theta$ is compact.

It follows from this theorem that there are plenty of compact weighted composition operators on L^2 of an atomic measure space as is shown in the following example.

Example 3.5. Let $X=N$ and $\lambda(n)=a^n, 0<a<1$. Then l_a^2 denotes the weighted sequence space. Define $T: X\to X$ by $T(n)=n+1$, if n is odd and $T(n)=n-1$, if n is even. Then $C_T \in B(l_a^2)$ and $f_0(n)=a^{n-1}(1+a)$. Hence C_T is compact. If M_θ is any multiplication operator on l_a^2, then $C_T M_\theta$ is always compact.

The following theorem characterizes compact weighted composition operators on l^2, the Hilbert space of square summable sequences of complex numbers on N, the set of natural numbers.

Theorem 3.6. *Let $C_T M_\theta \in B(l^2)$. Then $C_T M_\theta$ is compact if and only if $\{\theta(n)\}$ converges to zero.*

Proof. Suppose $C_T M_\theta$ is compact. Then $M_\delta^{\theta f_0}$ is finite dimensional and hence $N_\delta^{\theta f_0}$ contains finite number of elements of N. If N_δ^θ contains infinite number of elements of N, then $N_\delta^{f_0}$ must contain only finite number of elements of N. This shows that $f_0=0$ for all but finitely many elements of N and hence the range of T contains finitely many elements of N. By taking $T(N)=E$, we have $\lambda T^{-1}(E)\not\leqq$ $\not\leqq M\lambda(E)$ for any finite $M>0$. Hence by Theorem 1 of [3], C_T is not bounded. This proves that N_δ^θ contains finitely many elements of N. Hence $\{\theta(n)\}$ converges to zero.

Conversely, if $\{\theta(n)\}$ converges to zero, then M_θ is compact and hence $C_T M_\theta$ is compact.

Corollary 3.7. *No composition operator on l^2 is compact.*

Proof. The proof follows from Theorem 3.6, when $\theta(x)=1$.

Theorem 3.6 implies that the necessary condition for a weighted composition operator $C_T M_\theta$ on l^2 to be compact is that θ is not bounded away from zero. But this condition is not sufficient as is shown in the following example.

Example 3.8. Let $X=N$ and let $C_T \in B(l^2)$. Define $\theta\colon X \to \mathbf{C}$ by $\theta(1)=0$ and $\theta(n)=1$, if $n>2$. Hence θ is not bounded away from zero, but $C_T M_\theta$ is not compact.

Definition. A subalgebra $\mathscr{A}$ of $B(H)$ is said to be transitive if $\mathscr{A}$ is weakly closed, contains the identity operator and $\operatorname{Lat}\mathscr{A}=\{0, H\}$, where $\operatorname{Lat}\mathscr{A}= \bigcap_{A\in\mathscr{A}} \operatorname{Lat} A$. It has been proved in [2] that if $\mathscr{A}$ is a transitive algebra of $B(H)$ containing a compact operator, then $\mathscr{A}=B(H)$.

Let $\{w_n\}$ be a bounded sequence of non-zero complex numbers and let $\{e_n\}$ be an orthonormal basis of H. The operator W on H defined by the requirements $We_0=0$ and $We_n=w_n e_{n-1}$ $(n=1, 2, \ldots)$ is called a weighted unilateral (backward) shift with the weight sequence $\{w_n\}$.

Corollary 3.9. *The weighted shift W on l^2 is compact if and only if the sequence of weights $\{w_n\}$ converges to zero.*

Corollary 3.10. *If $\mathscr{A}$ is a transitive algebra of $B(l^2)$ containing a weighted composition operator $C_T M_\theta$ such that $\theta(n)\to 0$ as $n\to\infty$, then $\mathscr{A}=B(l^2)$.*

The following result of Yadav and Chattarjee [7] follows immediately from Theorem 3.6.

Corollary 3.11. *If $\mathscr{A}$ is a transitive algebra of $B(l^2)$ containing a weighted shift with weights $\{w_n\}$ such that*

$$\delta(n) = \sum_{k=0}^{\infty} w_{k+2} \ldots w_{k+n}/w_2 w_3 \ldots w_n$$

tends to zero as $n\to\infty$ (for $n\geqq 2$), then $\mathscr{A}=B(l^2)$.

Proof. Since the sequence $\{\delta(n)\}$ converges to zero, the corresponding sequence of weights $\{w_n\}$ converges to zero. Hence the weighted shift is compact. Thus the result follows (cf. [2]).

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Singular perturbations of singular systems

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Let $\Omega \subset \mathbf{R}^p$ $(p \in \mathbf{N})$ be a bounded open domain with C^2-smooth boundary and consider for $\varepsilon \neq 0$ the system

$$(1) \qquad -\varepsilon \Delta z - z^n = v, \quad v \in L^2(\Omega), \quad z \in H^2(\Omega) \cap H_0^1(\Omega).$$

This system is well-posed if $\varepsilon < 0$ and $n \in \{1, 3, \ldots\}$, not well-posed otherwise. Fixing $z_d \in L^{2n}(\Omega)$ and a number $N > 0$ arbitrarily, define

$$(2) \qquad J(v, z) = (1/2n) \|z - z_d\|^{2n}_{L^{2n}(\Omega)} + (N/2) \|v\|^2_{L^2(\Omega)}$$

and

$$(3) \qquad J_\varepsilon = \inf \{J(v, z) \mid (v, z) \text{ satisfies } (1)\}.$$

One can see easily (see [1]) that for any $\varepsilon \neq 0$ there exists (at least) a pair $(u_\varepsilon, y_\varepsilon)$ such that

$$(4) \qquad (u_\varepsilon, y_\varepsilon) \text{ satisfies (1) and } J(u_\varepsilon, y_\varepsilon) = J_\varepsilon.$$

The purpose of this paper is to investigate the behavior of the sequences (J_ε), (u_ε), (y_ε) when ε tends to 0.

In case $n = 3$ such investigations were done for $\varepsilon < 0$ by L. Tartar (see [1]), A. Haraux and F. Murat [4], [6] and for $\varepsilon > 0$ by A. Bensoussan [3]. All these considerations remain valid for any $n \in \{1, 3, \ldots\}$.

In the present paper, developing the method of A. Bensoussan, similar (and even stronger) results will be proved for the case $n \in \{2, 4, \ldots\}$. We shall also improve the results of Bensoussan in case $n \in \{1, 3, \ldots\}$.

Let us consider also the system

$$(5) \qquad -z^n = v, \quad v \in L^2(\Omega), \quad z \in L^{2n}(\Omega)$$

and put

$$(6) \qquad J_0 = \inf \{J(v, z) \mid (v, z) \text{ satisfies } (5)\}.$$

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One can see easily that there exists a unique pair (u_0, y_0) such that

(7) $$(u_0, y_0) \text{ satisfies (5) and } J(u_0, y_0) = J_0.$$

Let us introduce the polynomial

(8) $$p_{n,N}(x) = (1-x)^{2n} - x^{2n} + 2x^n \frac{(x+M)^n - x^n}{M} \quad \text{where} \quad M = (nN)^{1/(2n-1)}$$

of degree $2n-2$ and set

(9) $$N_n = \sup\{N > 0 \mid \inf_{x\in\mathbf{R}} p_{n,N}(x) > 0\}.$$

We shall prove the following two theorems:

Theorem 1. *Suppose* $N<N_n$. *Then*

(10) $$|J_\varepsilon - J_0| \to 0,$$

(11) $$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \to 0,$$

(12) $$\|y_\varepsilon - y_0\|_{L^{2n}(\Omega)} \to 0.$$

(10) *and* (11) *are valid for* $N=N_n<\infty$, *too.*

Theorem 2. *Suppose* $N<N_n$ *and*

(13) $$z_d,\ z_d^n \in H^2(\Omega)\cap H_0^1(\Omega).$$

Then

(14) $$|J_\varepsilon - J_0| = O(\varepsilon),$$

(15) $$\|u_\varepsilon - u_0\|_{L^2(\Omega)} = O(\sqrt{\varepsilon}),$$

(16) $$\|y_\varepsilon - y_0\|_{L^{2n}(\Omega)} = O(\sqrt[2n]{\varepsilon}).$$

(14) *and* (15) *are valid for* $N=N_n<\infty$, *too.*

Naturally, it is important to have some information on the numbers N_n:

Proposition. *We have*

(17) $$0 < N_n < \infty \quad \textit{if} \quad n\in\{3, 5, 7, \ldots\},$$

(18) $$N_n = +\infty \quad \textit{if} \quad n = 1 \quad \textit{and if} \quad n\in\{2, 4, 6, \ldots\}.$$

We turn to the proof of the theorems.

Lemma 1. *We have for all* $N>0$

(19) $$J_\varepsilon \leqq J_0 + o(1);$$

f condition (13) *is satisfied, we have also*

(20) $$J_\varepsilon \leqq J_0 + O(\varepsilon).$$

Proof. One can see by explicit calculation that

$$y_0 = z_d/\left(1+(nN)^{1/(2n-1)}\right) \tag{21}$$

and

$$u_0 = -y_0^n. \tag{22}$$

If condition (13) is satisfied then

$$J_\varepsilon \leqq J(-\varepsilon \Delta y_0 - y_0^n, y_0) = J(-y_0^n, y_0) + O(\varepsilon) = J_0 + O(\varepsilon)$$

whence (20) follows. In the general case fix a sequence $(z_m) \subset \mathscr{D}(\Omega)$ such that $\|z_m - y_0\|_{L^{2n}(\Omega)} \to 0$. Then for any fixed m

$$\overline{\lim}\, J_\varepsilon \leqq \overline{\lim}\, J(-\varepsilon \Delta z_m - z_m^n, z_m) = J(-z_m^n, z_m)$$

and

$$\overline{\lim}\, J_\varepsilon \leqq \lim J(-z_m^n, z_m) = J(-y_0^n, y_0) = J_0;$$

(19) is shown and the lemma is proved.

Now we fix for each $\varepsilon \neq 0$ a function $\bar{y}_\varepsilon$ such that

$$\bar{y}_\varepsilon,\ \bar{y}_\varepsilon^n \in H^2(\Omega) \cap H_0^1(\Omega), \tag{23}$$

$$\|\Delta \bar{y}_\varepsilon\|_{L^2(\Omega)} + \|\Delta(\bar{y}_\varepsilon^n)\|_{L^{2n/(2n-1)}(\Omega)} \leqq |\varepsilon|^{-1/2}, \tag{24}$$

$$\begin{aligned} \|\bar{y}_\varepsilon - y_0\|_{L^{2n}(\Omega)} \leqq |\varepsilon| + \inf\{&\|\bar{y} - y_0\|_{L^{2n}(\Omega)} \mid \bar{y},\ \bar{y}^n \in H^2(\Omega) \cap H_0^1(\Omega), \\ &\|\Delta \bar{y}\|_{L^2(\Omega)} + \|\Delta(\bar{y}^n)\|_{L^{2n/(2n-1)}(\Omega)} \leqq |\varepsilon|^{-1/2}\}. \end{aligned} \tag{25}$$

Furthermore, we put

$$\tilde{u}_\varepsilon = u_\varepsilon + \bar{y}_\varepsilon^n + \varepsilon \Delta \bar{y}_\varepsilon, \tag{26}$$

$$\tilde{y}_\varepsilon = y_\varepsilon - \bar{y}_\varepsilon, \tag{27}$$

$$\xi_\varepsilon = (\bar{y}_\varepsilon - z_d)^{2n-1} + nN\bar{y}_\varepsilon^{2n-1}. \tag{28}$$

Lemma 2. *We have*

$$\begin{aligned} J_\varepsilon = J(-\varepsilon \Delta \bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) + (N/2)\int_\Omega \tilde{u}_\varepsilon^2\, dx + \int_\Omega \xi_\varepsilon \tilde{y}_\varepsilon\, dx + \varepsilon N \int_\Omega -\tilde{u}_\varepsilon \Delta \bar{y}_\varepsilon + \tilde{y}_\varepsilon \Delta(\bar{y}_\varepsilon^n)\, dx + \\ + \int_\Omega \int_0^1 \int_0^1 \lambda \tilde{y}_\varepsilon^2 [(2n-1)(\bar{y}_\varepsilon - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + (n-1)nN\bar{y}_\varepsilon^n(\bar{y}_\varepsilon + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\, d\lambda\, d\mu\, dx. \end{aligned} \tag{29}$$

Proof. We recall that if $f\colon \mathbf{R} \to \mathbf{R}$ is a C^2-smooth function then

$$f(a+b) = f(a) + f'(a)b + \int_0^1 \int_0^1 \lambda b^2 f''(a+\lambda\mu b)\, d\lambda\, d\mu \tag{30}$$

for any $a, b \in \mathbf{R}$. Now using (1), (2), (4), (26), (27), (28), (30), we have the following three relations:

$$J_\varepsilon = J(u_\varepsilon, y_\varepsilon) = J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n + \tilde{u}_\varepsilon, \bar{y}_\varepsilon + \tilde{y}_\varepsilon) = J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) + \frac{N}{2}\int_\Omega \tilde{u}_\varepsilon^2\,dx +$$

$$(31)\qquad + N\int_\Omega (-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx + \int_\Omega (\bar{y}_\varepsilon - z_d)^{2n-1}\tilde{y}_\varepsilon\,dx +$$

$$+ \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2(2n-1)(\bar{y}_\varepsilon - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2}\,d\lambda\,d\mu\,dx,$$

$$\int_\Omega (\bar{y}_\varepsilon - z_d)^{2n-1}\tilde{y}_\varepsilon\,dx = \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx - N\int_\Omega \bar{y}_\varepsilon^n(n\bar{y}_\varepsilon^{n-1}\tilde{y}_\varepsilon)\,dx =$$

(32)

$$= \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx - N\int_\Omega \bar{y}_\varepsilon^n(y_\varepsilon^n - \bar{y}_\varepsilon^n)\,dx + \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2 n(n-1)N\bar{y}_\varepsilon^n(\bar{y}_\varepsilon + \lambda\mu\tilde{y}_\varepsilon)^{n-2}\,d\lambda\,d\mu\,dx,$$

$$-N\int_\Omega \bar{y}_\varepsilon^n(y_\varepsilon^n - \bar{y}_\varepsilon^n)\,dx + N\int_\Omega (-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx =$$

$$(33)\qquad = N\int_\Omega \bar{y}_\varepsilon^n(\varepsilon\Delta\tilde{y}_\varepsilon + \tilde{u}_\varepsilon)\,dx + N\int_\Omega (-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n)\tilde{u}_\varepsilon\,dx =$$

$$= \varepsilon N\int_\Omega \tilde{y}_\varepsilon\Delta(\bar{y}_\varepsilon^n)\,dx - \varepsilon N\int_\Omega \tilde{u}_\varepsilon\Delta\bar{y}_\varepsilon\,dx;$$

(31), (32) and (33) imply (29).

Lemma 3. *We have the following estimates for the terms of the formula* (29) *when* ε *tends to* 0:

$$(34)\qquad J(-\varepsilon\Delta\bar{y}_\varepsilon - \bar{y}_\varepsilon^n, \bar{y}_\varepsilon) = J_0 + o(1),$$

$$(35)\qquad (N/2)\int_\Omega \tilde{u}_\varepsilon^2\,dx \geqq 0,$$

$$(36)\qquad \int_\Omega \xi_\varepsilon\tilde{y}_\varepsilon\,dx = o(1),$$

$$(37)\qquad \varepsilon N\int_\Omega -\tilde{u}_\varepsilon\Delta\bar{y}_\varepsilon + \tilde{y}_\varepsilon\Delta(\bar{y}_\varepsilon^d)\,dx = o(1),$$

$$\int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(\bar{y}_\varepsilon - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)N\bar{y}_\varepsilon^n(\bar{y}_\varepsilon + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx =$$

(38)

$$= \int_\Omega\int_0^1\int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)Ny_0^n(y_0 + \lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx + o(1).$$

Proof. It follows from (23), (24), (25) that

(39) $$\|\tilde{y}_\varepsilon - y_0\|_{L^{2n}(\Omega)} = o(1),$$

(40) $$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = O(1),$$

(41) $$\varepsilon\|\Delta\tilde{y}_\varepsilon\|_{L^2(\Omega)} = o(1),$$

(42) $$\varepsilon\|\Delta(\tilde{y}_\varepsilon^n)\|_{L^{2n/(2n-1)}(\Omega)} = o(1);$$

(2), (5), (6), (23), (39), (40) and (41) imply (34).

(35) is obvious.

Using the obvious estimate $J(u_\varepsilon, y_\varepsilon) = J_\varepsilon \leqq J(0,0)$ and (2), we obtain

(43) $$\|u_\varepsilon\|_{L^2(\Omega)} = O(1),$$

(44) $$\|y_\varepsilon\|_{L^{2n}(\Omega)} = O(1);$$

(26), (27), (40), (41), (43) and (44) imply

(45) $$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} = O(1),$$

(46) $$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = O(1).$$

Furthermore we note that

(47) $$\|\xi_\varepsilon\|_{L^{2n/(2n-1)}(\Omega)} = o(1)$$

by (21), (28) and (39).

Now (36) follows from (47) and (46), (37) follows from (41), (42), (45), (46), finally (38) is a consequence of (39) and (46). The lemma is proved.

Lemma 4. *Putting*

(48) $$C_{n,N} = (2n)^{-1} \inf_{x\in\mathbf{R}} p_{n,N}(x),$$

we have

(49) $$\int_\Omega \int_0^1 \int_0^1 \lambda\tilde{y}_\varepsilon^2[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)Ny_0^n(y_0+\lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu\,dx \geqq C_{n,N}\int_\Omega \tilde{y}_\varepsilon^{2n}\,dx.$$

Proof. We show the stronger inequality

(50) $$\int_0^1\int_0^1 \lambda[(2n-1)(y_0 - z_d + \lambda\mu\tilde{y}_\varepsilon)^{2n-2} + n(n-1)\,Ny_0^n(y_0+\lambda\mu\tilde{y}_\varepsilon)^{n-2}]\,d\lambda\,d\mu \geqq C_{n,N}\,\tilde{y}_\varepsilon^{2n-2}.$$

This is obvious if $\tilde{y}_\varepsilon(x) = 0$. Otherwise, putting

(51) $$M = (nN)^{1/(2n-1)},$$

(52) $$y = M(y_0/\tilde{y}_\varepsilon)$$

and taking into account that

(53) $$z_d = (1+M)y_0$$

by (21), after integration we see that (50) is equivalent to

(54) $$p_{n,N}(y) \geqq 2nC_{n,N}.$$

The lemma is proved.

Lemma 5. *We have for all* $0<N<N_n$

(55) $$\inf_{x\in\mathbf{R}} p_{n,N}(x) > 0.$$

We prove Lemma 5 simultaneously with the Proposition (i.e. with (17) and (18)). First we note that

(56) $$p_{n,N}(x) = \big((1-x)^{2n} - x^{2n} + 2nx^{2n-1}\big) + (2/M)x^n\big((x+M)^n - x^n - nMx^{n-1}\big).$$

Consider first the case when $n=1$ or $n\in\{2,4,\ldots\}$. It suffices to show that

$$p_{n,N}(x) > 0 \quad \text{for all} \quad 0 < N < \infty \quad \text{and} \quad x\in\mathbf{R}.$$

This is obvious if $n=1$ because then $p_{n,N}(x)\equiv 1$. If $n\in\{2,4,\ldots\}$ then it follows from the formula (56), taking into account that $x^n\geqq 0$ and that the functions $t\mapsto t^{2n}$, $t\mapsto t^n$ are strictly convex.

Consider now the case $n\in\{3,5,\ldots\}$. One can see easily that

$$\lim_{N\to 0}\inf_{x\in\mathbf{R}} p_{n,N}(x) > 0$$

whence $N_n>0$. Now fix $0<\alpha<1$ such that

$$(1-\alpha)^n + \alpha^n - n\alpha^{n-1} > 0.$$

An easy computation shows that

$$M^{1-2n}p_{n,N}(-\alpha M) = -2\alpha^n\big((1-\alpha)^n + \alpha^n - n\alpha^{n-1}\big) + o(1) \quad (N\to\infty).$$

Therefore $N_n<\infty$ and (17) is proved.

To finish the proof of the lemma we show that for any fixed $x\in\mathbf{R}$ there exists a number $0<M_1\leqq\infty$ such that

$$p_{n,N}(x) > 0 \ \text{ if } \ M<M_1 \quad \text{and} \quad p_{n,N}(x) < 0 \ \text{ if } \ M > M_1.$$

Taking into account that $p_{n,N}(x)$ is a polynomial of degree $\geqq 1$ in M and that $\lim_{N\to 0} p_{n,N}(x)>0$, this would follow from the concavity of the function $f(M):=p_{n,N}(x)$ $(M>0)$. And f is concave because, applying the Taylor formula,

$$f''(M) = 2M^{-3}\left(x^n - (x+M)^n + n(x+M)^{n-1}M - \binom{n}{2}(x+M)^{n-2}M^2\right) =$$

$$= 2M^{-3}\binom{n}{3}\xi^{n-3}(-M)^3 = -2\binom{n}{3}\xi^{n-3} \leqq 0.$$

The lemma and the proposition are proved.

Proof of Theorem 1. It follows from Lemmas 1—5 that

(57) $$o(1) \geqq J_\varepsilon - J_0 \geqq (N/2)\|\tilde{u}_\varepsilon\|^2_{L^2(\Omega)} + C_{n,N}\|\tilde{y}_\varepsilon\|^{2n}_{L^{2n}(\Omega)} - o(1).$$

If $N \leqq N_n$ then $C_{n,N} \geqq 0$ and therefore (57) implies (10) and

(58) $$\|\tilde{u}_\varepsilon\|_{L^2(\Omega)} = o(1).$$

(11) follows from (58), (26), (39), (22) and (41). If $N < N_n$ then $C_{n,N} > 0$ and (57) implies also

(59) $$\|\tilde{y}_\varepsilon\|_{L^{2n}(\Omega)} = o(1).$$

(12) follows from (59), (27), (39) and the theorem is proved.

Proof of Theorem 2. In view of (13) and (21) we can put $\bar{y}_\varepsilon := y_0$; then (23), (24), (25) remain valid for $|\varepsilon|$ sufficiently small. Furthermore, in the estimates (39), (41), (42), (47) and therefore also in (34), (36), (37), (38) the term $o(1)$ can be replaced by $O(\varepsilon)$. (Moreover, in (39), (47), (36), (38) we can also write 0.) Therefore, repeating the proof of Theorem 1, we can change the terms $o(1)$ to $O(\varepsilon)$ in (57), (58), (59), too. Hence the theorem follows.

Remarks. (i) In case $n=3$ the condition $N < N_3$ is weaker than the original condition of Bensoussan:

(60) $$(0{,}3N)^{1/3}/(1+(3N)^{1/5}) < 4/3.$$

Indeed, up to decimals $N < N_3$ signifies $N < 5207$ while (60) signifies $N < 2841$.

(ii) All the results of this paper remain valid with the same proof if we replace in (1) the condition $v \in L^2(\Omega)$ by the more general condition $v \in K$ where K is a closed convex subset of $L^2(\Omega)$ such that

(61) $$(z_d/(1+(nN)^{1/(2n-1)}))^n \in \operatorname{int} K$$

(this is a problem with constraints).

(iii) A more general investigation of the influence of the different constraints is given by HARAUX and MURAT [4], [6]. A systematic study of the control of non-linear singular systems can be found in the book of J.-L. LIONS [1].

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On uniqueness and the Lifting Theorem

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It is the purpose of this note to present a simple proof of Theorem 1.1 in [1]. This theorem gives a necessary and sufficient condition for the existence of a unique intertwining dilation in the Lifting Theorem.

We follow the notation in [4]. If C is a contraction, then D_C is the positive square root of $I-C^*C$. The closure of the range of D_C is $\mathfrak{D}_C$.

A factorization C_1C_2 is regular [4] if

$$\mathfrak{D}_{C_1}\oplus D_{C_2} = \{D_{C_1}C_2h\oplus D_{C_2}h\colon h\in\mathfrak{H}\}^-.$$

Throughout T on $\mathfrak{H}$, T' on $\mathfrak{H}'$ and $A\colon \mathfrak{H}\to\mathfrak{H}'$ are contractions such that $T'A=AT$. The minimal isometric dilations of T on $\mathfrak{K}$ and T' on $\mathfrak{K}'$ are denoted by U and U', respectively. It is always assumed that U is in its matrix form with respect to the decomposition $\mathfrak{K}=\mathfrak{H}\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\oplus\mathfrak{D}_T\oplus\ldots$, i.e.,

$$U = \begin{bmatrix} T & 0 & 0 & \cdots \\ D_T & 0 & 0 & \\ 0 & I & 0 & \cdots \\ 0 & 0 & I & \\ 0 & 0 & 0 & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix}, \tag{1}$$

and analogously for U'. An operator B mapping $\mathfrak{K}$ into $\mathfrak{K}'$ is a contractive intertwining dilation (CID) of A if B is a contraction, and

$$U'B = BU \quad \text{and} \quad AP_{\mathfrak{H}} = P_{\mathfrak{H}'}B. \tag{2}$$

(The orthogonal projection onto $\mathfrak{H}$ is denoted by $P_{\mathfrak{H}}$.) The famous Lifting Theorem of Sz.-Nagy and Foiaş [3], [4] states that there exists a CID for A. The following shows when there is only one CID for A.

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Theorem 1 ([1]). *The contraction A has a unique CID if and only if AT or $T'A$ is a regular factorization.*

Proof. Let B be a CID for A. Matrix multiplication with (1) shows that B must be in the form:

$$B = \begin{bmatrix} A & 0 & 0 & 0 & 0 & \cdots \\ Z_1 & Y_1 & 0 & 0 & 0 & \ddots \\ Z_2 & Y_2 & Y_1 & 0 & 0 & \\ Z_3 & Y_3 & Y_2 & Y_1 & 0 & \cdots \\ Z_4 & Y_4 & Y_3 & Y_2 & Y_1 & \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{bmatrix} \tag{3}$$

where $Z_i\colon \mathfrak{H}\to\mathfrak{D}_{T'}$ and $Y_i\colon \mathfrak{D}_T\to\mathfrak{D}_{T'}$ are contractions for all $i\geqq 1$. Note the first row in B follows from the second equation in (2). Since $\|Bh\|^2\leqq\|h\|^2$ for all h in $\mathfrak{H}$, Equation (3) implies $\|Z_ih\|\leqq\|D_A h\|$. Thus, $Z_i=X_iD_A$ for $i\geqq 1$ where X_i is a contraction from $\mathfrak{D}_A$ into $\mathfrak{D}_{T'}$. Finally, using $U'Bh=BUh$ for h in $\mathfrak{H}$ with (3) gives:

$$[X_1, Y_1]\begin{bmatrix} D_ATh \\ D_Th \end{bmatrix} = D_{T'}Ah, \quad [X_{n+1}, Y_{n+1}]\begin{bmatrix} D_ATh \\ D_Th \end{bmatrix} = X_nD_A h \quad (n\geqq 1). \tag{4}$$

Assume AT is a regular factorization. This implies that the X_i's and Y_i's in (3) are uniquely determined by (4). Hence, B is unique. Now assume $T'A$ is a regular factorization. By Proposition VII.3.2 in [4] (or Lemma 3.1 in [2]) the factorization $A^*T'^*$ is regular. Therefore, A^* admits a unique CID. Lemma 2.1 in [1] shows that A has a unique CID if and only if A^* has a unique CID. Hence, A has a unique CID.

The other half of the proof follows from the one-step dilations for A in [2]. For completeness it is given. Assume $T'A$ and AT are not regular factorizations. By [2], there exist two different contractions

$$A_1 = \begin{bmatrix} A & 0 \\ Z_1 & Y_1 \end{bmatrix} \quad\text{and}\quad A_1' = \begin{bmatrix} A & 0 \\ Z_1' & Y_1' \end{bmatrix} \tag{5}$$

where $T_1'A_1=A_1T_1$ and $T_1'A_1'=A_1'T_1$. Here

$$T_1 = \begin{bmatrix} T & 0 \\ D_T & 0 \end{bmatrix} \quad\text{and}\quad T_1' = \begin{bmatrix} T' & 0 \\ D_{T'} & 0 \end{bmatrix}. \tag{6}$$

Applying the Lifting Theorem to (5) and (6) shows that A does not have a unique CID. The proof is now complete.

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Zur Symmetrisierung gewisser rationaler Eigenwertaufgaben

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1. Gegeben sei ein komplexer Hilbertraum $\mathfrak{H}$ mit innerem Produkt $(\cdot, \cdot)$ und Norm $\|\cdot\|$. Ferner seien gegeben komplexe Zahlen $a_k \in \mathbf{C}$, $k=1, 2, ..., K$, mit $a_k \neq a_j$ $(k \neq j)$, $a_k \neq 0$ für $k, j=1, 2, ..., K$. Dabei seien die Zahlen a_k so numeriert, daß für ein K_1, $0 \leqq K_1 \leqq K$, gilt: $a_1, a_2, ..., a_{K_1} \in \mathbf{R}$, $a_{K_1+1}, a_{K_1+2}, ..., a_K \in \mathbf{C} \setminus \mathbf{R}$ mit $a_{K_1+1} = \bar{a}_{K_1+2}, a_{K_1+3} = \bar{a}_{K_1+4}, ..., a_{K-1} = \bar{a}_K$. Weiterhin seien gegeben natürliche Zahlen $\varphi_j \in \mathbf{N}$, $j=1, 2, ..., K$, mit $\varphi_{K_1+2l-1} = \varphi_{K_1+2l}$, $l=1, 2, ..., (K-K_1)/2$, und lineare Operatoren A, H_k, $k=1, 2, ..., n-1$, $H_{k,j}$, $k=1, 2, ..., \varphi_j$, $j=1, 2, ..., K$, in $\mathfrak{H}$, die nicht alle gleich Null sind und wobei die Operatoren $A, H_1, ..., H_{n-1}$, $H_{k,j}$, $k=1, 2, ..., \varphi_j$, $j=1, 2, ..., K_1$, selbstadjungiert sind. Ferner gelte

$$H_{k, K_1+2l-1} = H^*_{k, K_1+2l}, \quad k = 1, 2, ..., \varphi_{K_1+2l-1}, \quad l = 1, 2, ..., (K-K_1)/2.$$

Schließlich seien die Operatoren H_k, $k=1, ..., n-1$, $H_{k,j}$, $k=1, 2, ..., \varphi_j$, $j=1, 2, ..., K$, endlichdimensional. Dann betrachten wir in dieser Arbeit die Operatorfunktion

$$\mathfrak{M}(\lambda) := I - \lambda A - \sum_{k=1}^{n-1} \lambda^{k+1} H_k + \sum_{j=1}^{K} \sum_{k=1}^{\varphi_j} \left(\lambda^2/(\lambda - a_j)^k\right) H_{k,j}$$

für $\lambda \in \mathbf{C}^* := \mathbf{C} \setminus \{a_1, ..., a_K\}$, wo I der Identitätsoperator in $\mathfrak{H}$ ist. Operatorfunktionen dieser Form treten bei Eigenwertaufgaben für gewöhnliche Differentialoperatoren mit Eigenwertparameter in den Randbedingungen auf (vgl. [2, 3]).

Eine Zahl $\lambda \in \mathbf{C}^*$ gehört zur Resolventenmenge von $\mathfrak{M}$, wenn der Operator $\mathfrak{M}(\lambda)^{-1}$ existiert als ein beschränkter, auf $\mathfrak{H}$ definierter Operator. Eine Zahl $\lambda \in \mathbf{C}^*$ heißt (regulärer) Eigenwert von $\mathfrak{M}$, wenn ein Element $0 \neq f \in \mathfrak{H}$ existiert, so daß $\mathfrak{M}(\lambda) f = 0$; f heißt dann ein zu λ gehöriges (reguläres) Eigenelement. $\lambda = a_l$ heißt

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ein singulärer Wert, wenn Elemente $f, g_1, g_2, \ldots, g_{\varphi_l} \in \mathfrak{H}$ existieren, so daß

$$f - a_l Af - \sum_{k=1}^{n-1} a_l^{k+1} H_k f + \sum_{\substack{j=1\\ j\neq l}}^{K} \sum_{k=1}^{\varphi_j} (a_l^2/(a_l - a_j)^k) H_{k,j} f - \sum_{k=1}^{\varphi_l} H_{k,l} g_k = 0,$$

$$H_{k,l} f = 0, \quad k = 1, \ldots, \varphi_l, \quad \|f\| + \sum_{k=1}^{\varphi_l} \|g_k\| > 0.$$

Ist $f \neq 0$, so heißt a_l singulärer Eigenwert von $\mathfrak{M}$. f heißt dann ein zu $\lambda = a_l$ gehörendes (singuläres) Eigenelement. In [2] wurde der Operatorfunktion $\mathfrak{M}$ ein linearer Operator zugeordnet (unter etwas allgemeineren Voraussetzungen) und damit gezeigt, daß $\mathfrak{M}$ höchstens abzählbar viele (reguläre und singuläre) Eigenwerte (endlicher) Vielfachheit ohne endlichen Häufungspunkt besitzt. Wir wollen in dieser Arbeit unter den etwas spezielleren Voraussetzungen zeigen, daß $\mathfrak{M}$ ein symmetrischer Operator in einem Pontryaginraum (vgl. [1]) zugeordnet werden kann, derart, daß die Eigenwerte von $\mathfrak{M}$ auch Eigenwerte dieses Operators sind.

Als Folgerung hieraus erhalten wir dann, daß $\mathfrak{M}$ höchstens endlich viele Eigenwerte in der Halbebene $\operatorname{Im} \lambda > 0$ und in der Halbebene $\operatorname{Im} \lambda < 0$ besitzt; beim Vorliegen abzählbar vieler Eigenwerte, gibt es also abzählbar viele reelle Eigenwerte.

2. Bei unserem Vorgehen benützen wir den in [2] $\mathfrak{M}$ zugeordneten linearen Operator, den wir hier zunächst zur einfacheren Handhabung in einer etwas anderen Form darstellen. Wir bezeichnen mit $P_{k,j}\colon \mathfrak{H} \to \{f \in \mathfrak{H} \mid H_{k,j} f = 0\}^{\perp}$, $k = 1, \ldots, \varphi_j$, $j = 1, \ldots, K$, die Projektionen mit $H_{k,j} = H_{k,j} P_{k,j}$, $k = 1, \ldots, \varphi_j$, $j = 1, \ldots, K$. Dann definieren wir lineare Operatoren $\mathfrak{A}\colon \mathfrak{H}^n \to \mathfrak{H}^n$, $\mathfrak{H}_{k,j}\colon \mathfrak{H}^k \to \mathfrak{H}^n$, $\mathfrak{P}_{k,j}\colon \mathfrak{H}^n \to \mathfrak{H}^k$, $\hat{\mathfrak{P}}_{k,j}\colon \mathfrak{H}^k \to \mathfrak{H}^k$, $k = 1, 2, \ldots, \varphi_j$, $j = 1, 2, \ldots, K$, durch

$$\mathfrak{A} := \begin{pmatrix} A & H_1 & H_2 \ldots H_{n-2} & H_{n-1} \\ I & 0 & 0 \ldots 0 & 0 \\ 0 & I & 0 \ldots 0 & 0 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ 0 & 0 & 0 \ldots 0 & 0 \\ 0 & 0 & 0 \ldots I & 0 \end{pmatrix};$$

$$\mathfrak{H}_{k,j} := \begin{pmatrix} \frac{1}{a_j} H_{k,j} & 0 \ldots 0 \\ 0 & 0 \ldots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \ldots 0 \end{pmatrix}.$$

$$\mathfrak{P}_{k,j} = \begin{pmatrix} \dfrac{(-1)^{k+1}}{a_j^{k-1}} P_{k,j} & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \vdots & \vdots \quad \vdots \\ 0 & 0 \dots 0 \end{pmatrix},$$

$$\hat{\mathfrak{P}}_{k,j} := \begin{pmatrix} \binom{k}{1} \dfrac{1}{a_j} P_{k,j} & -\binom{k}{2} \dfrac{1}{a_j^2} P_{k,j} \dots & & (-1)^{k+1} \binom{k}{k} \dfrac{1}{a_j^k} P_{k,j} \\ P_{k,j} & 0 & \dots & 0 \\ 0 & P_{k,j} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots P_{k,j} & 0 \end{pmatrix}$$

und definieren damit die Operatoren $\mathfrak{H}^{(j)}\colon \mathfrak{H}^{\varphi_j(\varphi_j+1)/2} \to \mathfrak{H}^n$, $\mathfrak{P}^{(j)}\colon \mathfrak{H}^n \to \mathfrak{H}^{\varphi_j(\varphi_j+1)/2}$, $\hat{\mathfrak{P}}^{(j)}\colon \mathfrak{H}^{\varphi_j(\varphi_j+1)/2} \to \mathfrak{H}^{\varphi_j(\varphi_j+1)/2}$ durch

$$\mathfrak{H}^{(j)} := (\mathfrak{H}_{1,j}, \mathfrak{H}_{2,j}, \dots, \mathfrak{H}_{\varphi_j,j}),$$

$$\mathfrak{P}^{(j)} := \begin{pmatrix} \mathfrak{P}_{1,j} \\ \mathfrak{P}_{2,j} \\ \vdots \\ \mathfrak{P}_{\varphi_j,j} \end{pmatrix}, \quad \hat{\mathfrak{P}}^{(j)} := \begin{pmatrix} \hat{\mathfrak{P}}_{1,j} & & \mathfrak{O} \\ & \hat{\mathfrak{P}}_{2,j} & \\ & & \ddots \\ \mathfrak{O} & & \hat{\mathfrak{P}}_{\varphi_j,j} \end{pmatrix}$$

für $j = 1, 2, \dots, K$. Schließlich können wir hiermit den Operator $\hat{\mathfrak{M}}\colon \mathfrak{H}^M \to \mathfrak{H}^M$ mit

$$M = n + (1/2) \sum_{j=1}^{K} \varphi_j(\varphi_j + 1)$$

erklären durch

$$\hat{\mathfrak{M}} := \begin{pmatrix} \mathfrak{A} & \mathfrak{H}^{(1)} & \mathfrak{H}^{(2)} \dots & \mathfrak{H}^{(K)} \\ \mathfrak{P}^{(1)} & \hat{\mathfrak{P}}^{(1)} & \mathfrak{O} \dots & \mathfrak{O} \\ \mathfrak{P}^{(2)} & \mathfrak{O} & \hat{\mathfrak{P}}^{(2)} \dots & \mathfrak{O} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ \mathfrak{P}^{(K)} & \mathfrak{O} & \mathfrak{O} \dots & \hat{\mathfrak{P}}^{(K)} \end{pmatrix}.$$

Dann besitzen $\mathfrak{M}$ und $\hat{\mathfrak{M}}$, $\tilde{\mathfrak{M}}(\lambda) := \mathfrak{E} - \lambda\hat{\mathfrak{M}}$, ($\mathfrak{E}$ Identitätsoperator in $\mathfrak{H}^M$) in $\mathbf{C}^*$ dieselben Resolventenmengen und dieselben Eigenwerte, während für einen Eigenwert a_j von $\hat{\mathfrak{M}}$ gleichzeitig a_j ein singulärer Wert von $\mathfrak{M}$ ist, und für einen singulären Eigenwert a_j von $\mathfrak{M}$ gleichzeitig a_j ein Eigenwert von $\hat{\mathfrak{M}}$ ist. Da

$\mathfrak{H}\mathfrak{K}^n$ ein kompakter operator ist, besitzt $\mathfrak{M}$ höchstens abzählbar viele Eigenwerte endlicher Vielfachheit ohne endlichen Häufungspunkt.

Wir betrachten nun den Vektorraum

$$\mathfrak{H}^M =: Z :=$$

$$:= \{(f_0, f_1, \ldots, f_{n-1}, f_{1,1,1}, f_{1,2,1}, f_{1,2,2}, \ldots, f_{1,\varphi_1,1}, \ldots, f_{1,\varphi_1,\varphi_1}, \ldots, f_{K,\varphi_K,\varphi_K})^{\mathrm{T}} \mid$$

$$f_k \in \mathfrak{H},\ k = 0, 1, \ldots, n-1,\ f_{j,k,l} \in \mathfrak{H},\ j = 1, 2, \ldots, K,\ k = 1, 2, \ldots, \varphi_j,\ l = 1, 2, \ldots, k\}$$

und wollen auf Z über einen selbstadjungierten Operator ein neues inneres Produkt einführen. Zur einfacheren Darstellung müssen wir dazu noch einige Abkürzungen einführen. Wir definieren lineare Operatoren $\mathfrak{K}_{\mathfrak{N}}$: $\mathfrak{H}^n \to \mathfrak{H}^n$, $\mathfrak{K}_{k,j}$: $\mathfrak{H}^k \to \mathfrak{H}^k$, $k=1,2,\ldots$ $\ldots, \varphi_j$, $j=1, 2, \ldots, K$, durch

$$\mathfrak{K}_{\mathfrak{N}} := \begin{pmatrix} I & 0 & 0 & \ldots & 0 & 0 \\ 0 & H_1 & H_2 & \ldots & H_{n-2} & H_{n-1} \\ 0 & H_2 & H_3 & \ldots & H_{n-1} & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & H_{n-2} & H_{n-1} & & & \\ 0 & H_{n-1} & 0 & \ldots & & 0 \end{pmatrix},$$

bzw.

$$\mathfrak{K}_{k,j} := \begin{pmatrix} K_{k,j}^{(0)} & 0 & 0 & \cdots & 0 \\ 0 & K_{k,j}^{(1)} & K_{k,j}^{(2)} & \cdots & K_{k,j}^{(k-1)} \\ 0 & K_{k,j}^{(2)} & & & 0 \\ \vdots & & & & \vdots \\ 0 & K_{k,j}^{(k-1)} & 0 & \cdots & 0 \end{pmatrix}$$

mit

$$K_{k,j}^{(0)} = (-1)^{k+1} a_j^{k-2} H_{k,j},$$

$$K_{k,j}^{(l)} = (-1)^{k+l-1} \binom{k}{l+1} a_j^{k-l-3} H_{k,j} \quad (l = 1, \ldots, k-1),$$

und erklären hiermit lineare Operatoren $\mathfrak{K}^{(j)}$: $\mathfrak{H}^{\varphi_j(\varphi_j+1)/2} \to \mathfrak{H}^{\varphi_j(\varphi_j+1)/2}$ durch

$$\mathfrak{K}^{(j)} := \begin{pmatrix} \mathfrak{K}_{1,j} & & \mathfrak{O} \\ & \mathfrak{K}_{2,j} & \\ \mathfrak{O} & & \ddots \ \mathfrak{K}_{\varphi_j,j} \end{pmatrix}.$$

Hiermit definieren wir nun den linearen Operator $\hat{\mathfrak{K}}\colon \mathfrak{H}^M \to \mathfrak{H}^M$ durch

$$\hat{\mathfrak{K}} := \begin{pmatrix} \mathfrak{K}_{\mathfrak{A}} & & & & & & & & & \\ & \mathfrak{K}^{(1)} & & & & & & & \mathfrak{O} & \\ & & \ddots & & & & & & & \\ & & & \mathfrak{K}^{(K_1)} & & & & & & \\ & & & & \mathfrak{O} & (\mathfrak{K}^{(K_1+1)})^* & & & & \\ & & & & \mathfrak{K}^{(K_1+1)} & \mathfrak{O} & & & & \\ & \mathfrak{O} & & & & & \ddots & & & \\ & & & & & & & & \mathfrak{O} & (\mathfrak{K}^{(K-1)})^* \\ & & & & & & & & \mathfrak{K}^{(K-1)} & \mathfrak{O} \end{pmatrix}$$

$\hat{\mathfrak{K}}$ ist offenbar ein selbstadjungierter beschränkter Operator in $\mathfrak{H}^M$ und mit Hilfe von $\hat{\mathfrak{K}}$ führen wir ein neues inneres Produkt $(\cdot,\cdot)_Z$ in $Z=\mathfrak{H}^M$ ein durch

$$(\mathfrak{f},\mathfrak{g})_{\hat{\mathfrak{K}}} := (\hat{\mathfrak{K}}\mathfrak{f},\mathfrak{g})$$

für $\mathfrak{f},\mathfrak{g}\in Z$. Dann ist das Paar $(Z,(\cdot,\cdot)_{\hat{\mathfrak{K}}})$ i.a. ein indefiniter Innenproduktraum, der darüber hinaus noch degeneriert ist. Der isotrope Teil Z^0 von Z ist gegeben durch

$$Z^0 = \Big\{\mathfrak{f}\in Z \,|\, f_0 = 0, \quad \sum_{j=1}^{n-k} H_{j+k-1} f_j = 0, \quad k = 1, 2, \ldots, n-1;$$

$$H_{k,j} f_{j,k,l} = 0, \quad j = 1, 2, \ldots, K, \; k = 1, 2, \ldots, \varphi_j, \; l = 1, 2, \ldots, k\Big\}.$$

Es gilt $Z^0 \neq \{0\}$ auf Grund unserer Annahmen. Ferner ist Z zerlegbar mit

$$Z = Z^0(\dot{+})Z^+(\dot{+})Z^-,$$

wobei $Z^+ = \mathfrak{R}(\hat{\mathfrak{P}}^+)$ und $Z^- = \mathfrak{R}(\hat{\mathfrak{P}}^-)$, wo $\hat{\mathfrak{P}}^+ = \mathfrak{E} - E_{\hat{\mathfrak{K}}}(0)$, $\hat{\mathfrak{P}}^- = E_{\hat{\mathfrak{K}}}(0-)$, wo $E_{\hat{\mathfrak{K}}}$ die Spektralschar von $\hat{\mathfrak{K}}$ bezeichnet und $\mathfrak{R}(\cdot)$ den Wertebereich des betreffenden Operators. Betrachten wir nun den Quotientenraum $Z/Z^0 := \mathfrak{Z}$, so erhalten wir einen nicht-degenerierten, zerlegbaren, indefiniten Innenproduktraum, wenn wir das innere Produkt in $\mathfrak{Z}$ durch $([\mathfrak{f}],[\mathfrak{g}])_z := (\mathfrak{f},\mathfrak{g})_{\hat{\mathfrak{K}}}$ definieren, wo $\mathfrak{f}\in[\mathfrak{f}]$, $\mathfrak{g}\in[\mathfrak{g}]$ Repräsentaten von $[\mathfrak{f}]$ bzw. $[\mathfrak{g}]$ sind. Es gilt $\mathfrak{Z} = \mathfrak{Z}^+(\dot{+})\mathfrak{Z}^-$ mit

$$\mathfrak{Z}^+ := \{[\mathfrak{f}]\in\mathfrak{Z} \,|\, \text{es ex. } \mathfrak{f}\in[\mathfrak{f}] \text{ mit } \mathfrak{f}\in Z^+\}, \quad \mathfrak{Z}^- := \{[\mathfrak{f}]\in\mathfrak{Z} \,|\, \text{es ex. } \mathfrak{f}\in[\mathfrak{f}] \text{ mit } \mathfrak{f}\in Z^-\}.$$

Das von dieser Zerlegung von $\mathfrak{Z}$ induzierte positiv definite Innenprodukt $[\cdot,\cdot]_{\mathfrak{Z}}$ auf $\mathfrak{Z}$ ist dann definiert durch

$$[[\mathfrak{f}],[\mathfrak{g}]]_{\mathfrak{Z}} := ([\hat{\mathfrak{P}}^+\mathfrak{f}],[\hat{\mathfrak{P}}^+\mathfrak{g}])_{\mathfrak{Z}} - ([\hat{\mathfrak{P}}^-\mathfrak{f}],[\hat{\mathfrak{P}}^-\mathfrak{f}])_{\mathfrak{Z}}$$

für $f\in[\mathfrak{f}]\in\mathfrak{Z}$, $\mathfrak{g}\in[\mathfrak{g}]\in\mathfrak{Z}$. Die zugehörige Norm auf $\mathfrak{Z}$ bezeichnen wir mit $\|\cdot\|_{\mathfrak{Z}}$. Die vollständige Hülle von $\mathfrak{Z}^+$ in dieser Norm sei $\overline{\mathfrak{Z}^+}$. Unter unseren Voraussetzungen ist dann $\overline{\mathfrak{Z}^+}\times\mathfrak{Z}^-$ ein Pontryaginraum; dieser sei $\overline{\mathfrak{Z}}$. $\mathfrak{Z}$ ist isometrisch isomorph zu einem dichten Teilraum von $\overline{\mathfrak{Z}}$. In $\mathfrak{Z}$ definieren wir die Abbildung $[\hat{\mathfrak{M}}]$ durch $[\hat{\mathfrak{M}}][\mathfrak{f}]:=[\hat{\mathfrak{M}}\mathfrak{f}]$ für $\mathfrak{f}\in[\mathfrak{f}]\in\mathfrak{Z}$.

Lemma. *$[\hat{\mathfrak{M}}]$ ist ein symmetrischer Operator in $\mathfrak{Z}$ und $\overline{\mathfrak{Z}}$. Jeder Eigenwert λ mit zugehörigem Eigenelement $\mathfrak{f}$ von $\hat{\mathfrak{M}}$ ist auch Eigenwert von $[\hat{\mathfrak{M}}]$ mit zugehörigem Eigenelement* $[\mathfrak{f}]$.

Beweis. Für $\mathfrak{f}\in[\mathfrak{f}]\in\mathfrak{Z}$, $\mathfrak{g}\in[\mathfrak{g}]\in\mathfrak{Z}$ gilt

$$([\hat{\mathfrak{M}}][\mathfrak{f}],[\mathfrak{g}])_{\mathfrak{Z}}=(\hat{\mathfrak{K}}\hat{\mathfrak{M}}\mathfrak{f},\mathfrak{g}),$$

d. h. es genügt zu zeigen, daß der Operator $\hat{\mathfrak{K}}\hat{\mathfrak{M}}$ symmetrisch ist. Wir setzen

$$\mathfrak{K}_{\mathfrak{A}}\mathfrak{A}:=\tilde{\mathfrak{K}}^{(0)},\quad \tilde{\mathfrak{K}}^{(j)}:=\mathfrak{K}_{\mathfrak{A}}\mathfrak{H}^{(j)},\quad j=1,2,\dots,K,$$

$$\tilde{\tilde{\mathfrak{K}}}^{(j)}:=\mathfrak{K}^{(j)}\mathfrak{P}^{(j)},\quad j=1,2,\dots,K_1.$$

$$\tilde{\tilde{\mathfrak{K}}}^{(K_1+2l-1)}:=(\mathfrak{K}^{(K_1+2l-1)})^*\,\mathfrak{P}^{(K_1+2l)},$$

$$\tilde{\tilde{\mathfrak{K}}}^{(K_1+2l)}:=\mathfrak{K}^{(K_1+2l-1)}\mathfrak{P}^{(K_1+2l-1)},\quad l=1,2,\dots,(K-K_1)/2,$$

$$\hat{\tilde{\tilde{\mathfrak{K}}}}^{(j)}:=\mathfrak{K}^{(j)}\hat{\mathfrak{P}}^{(j)},\quad j=1,2,\dots,K_1,$$

$$\hat{\tilde{\tilde{\mathfrak{K}}}}^{(K_1+2l-1)}:=(\mathfrak{K}^{(K_1+2l-1)})^*\,\hat{\mathfrak{P}}^{(K_1+2l)},$$

$$\hat{\tilde{\tilde{\mathfrak{K}}}}^{(K_1+2l)}:=\mathfrak{K}^{(K_1+2l-1)}\hat{\mathfrak{P}}^{(K_1+2l-1)},\quad l=1,2,\dots,(K-K_1)/2.$$

Dann gilt

$$\hat{\mathfrak{K}}\hat{\mathfrak{M}}=\begin{pmatrix}
\tilde{\mathfrak{K}}^{(0)} & \tilde{\mathfrak{K}}^{(1)} & \tilde{\mathfrak{K}}^{(2)} & \dots & \tilde{\mathfrak{K}}^{(K_1)} & \tilde{\mathfrak{K}}^{(K_1+1)} & \tilde{\mathfrak{K}}^{(K_1+2)} & \dots & \tilde{\mathfrak{K}}^{(K-1)} & \tilde{\mathfrak{K}}^{(K)}\\
\tilde{\tilde{\mathfrak{K}}}^{(1)} & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(1)} & \mathfrak{O} & \dots & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} & \dots & \mathfrak{O} & \mathfrak{O}\\
\tilde{\tilde{\mathfrak{K}}}^{(2)} & \mathfrak{O} & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(2)} & \dots & \mathfrak{O} & \mathfrak{O} & \mathfrak{O} & \dots & \mathfrak{O} & \mathfrak{O}\\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\
\tilde{\tilde{\mathfrak{K}}}^{(K_1)} & \mathfrak{O} & \mathfrak{O} & \dots & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(K_1)} & \mathfrak{O} & \mathfrak{O} & & & \\
\tilde{\tilde{\mathfrak{K}}}^{(K_1+1)} & \mathfrak{O} & \mathfrak{O} & \dots & \mathfrak{O} & \mathfrak{O} & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(K_1+1)} & & & \\
\tilde{\tilde{\mathfrak{K}}}^{(K_1+2)} & \mathfrak{O} & \mathfrak{O} & \dots & \mathfrak{O} & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(K_1+2)} & \mathfrak{O} & & & \\
\vdots & \vdots & \vdots & & & & & \ddots & \ddots & \\
\tilde{\tilde{\mathfrak{K}}}^{(K-1)} & \mathfrak{O} & \mathfrak{O} & \dots & & & & & \mathfrak{O} & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(K-1)}\\
\tilde{\tilde{\mathfrak{K}}}^{(K)} & \mathfrak{O} & \mathfrak{O} & \dots & & & & & \hat{\tilde{\tilde{\mathfrak{K}}}}^{(K)} & \mathfrak{O}
\end{pmatrix}$$

Da

$$\mathfrak{K}_{\mathfrak{A}}\mathfrak{A} = \begin{pmatrix} A & H_1 & H_2 \dots H_{n-2} & H_{n-1} \\ H_1 & H_2 & H_3 \dots H_{n-1} & 0 \\ H_2 & H_3 & H_4 \dots \;\; 0 & 0 \\ \vdots & \vdots & \vdots \quad\;\; \vdots & \vdots \\ H_{n-2} & H_{n-1} & 0 \;\dots\;\; 0 & 0 \\ H_{n-1} & 0 & 0 \;\dots\;\; 0 & 0 \end{pmatrix},$$

$$\mathfrak{K}_{\mathfrak{A}}\mathfrak{H}^{(j)} = \mathfrak{H}^{(j)} = (\mathfrak{H}_{1,j}, \mathfrak{H}_{2,j}, \dots, \mathfrak{H}_{\varphi_j,j}), \quad j = 1, 2, \dots K,$$

$$\mathfrak{K}^{(j)}\mathfrak{P}^{(j)} = (\mathfrak{H}^{(j)})^* = (\mathfrak{H}_{1,j}, \mathfrak{H}_{2,j}, \dots, \mathfrak{H}_{\varphi_j,j})^{\mathrm{T}}, \quad j = 1, 2, \dots, K_1,$$

$$(\mathfrak{K}^{(K_1+2l-1)})^*\mathfrak{P}^{(K_1+2l)} = (\mathfrak{H}^*_{1,K_1+2l-1}, \mathfrak{H}^*_{2,K_1+2l-1}, \dots, \mathfrak{H}^*_{\varphi_{K_1+2l-1},K_1+2l-1})^{\mathrm{T}} =$$
$$= (\mathfrak{H}^{(K_1+2l-1)})^*,$$

$$\mathfrak{K}^{(K_1+2l-1)}\mathfrak{P}^{(K_1+2l-1)} = (\mathfrak{H}^*_{1,K_1+2l}, \mathfrak{H}^*_{2,K_1+2l}, \dots, \mathfrak{H}^*_{\varphi_{K_1+2l},K_1+2l})^{\mathrm{T}} = (\mathfrak{H}^{(K_1+2l)})^*,$$

$$l = 1, 2, \dots, (K-K_1)/2,$$

und

$$\mathfrak{K}^{(j)}\hat{\mathfrak{P}}^{(j)} = \begin{pmatrix} \mathfrak{K}_{1,j}\hat{\mathfrak{P}}_{1,j} & & \mathfrak{O} \\ & \mathfrak{K}_{2,j}\hat{\mathfrak{P}}_{2,j} & \\ & & \ddots \\ \mathfrak{O} & & \mathfrak{K}_{\varphi_j,j}\hat{\mathfrak{P}}_{\varphi_j,j} \end{pmatrix}, \quad j = 1, 2, \dots, K_1,$$

mit

$$\mathfrak{K}_{k,j}\hat{\mathfrak{P}}_{k,j} = \begin{pmatrix} (-1)^{k+1}\binom{k}{1}a_j^{k-3}H_{k,j} & (-1)^{k+2}\binom{k}{2}a_j^{k-4}H_{k,j} & \dots & (-1)^{2k}\binom{k}{k}a_j^{-2}H_{k,j} \\ (-1)^{k}\binom{k}{2}a_j^{k-4}H_{k,j} & (-1)^{k+1}\binom{k}{3}a_j^{k-5}H_{k,j} & & 0 \\ \vdots & \vdots & & \vdots \\ (-1)^{2k-2}\binom{k}{k}a_j^{-2}H_{k,j} & 0 & \dots & 0 \end{pmatrix}$$

$k=1, 2, \dots, \varphi_j, j=1, 2, \dots, K_1$, und ferner (wie man aus dem letzteren sieht)

$$(\mathfrak{K}^{(K_1+2l-1)})^*\hat{\mathfrak{P}}^{(K_1+2l)} = (\mathfrak{K}^{(K_1+2l-1)}\hat{\mathfrak{P}}^{(K_1+2l-1)})^*$$

für $l=1, 2, \dots, (K-K_1)/2$ gilt, folgt die Symmetrie von $\hat{\mathfrak{K}}\hat{\mathfrak{M}}$ unmittelbar.

Zum beweis der zweiten Behauptung gelte $(\mathfrak{E}-\lambda\hat{\mathfrak{M}})\mathfrak{f}=\mathfrak{O}$, $\mathfrak{f}\neq\mathfrak{O}$. Dann folgt $[\mathfrak{O}]=[(\mathfrak{E}-\lambda\hat{\mathfrak{M}})\mathfrak{f}]=[\mathfrak{f}]-\lambda[\hat{\mathfrak{M}}][\mathfrak{f}]$. Da $\mathfrak{f}\neq\mathfrak{O}$ ist, gilt auch $f_0\neq 0$ (vgl. [2]); damit ist $[\mathfrak{f}]\neq\mathfrak{O}$ ein Eigenelement um Eigenwert λ von $[\hat{\mathfrak{M}}]$.

$[\hat{\mathfrak{M}}]$ ist also der gesuchte, $\mathfrak{M}$ zugeordnete, symmetrische Operator in dem Pontryaginraum $\mathfrak{Z}$.

Aus den Eigenschaften symmetrischer Operatoren in Pontryaginräumen (vgl. Bognár [1]) erhalten wir nun als Folgerung für die Eigenwerte von $\mathfrak{M}$ den folgenden

Satz. *$\mathfrak{M}$ besitzt höchstens abzählbar viele Eigenwerte endlicher Vielfachheit ohne endlichen Häufungspunkt. Von diesen liegen höchstens* dim $\mathfrak{Z}^-$ *in der oberen Halbebene* (Im $\lambda>0$) *und höchstens* dim $\mathfrak{Z}^-$ *in der unteren Halbebene* (Im $\lambda<0$), *alle anderen sind reell.*

Da dim $\mathfrak{Z}^-$ gleich der Anzahl der negativen Eigenwerte von $\mathfrak{K}$ ist (Vielfachheiten mitgezählt) ist dim $\mathfrak{Z}^->0$ auf jeden Fall, wenn für ein a_k entweder $\varphi_K\geqq 2$ oder $a_k\in\mathbf{C}\setminus\mathbf{R}$ oder wenn $n>1$ ist (falls die zugehörigen Operatoren nicht Null sind).

Es ist klar, daß der obige Satz auch auf Eigenwertaufgaben für gewöhnliche Differentialgleichungen mit Eigenwertparameter in den Randbedingungen angewendet werden kann (vgl. [2, 3]). Wir wollen hier nicht mehr genauer darauf eingehen.

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Espaces de James généralisés et espaces de type E.S.A.

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Introduction. A partir d'un espace engendré par une suite I.S. (« invariant under spreading » c'est-à-dire invariante par étalement, voir paragraphe 1), monotone inconditionnelle, A. BRUNEL et L. SUCHESTON ont construit dans [4] un espace de type E.S.A. (« equal signs additive ») défini comme suit:

Définition 1. Soit $(F, |\cdot|)$ un espace de Banach engendré par une suite (x_n) normalisée, I.S. monotone inconditionnelle, pour $a=\sum_{i=1}^{n} a_i x_i$, posons:

$$M(a) = \sup_{\pi} \Big|\big(\sum_{j\in I_1} a_j\big)x_1+\ldots+\big(\sum_{j\in I_k} a_j\big)x_k\Big|$$

où π désigne l'ensemble de toutes les partitions de $\{1, ..., n\}$ en intervalles consécutifs disjoints: $I_1, ..., I_k$ $(1\leqq k\leqq n)$. L'espace engendré par (x_n) pour la norme M est de type E.S.A. (il sera noté G).

Nous allons établir que cet espace se rattache à une classe d'espaces de James.

Définition 2. [9] Soit $(F, |\cdot|)$ un espace de Banach engendré par une suite (x_n) basique, monotone, normalisée; pour une suite de réels $a=(a_1, a_2, ...)$ posons:

$$\|a\|_J = \sup \Big|\sum_{k=1}^{n} (a_{p_{2k-1}}-a_{p_{2k}})x_k+a_{p_{2n+1}}x_{n+1}\Big|$$

où le supremum est pris sur tous les naturels n et les suites croissantes d'entiers $p_1, ..., p_{2n+1}$.

L'espace de James généralisé J est l'espace de Banach de tous les a tels que $\|a\|_J$ est finie et $\lim_n a_n=0$.

Notons que si l'on choisit pour F l'espace l^2 et pour (x_n) sa base canonique, la définition 2 fournit l'espace de Banach J initialement introduit par R. C. JAMES [7]. P. G. CASAZZA et R. H. LOHMAN ont obtenu les résultats suivants [9]: La suite (e_n) des vecteurs unités est une suite basique dans J. Si (x_n) est symétrique

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et complète (« boundedly complete ») dans F, (e_n) est une base de J. Si F est réflexif et (x_n) symétrique, « p-Hilbertienne sur des blocs » $(1<p<\infty)$ dans F, (e_n) est une base contractante (« shrinking ») de J et J est quasi-réflexif d'ordre 1.

Dans cet article, nous étendons ces résultats aux bases (normalisées, monotones) sous-symétriques. Nous étudions entre autres la question: quand la suite (e_n) est-elle une base de J? P. G. Casazza et R. H. Lohman y ont répondu mais de façon partielle en posant comme condition: (x_n) complète dans F. Nous avons voulu affiner ce résultat et nous sommes arrivés à la conclusion que (e_n) est également une base de J si (x_n) est équivalente à la base canonique de c_0. Nous montrons qu'en fait hormis ces deux cas (hypothèse de P. G. Casazza et R. H. Lohman et hypothèse citée ci-dessus) (e_n) n'est pas une base de J (théorème 1).

Nous supposons la base (x_n) de F sous-symétrique et nous smontrons que l'espace de James J associé est (lorsque (e_n) est une base de J) isomorphe à l'espace de A. Brunel et L. Sucheston G. Ceci précise donc l'affirmation de A. Brunel et L. Sucheston [4, introduction] selon laquelle on obtiendrait le dual de l'espace construit par R. C. James [7] si l'on choisit pour la norme inconditionnelle de la définition 1 la norme de l^2.

Nous examinons cet espace G dans l'intention d'apporter une réponse aux questions suivantes: quand G est-il isomorphe à c_0, l^1?, quand contient-il un sous-espace isomorphe à c_0, l^p? Nous parvenons aux résultats: G est isomorphe à c_0 (respectivement l^1) si et seulement si F est isomorphe à c_0 (respectivement l^1). G contient c_0 si et seulement si F contient c_0.

Si F contient l^p, il en va de même pour G; mais nous ne savons pas si « G contient l^p » implique « F contient l^p ». Pour $p=1$: P. G. Casazza et R. H. Lohman ont imposé la condition: (x_n) est q-Hilbertienne sur des blocs $(1<q<\infty)$; ils en ont déduit que J a une base contractante. De notre côté, nous considérons la propriété $(\mathscr{P})$ pour une base (x_n): « toute suite basique bornée de blocs sur (x_n) converge en moyenne de Cesàro vers zéro »; et nous établissons le résultat suivant (théorème 4): si (x_n) a la propriété $(\mathscr{P})$ dans F, G a une base contractante: $(x'_n)_{n\geq 1}=(x_1, x_2, -x_1, \ldots, x_n-x_{n-1}, \ldots)$ et G est quasi-réflexif d'ordre 1 si de plus F ne contient pas c_0. La condition: « (x_n) est p-Hilbertienne sur des blocs $(1<p<\infty)$ » de P. G. Casazza et R. H. Lohman implique la nôtre: « (x_n) a la propriété $(\mathscr{P})$ ».

Nous étudions ces deux propriétés et nous montrons entre autres que

$(x_n)_{n\geq 1}$ a la propriété $(\mathscr{P})$ dans F si $(x'_n)_{n\geq 1}$ a la propriété $(\mathscr{P})$ dans G.

$(x_n)_{n\geq 1}$ a la propriété $(\mathscr{P})$ dans F implique

1⁰) F ne contient pas de sous-espace isomorphe à l^1 (la réciproque n'étant pas vraie);

2⁰) F a la propriété de Banach—Saks faible (la réciproque n'étant pas vraie).

Finalement, nous obtenons: (x_n) est p-Hilbertienne sur des blocs $(1<p\leq\infty)$ si

et seulement si (x_n) est B-convexe sur des blocs ou encore si et seulement si l^1 n'est pas finiment représentable en blocs dans (x_n).

En terminant cette introduction, nous avons à coeur de remercier chaleureusement Messieurs B. Beauzamy et G. Noël qui l'un par ses conseils, remarques et suggestions, l'autre par sa compréhension ont permis la réalisation de ce travail.

1. Préliminaires. Soit S l'espace vectoriel des suites réelles à support fini; nous rappelons qu'une base $(x_n)_{n\geqq 1}$ d'un espace de Banach $(X, \|\cdot\|)$ est *monotone* si sa constante de base est 1; *monotone inconditionnelle* si pour tout A, B, ensembles finis de naturels, et pour tout $a\in S$, $A\subset B$ implique:

$$\|\sum_{i\in A} a_i x_i\| \leqq \|\sum_{i\in B} a_i x_i\|;$$

sous-symétrique si elle est inconditionnelle et si pour toute suite croissante d'entiers $(n_i)_{i\geqq 1}$, $(x_{n_i})_{i\geqq 1}$ est équivalente à $(x_n)_{n\geqq 1}$; *symétrique* si pour toute permutation π des entiers, $(x_n)_{n\geqq 1}$ est équivalente à $(x_{\pi(n)})_{n\geqq 1}$; *p-Hilbertienne sur des blocs* $(1\leqq p<\infty)$ s'il existe une constante K telle que pour toute suite basique bornée de blocs $(w_n)_{n\geqq 1}$ sur la base $(x_n)_{n\geqq 1}$ et toute suite $(a_n)_{n\geqq 1}$ de scalaires:

$$\|\sum_{n=1}^{k} a_n w_n\| \leqq (K\cdot \sup_n \|w_n\|)\Big(\sum_{n=1}^{k} |a_n|^p\Big)^{1/p} \quad (k = 1, 2, \ldots);$$

complète (« boundedly complete ») si pour toute suite de scalaires $(a_n)_{n\geqq 1}$ telle que le supremum $\sup_n \|\sum_{i=1}^{n} a_i x_i\|$ soit fini, la série $\sum_{n=1}^{+\infty} a_n x_n$ converge; *k-contractante* (« k-shrinking ») avec k dans **N** si l'adhérence de l'espace engendré par les « applications coordonnées » dans le dual X^* de X est de codimension k; *contractante* (« shrinking ») si $(x_n)_{n\geqq 1}$ est 0-contractante, c'est-à-dire si l'adhérence de l'espace engendré par les « applications coordonnées » coïncide avec X^*.

Un espace de Banach X est *quasi-réflexif (d'ordre n)* si le plongement canonique de X dans son bidual X^{**} est de codimension finie (de codimension n). Soit $(x_n)_{n\geqq 1}$ une suite normalisée qui engendre un espace de Banach X. $(x_n)_{n\geqq 1}$ est de type I.S. (« invariant under spreading ») si pour tout $a\in S$ et pour toute suite strictement croissante d'entiers (k_i),

$$\|\sum_i a_i x_i\| = \|\sum_i a_i x_{k_i}\|;$$

$(x_n)_{n\geqq 1}$ est de type E.S.A. (« equal signs additive ») si pour tout $a\in S$ et pour tout entier $k\geqq 1$ tel que $a_k\cdot a_{k+1}\geqq 0$,

$$\|\sum_i a_i x_i\| = \|\sum_{i=1}^{k-1} a_i x_i + (a_k+a_{k+1})x_k + \sum_{i\geqq k+2} a_i x_i\|.$$

2. Dans ce paragraphe nous rappelons et étudions certaines propriétés des suites E.S.A. Soient X un espace de Banach et $\|\cdot\|$ sa norme. On considère une suite normalisée $(x_n)_{n\geqq 1}$ qui engendre X. Si $(x_n)_{n\geqq 1}$ est E.S.A., on sait que $(x_n)_{n\geqq 1}$ est une base de X.

Proposition 1. *Si (x_n) est E.S.A.,*

1. *(x_n) est inconditionnelle si et seulement si X est isomorphe à l^1.*
2. *(x_n) n'est jamais faiblement convergente.*
3. *(x_n) est équivalente à la base sommante de c_0 si et seulement si X est isomorphe à c_0.*
4. *Il existe une constante K positive telle que pour tout naturel n, on ait: $\left\|\sum_{k=1}^{n}(-1)^k x_k\right\| \leqq K$ si et seulement si X est isomorphe à c_0.*

Démonstration. 1. Si (x_n) est inconditionnelle, il existe une constante C telle que pour tout $a\in S$, on ait:

$$\left\|\sum_n a_n x_n\right\| \geqq C\left\|\sum_n |a_n| x_n\right\|.$$

Et donc, puisque (x_n) est E.S.A.:

$$\left\|\sum_n a_n x_n\right\| \geqq C\sum_n |a_n|.$$

(x_n) est ainsi équivalente à la base canonique de l^1. D'autre part, si X est isomorphe à l^1, alors (x_n) (qui est I.S.) est équivalente à la base canonique de l^1 [2] et donc inconditionnelle.

2. La suite (x_n) étant basique, il suffit de montrer qu'elle ne converge pas faiblement vers zéro. Or, une suite I.S. tend faiblement vers zéro si et seulement si elle est basique inconditionnelle, non équivalente à la base canonique de l^1 [1].

3. L'espace c_0 possède exactement deux suites basiques I.S. non équivalentes= la base canonique et la base sommante; et la base canonique n'est pas E.S.A.

4. Supposons qu'il existe une constante K positive telle que pour tout naturel n, on ait: $\left\|\sum_{k=1}^{n}(-1)^k x_k\right\| \leqq K$; considérons alors la suite:

$$x_1' = x_1,\ x_2' = x_2 - x_1,\ x_3' = x_3 - x_2,\ \dots .$$

$(x_n')_{n\geqq 1}$ est une base de X (de constante de base 1) [4]. On a: $x_n = \sum_{i=1}^{n} x_i'$. Les suites (x_{2n}') et (x_{2n+1}') sont inconditionnelles [1]; elles sont donc, par hypothèse, toutes deux équivalentes à la base canonique de c_0. Ainsi pour tout $a\in S$:

$$\left\|\sum_{i=1}^{+\infty} a_i x_i'\right\| \leqq |a_1|\,\|x_1'\| + \left\|\sum_{i=1}^{+\infty} a_{2i} x_{2i}'\right\| + \left\|\sum_{i=1}^{+\infty} a_{2i+1} x_{2i+1}'\right\| \leqq C \sup |a_i|,$$

On déduit de la basicité de la suite (x'_n) l'existence d'une constante K telle que pour tout $a \in S$

$$\left\|\sum_{i=1}^{+\infty} a_i x'_i\right\| \geqq K \sup |a_i|.$$

(x'_n) est donc équivalente à la base caninoque de c_0 et (x_n) est équivalente à la base sommante de c_0.

Remarque. Le point 2 de la proposition entraîne que la suite (x_n) n'est jamais contractante et X n'est pas réflexif.

3. La norme M définie dans l'introduction possède les propriétés suivantes:

1. Si $a=\sum_{i=1}^{n} a_i x_i$, alors $|a| \leqq M(a)$ et $\left|\sum_{i=1}^{n} a_i\right| \leqq M(a)$.

2. Soit $u_1=x_2-x_1, u_2=x_4-x_3, \dots$. Si $a=\sum_{i=1}^{n} a_i u_i$, on a: $\left|\sum_{i=1}^{n} a_i x_i\right| \leqq |a|$ puisque (x_n) est monotone inconditionnelle.

De la définition de la norme M et de la monotone inconditionnalité de (x_n), on déduit que ([4])

$$M(a) \leqq 2\left|\sum_{i=1}^{n} a_i x_i\right|.$$

Ainsi $|a| \leqq M(a) \leqq 2|a|$. Et les normes M et $|\cdot|$ sont équivalentes sur l'espace engendré par les vecteurs $(u_i)_{i \geqq 1}$.

Notons que contrairement au modèle E.S.A. obtenu à partir d'un modèle étalé [5], l'espace G n'est pas finiment représentable dans $F=$ en effet si l'on prend pour F l'espace l^2, celui-ci est super-réflexif mais G n'est pas réflexif comme nous l'avons vu au paragraphe 2.

4. Passons maintenant aux espaces de James généralisés définis dans l'introduction. Soit (x_i) une base monotone normalisée de $(F, |\cdot|)$, si (x_i) est symétrique et complète, R. H. Lohman et P. G. Casazza ont montré [9] que (e_i) est alors une base de J. Ce résultat s'étend aux bases sous-symétriques.

D'autre part, on obtient facilement la

Proposition 2. *Si (x_i) est équivalente à la base canonique de c_0, alors (e_i) est une base de J.*

Preuve. Soit $\alpha=(\alpha_n) \in J$, considérons: $\alpha-P_k\alpha=(\alpha_i^{(1)})$

$$\|\alpha-P_k\alpha\|_J = \sup\left|\sum_{i=1}^{n}(\alpha_{p_{2i-1}}^{(1)}-\alpha_{p_{2i}}^{(1)})x_i+\alpha_{p_{2n+1}}^{(1)}x_{n+1}\right|.$$

(Ce supremum étant pris sur tous les n et les suites croissantes d'entiers (p_i)). Il existe une constante C telle que

$$\|\alpha - P_k\alpha\|_J \leqq C \sup_{i>k} |\alpha_i|$$

(puisque (x_i) est équivalente à la base canonique de c_0). Comme $\alpha \in J$, $\lim_n |\alpha_n| = 0$ et $\|\alpha - P_k\alpha\|_J$ tend vers zéro lorsque k tend vers l'infini. De plus, si (x_i) est équivalente à la base canonique de c_0, il existe deux constantes m et M telles que pour tout $a \in S$:

$$m \sup_i |a_i| \leqq \left|\sum_i a_i x_i\right| \leqq M \sup_i |a_i|.$$

De la définition de la norme dans J, on déduit qu'il existe une constante C telle que:

$$\left\|\sum_i a_i e_i\right\|_J \leqq C \sup_i |a_i|.$$

Et puisque (e_i) est une suite basique dans J, il existe une constante C_1 telle que:

$$\left\|\sum_i a_i e_i\right\|_J \geqq C_1 \sup_i |a_i|.$$

Ceci prouve que (e_i) est équivalente à la base canonique de c_0.

Cette démonstration montre aussi que:

Proposition 3. *Si F est isomorphe à c_0, alors J est isomorphe à c_0.*

Il suffit, en effet, de rappeler que F est isomorphe à c_0 si et seulement si (x_i) est équivalente à la base canonique de c_0 [6].

Nous verrons (proposition 4, paragraphe 6) qu'en fait F est isomorphe à c_0 si et seulement si J est isomorphe à c_0.

Théorème 1. *Soit (x_i) une base (normalisée, monotone) sous-symétrique de F; (e_i) est une base de J si et seulement si (x_i) est complète dans F ou équivalente à la base canonique de c_0.*

Démonstration. La condition suffisante se déduit immédiatement de ce qui précède.

Quant à la condition nécessaire = nous supposerons (x_i) I.S., monotone inconditionnelle. Considérons (a_i) une suite de scalaires qui converge vers zéro et telle que le supremum

$$\sup_n \left|\sum_{i=1}^n a_i x_i\right|$$

soit fini. Soit alors $a' = (0, a_1, 0, a_2, 0, \ldots)$. On a:

$$\left|\sum_{i=1}^n a_i x_i\right| \leqq \left\|\sum_{i=1}^n a_i e_{2i}\right\|_J \leqq 3\left|\sum_{i=1}^n a_i x_i\right|. \tag{1}$$

Puisque (e_i) est une base de J, la série $\sum_{i=1}^{+\infty} a_i e_{2i}$ converge vers a' (dans J), donc par la première inégalité de (1), la série $\sum_{i=1}^{+\infty} a_i x_i$ converge dans F.

Nous avons ainsi montré que si (a_i) est une suite de scalaires convergeant vers zéro et telle que le supremum $\sup_n \left|\sum_{i=1}^{n} a_i x_i\right|$ soit fini, alors la série $\sum_{i=1}^{\infty} a_i x_i$ converge dans F ce qui n'est possible que si (x_n) est complète ou équivalente à la base canonique de c_0. En effet, supposons que (x_n) ne soit pas équivalente à la base canonique de c_0; pour prouver que (x_n) est alors complète il suffit, par ce qui précède, de considérer le cas où (a_i) est une suite de scalaires ne convergeant pas vers zéro. Il existe alors une constante C et une suite strictement croissante d'entiers (n_k) telles que pour tout naturel m, on ait:

$$\left|\sum_{k=1}^{m} a_{n_k} x_{n_k}\right| > C \left|\sum_{k=1}^{m} x_{n_k}\right|$$

((x_n) étant inconditionnelle dans F). Ainsi, puisque (x_n) est I.S., on a:

$$\left|\sum_{k=1}^{m} a_{n_k} x_{n_k}\right| > C \left|\sum_{k=1}^{m} x_k\right|.$$

Mais on a supposé (x_n) non équivalente à la base canonique de c_0, donc le supremum $\sup_m \left|\sum_{k=1}^{m} x_k\right|$ est infini puisque (x_n) est inconditionnelle. Et le supremum $\sup_m \left|\sum_{k=1}^{m} a_{n_k} x_{n_k}\right|$ est infini, il en est de même pour le supremum $\sup_m \left|\sum_{k=1}^{m} a_k x_k\right|$ ce qui achève la démonstration du théorème.

5. Dans ce paragraphe, nous établissons l'isomorphisme entre l'espace de Brunel et Sucheston G et un espace de James généralisé. Pour $a=(a_1, a_2, \ldots)$ une suite de salaires, soit

$$|||a|||_J = \sup \left|\sum_{i=1}^{n} (a_{p_i} - a_{p_{i+1}}) x_i\right|,$$

où le supremum est pris sur tous les naturels n et toutes les suites croissantes de naturels $p_1, \ldots, p_{n+1}$. Si l'on suppose (x_i) sous-symétrique, les normes $\|\cdot\|_J$ et $|||\cdot|||_J$ sont équivalentes.

Lemma 1. *Si $(x_i)_{i\geq 1}$ est une base sous-symétrique, alors la suite basique des vecteurs unités (e_i) de J est équivalente à la base $(x_i')_{i\geq 1}=(x_1, x_2-x_1, \ldots, x_n-x_{n-1}, \ldots)$ de G.*

Démonstration. Comme toujours, nous supposerons (x_i) I.S. monotone inconditionnelle, montrons que pour tous réels $a_1, \ldots, a_n$:

$$M\Big(\sum_{i=1}^{n} a_i x_i'\Big) = |||(a_1, \ldots, a_n, 0, \ldots)|||_J;$$

$$M\Big(\sum_{i=1}^{n} a_i x_1'\Big) = M\Big(\sum_{i=1}^{n} a_i(x_i - x_{i-1})\Big) = M((a_1 - a_2)x_1 + \ldots + (a_{n-1} - a_n)x_{n-1} + a_n x_n) =$$

(1)

$$= \sup\Big|(a_1 - a_{p_1})x_1 + \sum_{i=1}^{m}(a_{p_i} - a_{p_{i+1}})x_{i+1} + a_{p_{m+1}}x_{m+2}\Big|.$$

D'autre part:

(2) $$|||(a_1, \ldots, a_n, 0, 0, \ldots)|||_J = \sup\Big|\sum_{i=1}^{i}(a_{q_i} - a_{q_{i+1}})x_i\Big|.$$

(Ce supremum peut être calculé en utilisant des indices q_i dans l'intervalle $[1, n+1]$).

Si dans l'égalité (1) le supremum est atteint pour une suite $p_1, p_2, \ldots, p_{m+1}$, soit alors $q_1 = 1$, $q_{i+1} = p_i$ $(i = 1, \ldots, m+1)$ et $q_{m+3} > n$, on obtient ainsi:

$$M\Big(\sum_{i=1}^{n} a_i x_i'\Big) \leqq |||(a_1, \ldots, a_n, 0, \ldots)|||_J.$$

D'autre part, si le supremum est atteint dans (2) pour une suite $q_1, q_2, \ldots, q_{l+1}$

$$|||(a_1, \ldots, a_n, 0, \ldots)|||_J = |(a_{q_1} - a_{q_2})x_1 + (a_{q_2} - a_{q_3})x_2 + \ldots + (a_{p_l} - a_{p_{l+1}})x_l|$$

avec $1 \leqq q_i \leqq n+1$.

Le supremum est atteint lorsque $q_1 = 1$ ((x_i) est monotone inconditionnelle). Si $q_{l+1} > n$, alors

$$|||(a_1, \ldots, a_n, 0, \ldots)|||_J \leqq M\Big(\sum_{i=1}^{n} a_i x_i'\Big).$$

Sinon, comme (x_n) est monotone inconditionnelle, on a:

$$|||(a_1, \ldots, a_n, 0, \ldots)|||_J \leqq |(a_1 - a_{q_2})x_1 + (a_{q_2} - a_{q_3})x_2 + \ldots + (a_{q_l} - a_{q_{l+1}})x_l + a_{q_{l+1}}x_{l+1}| \leqq$$

$$\leqq M\Big(\sum_{i=1}^{n} a_i x_i'\Big).$$

Ce qui achève la démonstration du lemme.

Ainsi est établi le

Théorème 2. *Soit (x_n) une base (normalisée) sous-symétrique de F. G est isomorphe à J lorsque (e_n) est une base de J.*

(C'est-à-dire si (x_n) est complète dans F ou équivalente à la base canonique de c_0.)

6. Nous allons maintenant étudier les questions suivantes: quand G est-il isomorphe à c_0 ou l^1? Dans ce qui suit, nous supposons la base (x_n), de F, I.S. monotone inconditionnelle.

Proposition 4. *G est isomorphe à c_0 si et seulement si F est isomorphe à c_0.*

Démonstration. D'après la proposition 1. (4), G est isomorphe à c_0 si et seulement si il existe une constante K positive telle que pour tout naturel n:

$$M\Big(\sum_{i=1}^{n}(-1)^i x_i\Big) \leqq K;$$

si et seulement si il existe une constante K positive telle que pour tout naturel n:

$$\Big|\sum_{i=1}^{n}(-1)^i x_i\Big| \leqq K$$

(par l'équivalence des normes M et $|\cdot|$ sur $(x_{2i-1}-x_{2i})$);

si et seulement si il existe une constante K positive telle que pour tout naturel n:

$$\text{(1)} \qquad \Big|\sum_{i=1}^{n} a_i x_i\Big| \leqq K \sup |a_i|$$

(2) si et seulement si F est isomorphe à c_0

((1) implique (2) puisque (x_i) est basique).

Ce qui établit la proposition.

Passons maintenant à la caractérisation des espaces G isomorphes à l^1. La norme M est E.S.A., donc I.S., ainsi d'après [1], l'espace G est isomorphe à l^1 si et seulement si il existe $K>0$ tel que, pour tout $n \in \mathbf{N}^*$, on ait:

$$\frac{1}{n} M\Big(\sum_{i=1}^{n}(-1)^i x_i\Big) \geqq K.$$

Par un raisonnement analogue à celui fait à la proposition précédente, on obtient la

Proposition 5. *G est isomorphe à l^1 si et seulement si F est isomorphe à l^1.*

7. Dans ce qui suit, nous nous intéressons au problème de la présence de sous-espaces isomorphes à l^p ou c_0 dans G et dans F. Il suffit, en fait, de considérer les sous-espaces engendrés par les suites de blocs consécutifs disjoints sur la base (x_n).

Lemme 2. *F contient un sous-espace isomorphe à c_0 si et seulement si on peut trouver une suite de blocs sur $(x_{2n-1}-x_{2n})$ dans G, équivalente à la base canonique de c_0.*

Démonstration. La condition suffisante est évidente. Maintenant, si F contient c_0, il existe une suite de blocs (u_n) sur (x_n) équivalente à la base canonique de c_0. Soit

$$u_n = \sum_{i=m_{n-1}+1}^{m_n} a_i x_i.$$

On considère alors

$$u_n' = \sum_{i=m_{n-1}+1}^{m_n} a_i(x_{2i-1}-x_{2i}).$$

De l'équivalence des normes M et $|\cdot|$ sur le sous-espace engendré par les vecteurs $(x_{2n-1}-x_{2n})_{n\geq 1}$ on déduit que la suite (u_n') est bornée dans G, ainsi que l'existence d'une constante C positive telle que pour tout $b\in S$

$$M(\sum_i b_i u_i') \leq C\left|\sum_i b_i u_i'\right| \leq 2C\left|\sum_i b_i u_i\right|.$$

La dernière inégalité résulte de l'inégalité triangulaire et de l'invariance par étalement de la suite (x_n). Donc il existe une constante K positive telle que

$$M(\sum_i b_i u_i') \leq K \sup |b_i| \quad \text{pour tout} \quad b\in S.$$

On a aussi

$$M(\sum_i b_i u_i') \geq \frac{1}{K}\sup |b_i|$$

car (u_i') est une suite basique (suite de blocs construite sur la suite basique $(x_{2n}-x_{2n-1})$). Et (u_i') est une suite basique de blocs bornée équivalente à la base canonique de c_0.

Théorème 3. *F contient un sous-espace isomorphe à c_0 si et seulement si G contient un sous-espace isomorphe à c_0.*

Démonstration. Le lemme 2 démontre la condition nécessaire. La condition est suffisante: si G contient c_0, il existe une suite basique de blocs (u_n) bornée dans G, équivalente à la base canonique de c_0. Soit

$$u_n = \sum_{i=p_n+1}^{p_{n+1}} a_i x_i.$$

Il existe une constante K positive telle que pour tout $b\in S$:

$$\frac{1}{K}\sup_i |b_i| \leq M(\sum_i b_i u_i) \leq K \sup_i |b_i|.$$

Par définition de la norme M, il existe une suite d'entiers strictement croissante: $p_1=j_1^{(1)}<j_2^{(1)}<\ldots<j_{m_1+1}^{(1)}=p_2$ telle que:

$$M(u_1) = |(a_{j_1^{(1)}+1}+\ldots+a_{j_2^{(1)}})x_1+(a_{j_2^{(1)}+1}+\ldots+a_{j_3^{(1)}})x_2+\ldots$$
$$\ldots+(a_{j_{m_1}^{(1)}+1}+\ldots+a_{j_{m_1+1}^{(1)}})x_{m_1}|.$$

Posons alors

$$y_1 = (a_{j_1^{(1)}+1}+\ldots+a_{j_2^{(1)}})x_1+\ldots+(a_{j_{m_1}^{(1)}+1}+\ldots+a_{j_{m_1+1}^{(1)}})x_{m_1}.$$

On a $M(u_1)=|y_1|$. Il existe une suite d'entiers $p_2=j_1^{(2)}<\ldots<j_{m_2+1}^{(2)}=p_3$ telle que

$$M(u_2) = |(a_{j_1^{(2)}+1}+\ldots+a_{j_2^{(2)}})x_1+\ldots+(a_{j_{m_2}^{(2)}+1}+\ldots+a_{j_{m_2+1}^{(2)}})x_{m_2}|.$$

Posons

$$y_2 = (a_{j_1^{(2)}+1}+\ldots+a_{j_2^{(2)}})x_{m_1+1}+\ldots+(a_{j_{m_2}^{(2)}+1}+\ldots+a_{j_{m_2+1}^{(2)}})x_{m_1+m_2}.$$

On a par invariance par étalement $M(u_2)=|y_2|$. Par récurrence, on obtient pour tout $n\in\mathbf{N}^*$ une suite d'entiers $j_1^{(n)}, j_2^{(n)}, \ldots, j_{m_n+1}^{(n)}$ strictement croissante telle que: $j_1^{(n)}=p_n$ et $j_{m_n+1}^{(n)}=p_{n+1}$ avec la propriété que, pour chaque n, si l'on pose

$$y_n = \sum_{k=1}^{m_n}\Big(\sum_{i=j_k^{(n)}+1}^{j_{k+1}^{(n)}} a_i\Big)x_{m_0+\ldots+m_{n-1}+k} \quad (\text{avec } m_0 = 0),$$

on a $M(u_n)=|y_n|$. On a évidemment pour tout $b\in S$:

$$\Big|\sum_i b_i y_i\Big| \leqq M\Big(\sum_i b_i u_i\Big).$$

Donc

$$\Big|\sum_i b_i y_i\Big| \leqq K \sup_i |b_i|.$$

(y_i) est une suite basique bornée de blocs donc

$$\Big|\sum_i b_i y_i\Big| \geqq M \sup_i |b_i|.$$

Donc (y_i) est équivalente à la base canonique de c_0.

Considérons maintenant le cas des sous-espaces isomorphes à l^p. Démontrons l'analogue du lemme 2.

Lemme 3. *F contient un sous-espace isomorphe à l^p $(1\leqq p<\infty)$ si et seulement si on peut trouver une suite basique de blocs sur $(x_{2n}-x_{2n-1})$ dans G équivalente à la base canonique de l^p $(1\leqq p<\infty)$.*

Démonstration. Un sens est trivial. D'autre part, s'il existe un $p\in[1, +\infty[$ tel que F contienne l^p, on peut trouver une suite normalisée de blocs (u_n) consécutifs disjoints sur (x_n) équivalente à la base canonique de l^p. Soit

$$u_n = \sum_{i=m_{n-1}+1}^{m_n} a_i x_i, \quad |u_n| = 1 \quad (n = 1, 2, \ldots)$$

Il existe donc deux constantes m et M positives telles que pour tout $b \in S$:

$$m\left(\sum_i |b_i|^p\right)^{1/p} \leqq \left|\sum_i b_i u_i\right| \leqq M\left(\sum_i |b_i|^p\right)^{1/p}.$$

Considérons alors

$$u_n' = \sum_{i=m_{n-1}+1}^{m_n} a_i(x_{2i} - x_{2i-1}) \quad (n = 1, 2, \ldots).$$

La suite (u_n') est bornée dans G. Pour tout $s \in S$, on a:

$$M\left(\sum_i b_i u_i'\right) \geqq \left|\sum_i b_i u_i'\right|$$

(par définition de la norme M), ainsi comme (x_n) est monotone inconditionnelle dans F, on a aussi:

$$M\left(\sum_i b_i u_i'\right) \geqq \left|\sum_i b_i u_i\right|.$$

Donc pour tout $b \in S$:

$$M\left(\sum_i b_i u_i'\right) \geqq m\left(\sum_i |b_i|^p\right)^{1/p}.$$

Puisque les normes M et $|\cdot|$ sont équivalentes sur le sous-espace engendré par les vecteurs $(x_{2i} - x_{2i-1})$, il existe une constante $c > 0$ telle que pour tout $b \in S$:

$$M\left(\sum_i b_i u_i'\right) \leqq c\left|\sum_i b_i u_i'\right| \leqq 2c\left|\sum_i b_i u_i\right|.$$

Donc

$$M\left(\sum_i b_i u_i'\right) \leqq 2cM\left(\sum_i |b_i|^p\right)^{1/p}.$$

Ce qui montre bien que (u_i') est une suite basique bornée de blocs dans G équivalente à la base canonique de l^p.

Nous abordons maintenant le problème suivant: si G contient l^1, F contient-il l^1? Soit (x_n) une base d'un espace de Banach F, nous dirons que (x_n) a la propriété $(\mathscr{P})$ si et seulement si toute suite bornée de blocs consécutifs disjoints sur (x_n) converge vers zéro en moyenne de Cesàro dans F.

Théorème 4. *Si (x_n) a la propriété $(\mathscr{P})$ dans F, alors la base $(x_n')_{n\geqq 1} = (x_n - x_{n-1})_{n\geqq 1}$ est contractante dans G.*

Démonstration. Supposons au contraire que (x_n') ne soit pas contractante dans G. Il existe alors $f \in G^*$, $\delta > 0$ et des blocs u_k consécutifs disjoints, normalisés tels que

$$u_k = \sum_{j=m_{k-1}+1}^{m_k} a_j x_j' \quad \text{et} \quad \forall k, \quad f(u_k) \geqq \delta.$$

Alors, pour tout $n \in \mathbf{N}$,

$$f\left(\frac{1}{2^n}\sum_{i=2^n}^{2^{n+1}-1} u_i\right) \geqq \delta.$$

On peut supposer $\|f\| \leqq 1$, ainsi pour tout $n \in \mathbf{N}$, on a:

$$M\left(\frac{1}{2^n}\sum_{i=2^n}^{2^{n+1}-1} u_i\right) \geqq \delta. \tag{1}$$

Calculons pour n non nul:

$$\frac{1}{2^n} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) = \frac{1}{2^n} M\big(a_{m_{2^n-1}+1}(x_{m_{2^n-1}+1} - x_{m_{2^n-1}}) +$$

$$+ a_{m_{2^n-1}+2}(x_{m_{2^n-1}+2} - x_{m_{2^n-1}+1}) + \ldots + a_{m_{2^{n+1}-1}}(x_{m_{2^{n+1}-1}} - x_{m_{2^{n+1}-1}-1})\big) =$$

$$= \frac{1}{2^n} M\big(-a_{m_{2^n-1}+1} x_{m_{2^n-1}} + (a_{m_{2^n-1}+1} - a_{m_{2^n-1}+2}) x_{m_{2^n-1}+1} +$$

$$+ (a_{m_{2^n-1}+2} - a_{m_{2^n-1}+3}) x_{m_{2^n-1}+2} + \ldots + (a_{m_{2^{n+1}-1}-1} - a_{m_{2^{n+1}-1}}) x_{m_{2^{n+1}-1}-1} +$$

$$+ a_{m_{2^{n+1}-1}} x_{m_{2^{n+1}-1}}\big).$$

Supposons que pour chaque n non nul, le supremum soit atteint pour une suite de naturels $p_1^{(n)}, \ldots, p_{l(n)+1}^{(n)}$ strictement croissante, alors:

$$\frac{1}{2^n} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) = \frac{1}{2^n}\Big|-a_{p_1^{(n)}} x_1 + \sum_{i=1}^{l(n)} (a_{p_i^{(n)}} - a_{p_{i+1}^{(n)}}) x_{i+1} + a_{p_{l(n)+1}^{(n)}} x_{l(n)+2}\Big|.$$

On peut écrire:

$$\frac{1}{2^n} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) = \frac{1}{2^n}\Big|\sum_{i=2^n}^{2^{n+1}-1} v_n^{(i)} + \sum_{i=2^n}^{2^{n+1}-1} w_i^{(n)} + a_{p_{l(n)+1}^{(n)}} x_{l(n)+2}\Big|.$$

Où chaque $v_i^{(n)}$ est 0 ou une somme de termes de la forme

$$\sum (a_{p_j^{(n)}} - a_{p_{j+1}^{(n)}}) x_{j+1}$$

avec chaque a_k provenant du même bloc u_i. Où chaque $w_i^{(n)}$ est 0 ou de la forme $-a_k x_j$ ou $(a_{p_j^{(n)}} - a_{p_{j+1}^{(n)}}) x_{j+1}$ avec les a_k provenant de deux blocs u_j différents.

Faisons varier n:

pour $n=1$, on obtient deux blocs consécutifs disjoints sur $(x_n) = v_2^{(1)}, v_3^{(1)}$ dont les coefficients proviennent de u_2 et u_3 respectivement, ainsi $|v_2^{(1)}| \leqq M(u_2) = 1$ et $|v_3^{(1)}| \leqq M(u_3) = 1$; pour $n=2$, on obtient quatre blocs consécutifs disjoints sur $(x_n) = v_4^{(2)}, v_5^{(2)}, v_6^{(2)}, v_7^{(2)}$ et $|v_i^{(2)}| \leqq M(u_i) = 1$ pour $i = 4, \ldots, 7$.

Et ainsi de suite, pour un n quelconque non nul, on obtient 2^n blocs sur (x_n): $=v_{2^n}^{(n)}, v_{2^n+1}^{(n)}, \ldots, v_{2^{n+1}-1}^{(n)}$ dont les coefficients proviennent respectivement de $u_{2^n}, u_{2^n+1}, \ldots, u_{2^{n+1}-1}$, ainsi $|v_i^{(n)}| \leqq M(u_i) = 1$ pour tout $i=2^n, \ldots, 2^{n+1}-1$.

Formons la suite:

$$(v_2^{(1)}, v_3^{(1)}, v_4^{(2)}, v_5^{(2)}, v_6^{(2)}, v_7^{(2)}, \ldots)$$

suite que nous noterons:

$$(v_2, v_3, v_4, v_5, v_6, v_7, \ldots) = (v_i)_{i \geqq 2}.$$

Comme (x_n) est invariante par étalement dans F, on peut supposer que les v_i sont des blocs consécutifs disjoints sur (x_n). La suite $(v_i)_{i\geqq 2}$ est bornée (par 1). De même, formons la suite

$$(w_2^{(1)}, w_3^{(1)}, w_4^{(2)}, w_5^{(2)}, w_6^{(2)}, w_7^{(2)}, \ldots).$$

Nous la noterons:

$$(w_2, w_3, w_4, w_5, w_6, w_7, \ldots) = (w_i)_{i \geqq 2}.$$

Nous supposerons aussi que ces blocs sont consécutifs disjoints. Comme pour tout i et $k=m_{i-1}+1, \ldots, m_i$

$$|a_k| \leqq M(u_i) = 1,$$

la suite $(w_i)_{i\geqq 2}$ est bornée (par 2). Par hypothèse, les suites $(v_i)_{i\geqq 2}$ et $(w_i)_{i\geqq 2}$ convergent en moyenne de Cesàro vers zéro, on peut donc trouver un naturel N tel que pour tout $n \geqq N$, on ait:

$$\frac{1}{2^{n+1}}\left|\sum_{i=2}^{2^{n+1}+1} v_i\right| \leqq \frac{\delta}{10}, \quad \frac{1}{2^{n+1}}\left|\sum_{i=2}^{2^{n+1}+1} w_i\right| \leqq \frac{\delta}{10}$$

et

$$\frac{1}{2^n}\left|a_{p_{l(n)}^{(n)}+1}\, x_{l(n)+2}\right| \leqq \frac{1}{2^n} \leqq \frac{\delta}{5}.$$

Ainsi:

$$\frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)}\right| = \frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i\right|.$$

Et par la monotone inconditionnalité de (x_n) dans F, on a:

$$\frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)}\right| \leqq \frac{1}{2^n}\left|\sum_{i=2}^{2^{n+1}+1} v_i\right|.$$

Et pour $n \geqq N$:

$$\frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)}\right| \leqq \frac{\delta}{5}.$$

De même:

$$\frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} w_i^{(n)}\right| = \frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} w_i\right| \leqq \frac{1}{2^n}\left|\sum_{i=2}^{2^{n+1}+1} w_i\right|.$$

Et pour $n \geqq N$:

$$\frac{1}{2^n}\left|\sum_{i=2^n}^{2^{n+1}-1} w_i^{(n)}\right| \leqq \frac{\delta}{5}.$$

Ainsi, pour $n \geqq N$, on aurait

$$\frac{1}{2^n} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) \leqq \frac{3\delta}{5}$$

ce qui contredit (1) donc (x_n') est une base contractante de G.

La méthode utilisée dans la démonstration du théorème 4 permet d'obtenir également des renseignements sur la présence de sous-espaces isomorphes à l^p $(1 \leqq p < \infty)$:

Théorème 5. *S'il existe un réel $p \in [1, +\infty[$ tel que toute suite bornée de blocs w_i consécutifs disjoints sur (x_n) a dans F la propriété suivante:*

$$\lim_{n \to +\infty} n^{-1/p}\left|\sum_{i=1}^{n} w_i\right| = 0$$

alors, quel que soit $q \in [1, p]$, G ne contient aucun sous-espace isomorphe à l^q.

Démonstration. Supposons le contraire: il existerait un réel $q \in [1, p]$ et des blocs normalisés u_n dans G consécutifs disjoints, équivalents à la base canonique de l^q. On pourrait donc trouver deux constantes m, M positives telles que, pour tout $n \in \mathbf{N}$:

$$m \leqq M\Big(2^{-n/q} \sum_{i=2^n}^{2^{n+1}-1} u_i\Big) \leqq M. \tag{1}$$

On peut écrire:

$$2^{-n/q} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) = 2^{-n/q}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)} + \sum_{i=2^n}^{2^{n+1}-1} w_i^{(n)} + a_{p_{l(n)}^{(n)}+1}\, x_{l^{(n)}+2}\right|.$$

où les $v_i^{(n)}$ et $w_i^{(n)}$ sont définis comme dans la démonstration du théorème 4.

On obtient, de même, deux suites bornées de blocs consécutifs disjoints sur (x_n):

$$(v_2^{(1)}, v_3^{(1)}, v_4^{(2)}, v_5^{(2)}, \ldots) = (v_i)_{i \geqq 2}, \quad (w_2^{(1)}, w_3^{(1)}, w_4^{(2)}, w_5^{(2)}, \ldots) = (w_i)_{i \geqq 2}.$$

Ainsi, par hypothèse, comme $q \leqq p$, on a:

$$\lim_{n \to +\infty} n^{-1/q}\left|\sum_{i=2}^{n+1} v_i\right| = 0 \quad \text{et} \quad \lim_{n \to +\infty} n^{-1/q}\left|\sum_{i=2}^{n+1} w_i\right| = 0.$$

On peut donc trouver un naturel N tel que pour tout $n \geqq N$, on ait

$$2^{-(n+1)/q}\left|\sum_{i=2}^{2^{n+1}+1} v_i\right| \leqq \frac{m}{10} \quad \text{et} \quad 2^{-(n+1)/q}\left|\sum_{i=2}^{2^{n+1}+1} w_i\right| \leqq \frac{m}{10}$$

et

$$2^{-n/q}|a_{p^{(n)}_{j(n)}+1}| \leqq \frac{m}{5}.$$

Ainsi:

$$2^{-n/q}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)}\right| \leqq 2^{-n/q}\left|\sum_{i=2}^{2^{n+1}+1} v_i\right|.$$

Pour $n \geqq N$:

$$2^{-n/q}\left|\sum_{i=2^n}^{2^{n+1}-1} v_i^{(n)}\right| \leqq 2^{1/q}\frac{m}{10} \leqq \frac{m}{5}, \quad 2^{-n/q}\left|\sum_{i=2^n}^{2^{n+1}-1} w_i^{(n)}\right| \leqq \frac{m}{5}.$$

Ainsi, pour $n \geqq N$, on aurait:

$$2^{-n/q} M\Big(\sum_{i=2^n}^{2^{n+1}-1} u_i\Big) \leqq \frac{3}{5} m.$$

Ce qui contredit (1).

Remarque. S'il existe un réel $p \in]1, +\infty[$, tel que (x_n) soit p-Hilbertienne sur des blocs dans F, (x_n) vérifie la condition:

$$\lim_{n\to+\infty} n^{-1/q}\left|\sum_{i=1}^{n} w_i\right| = 0,$$

pour tout $q \in [1, p[$ et pour toute suite bornée de blocs (w_i) consécutifs disjoints sur (x_n).

Donc G ne contient aucun sous-espace isomorphe à l^q.

De la démonstration du théorème 2 de [4], on déduit le

Théorème 6. *Si (x_n) a la propriété $(\mathscr{P})$ dans F, alors (x_n) est une base* 1-*contractante de G.*

Démonstration. Soient dans $G^*, x_1^*, x_2^*, \ldots$ (respectivement $f_1, f_2, \ldots$) les fonctionnelles biorthogonales de $x_1, x_2, \ldots$ (respectivement $x_1', x_2', \ldots$). Puisque (x_n') est une base contractante de G (théorème 4), on a $G^* = \overline{\text{span}}\,\{f_n\}_{n\geqq 1}$, de plus (x_n) étant une base E.S.A. dans G, $(f_n)_{n\geqq 2}$ est une base E.S.A. de $\overline{\text{span}}\,\{f_n\}_{n\geqq 2}$ [4] et $(f_2, f_2-f_3, f_3-f_4, \ldots)$ est une suite basique [4]. Donc $(f_1, f_1-f_2, \ldots) = (f_1, x_1^*, \ldots)$ est une base de G^* ce qui signifie que (x_n) est 1-contractante dans G.

Théorème 7. *Si F ne contient pas de sous-espace isomorphe à c_0, si (x_n) a la propriété $(\mathscr{P})$ dans F, alors G est quasi-réflexif d'ordre* 1.

Démonstration. La première hypothèse faite sur F implique que (x_n) est complète et la seconde que (x_n) est 1-contractante donc G est quasi-réflexif d'ordre 1 [11].

Nous allons maintenant étudier la propriété $(\mathscr{P})$.

1. *Si (x_n) a la propriété $(\mathscr{P})$, F ne contient pas l^1.* Par contre, il existe des espaces ne contenant pas l^1 et dont la base n'a pas la propriété $(\mathscr{P})$. Nous donnons deux exemples; dans le premier, F est muni d'une base monotone inconditionnelle, dans le second d'une base symétrique,

Exemple 1. Soit F l'espace dual T de l'espace de TSIRELSON [8], les vecteurs unités (t_n) forment une base monotone inconditionnelle de T, celui-ci est réflexif, il ne contient donc pas l^1 et (t_n) n'a pas la propriété $(\mathscr{P})$ dans T (il existe en effet une constante C positive telle que pour tout k on ait

$$\Big\|\sum_{h=k+1}^{2k} w_k\Big\|_T \geqq C\cdot k$$

où (w_k) est une suite basique de blocs bornée sur (t_n)).

Exemple 2. Choisissons pour F l'espace Y dû à ALTSHULER [8] (obtenu en modifiant l'exemple de Tsirelson); les vecteurs unités (e_n) forment une base symétrique (constante de symétrie 1), Y est réflexif et (e_n) n'a pas la propriété $(\mathscr{P})$: par la proposition 3.b.4. de [8], on voit qu'il existe dans Y une suite basique de blocs (u_j) équivalente à la base (t_n) de T, ainsi

$$\frac{1}{n}\sum_{j=n+1}^{2n} u_j$$

ne tend pas vers zéro dans Y quand n tend vers l'infini.

Supposons (x_n) I.S. inconditionnelle dans F.

2. *Si (y_k) est une suite bornée de blocs sur (x_n) dans F, et si*

$$\frac{1}{n}\sum_{k=1}^{n} y_k$$

converge dans F, alors c'est vers zéro. En effet, soit

$$s_n = \frac{1}{n}\sum_{k=1}^{n} y_k;$$

on a $s_n - s_{2n} \to 0$,

$$|s_n - s_{2n}| = \left|\frac{1}{2n}\Big(\sum_{k=1}^{n} y_k - \sum_{k=n+1}^{2n} y_k\Big)\right|$$

et

$$|s_n - s_{2n}| \geqq \frac{C}{2n}\Big|\sum_{k=1}^{n} y_k + \sum_{k=n+1}^{2n} y_k\Big| = \frac{C}{2n}\Big|\sum_{k=1}^{2n} y_k\Big|$$

puisque (x_n) est inconditionnelle. Ainsi $s_{2n} \to 0$. Et

$$|s_n - s_{2n}| \geqq \frac{C}{2n}\Big|\sum_{k=1}^{2n+1} y_k - y_{2n+1}\Big| \geqq \frac{C(2n+1)}{2n}|s_{2n+1}| - \frac{C}{2n}\sup_n |y_n|.$$

Et $s_{2n+1} \to 0$.

3. Si (x_n') a la propriété $(\mathscr{P})$ dans G, (x_n') est contractante dans G.

Théorème 8. *Si (x_n') a la propriété $(\mathscr{P})$ dans G, (x_n) a la propriété $(\mathscr{P})$ dans F.*

Démonstration. Soit une suite normalisée de blocs dans F:

$$u_j = \sum_{k=m_{j-1}+1}^{m_j} a_k x_k,$$

il existe une constante C positive telle que:

$$\left|\frac{1}{n}\sum_{j=1}^{n} u_j\right| \leqq \frac{C}{n} M\Big(\sum_{j=1}^{n}\Big(\sum_{k=m_{j-1}+1}^{m_j} a_k(x_{2k-1}-x_{2k})\Big)\Big) \leqq \frac{C}{n} M\Big(\sum_{j=1}^{n} w_j\Big)$$

avec

$$w_j = \sum_{k=m_{j-1}+1}^{m_j} a_k(x_{2k-1}-x_{2k}).$$

On peut considérer les w_j comme des blocs sur (x_n'), c'est une suite bornée dans G et

$$\left|\frac{1}{n}\sum_{j=1}^{n} u_j\right| \to 0.$$

4. Si F est B-convexe, alors (x_n) a la propriété $(\mathscr{P})$; mais c_0 n'est pas B-convexe et cependant la base canonique de c_0 a la propriété $(\mathscr{P})$.

5. Si (x_n) a la propriété $(\mathscr{P})$ dans F, alors (x_n) a la propriété de Banach—Saks faible: en effet, soit (z_n) une suite dans F qui tend faiblement vers zéro, il existe alors une sous-suite (z_n') de (z_{n_n}) équivalente à une suite de blocs sur (x_n), ainsi

$$\frac{1}{n}\sum_{i=1}^{n} z_i'$$

tend vers zéro dans F. Mais l^1 a la propriété de Banach—Saks faible et la base canonique de l^1 n'a pas la propriété $(\mathscr{P})$.

P. G. Casazza et R. H. Lohman ont montré dans [9] que si (x_n) est une base symétrique, p-Hilbertienne « sur des blocs » dans F $(1 \leqq p < \infty)$ alors (e_n) est p-Hilbertienne « sur des blocs » dans J et donc (e_n) est une suite (basique) contractante dans J. Nous allons maintenant nous intéresser à la propriété: « être p-Hilbertienne sur des blocs » pour une base. Remarquons d'abord que si (x_n) est p-Hilbertienne sur des blocs pour un $p \in]1, +\infty[$ alors (x_n) a la propriété $(\mathscr{P})$.

Dans ce qui suit, nous supposerons toujours (x_n) I.S., monotone inconditionnelle. Les résultats obtenus s'étendent au cas des suites sous-symétriques. Nous aurons besoin de la notion de « B-convexité sur des blocs »:

Définition. Nous dirons que (x_n) est B-convexe sur des blocs si, et seulement si, il existe un naturel k (non nul), $\varepsilon\in]0, 1]$ tels que pour tout k-uple $(w_1, \dots, w_k)$ de blocs consécutifs disjoints sur (x_n), on ait:

$$\left|\sum_{i=1}^{k} w_i\right| \leqq k(1-\varepsilon)\sup_i |w_i|.$$

Pour démontrer le théorème 9, nous utilisons la méthode de B. MAUREY et G. PISIER dans [10]; les démonstrations de ce qui suit n'étant que des adaptations de celles de [10], nous n'en donnerons pas les détails.

Théorème 9. *Il existe $p\in]1, +\infty[$ tel que (x_n) est p-Hilbertienne «sur des blocs» si et seulement si (x_n) est B-convexe «sur des blocs».*

Démonstration. Pour cela, nous définissons, pour tout entier k, le nombre λ_k comme la plus petite constante positive λ, vérifiant, pour tout k-uple $(w_1, \dots, w_k)$ de blocs consécutifs disjoints sur (x_n)

$$\left|\sum_{i=1}^{k} w_i\right| \leqq \lambda\cdot k\cdot\sup_i |w_i|.$$

On a alors la

Proposition 6. (1) $\forall k\in\mathbf{N}_*$: $0\leqq\lambda_k\leqq 1$;
(2) $\forall k\in\mathbf{N}_*$: $\lambda_k\geqq 1/k$;
(3) $\forall n, k\in\mathbf{N}_*$: $(n+k)\lambda_{n+k}\leqq n\lambda_n+k\lambda_k$;
(4) $\forall n, m\in\mathbf{N}_*$: $n\leqq m\Rightarrow n\lambda_n\leqq m\lambda_m$.

De plus, l'application: $n\to\lambda_n$ est sous-multiplicative:

Lemme 4. $\forall k, n\in\mathbf{N}_*$: $\lambda_{nk}\leqq\lambda_n\lambda_k$.

Preuve. Soit $(w_j)_{1\leqq j\leqq nk}$ un nk-uple de blocs consécutifs disjoints sur (x_n). Pour chaque $i\in\{1, \dots, n\}$, on pose

$$W_i = \sum_{(i-1)k<j\leqq ik} w_j.$$

Et

$$\left|\sum_{j=1}^{nk} w_j\right| = \left|\sum_{i=1}^{n} W_i\right| \leqq n\lambda_n \sup_{1\leqq i\leqq n} |W_i| \leqq n\lambda_n k\lambda_k \sup_{1\leqq j\leqq nk} |w_j|.$$

Ce qui entraîne bien: $\lambda_{nk}\leqq\lambda_n\cdot\lambda_k$.

Remarque. De ce lemme, on déduit que (x_n) est B-convexe sur des blocs si et seulement si $\lambda_n\to 0$.

Lemme 5. *Si $\lambda_N=1/N^{1/p'}$ pour un entier $N>1$ et un réel p' dans $[1, +\infty[$ alors (x_n) est q-Hilbertienne sur des blocs pour tout $q<p$, où p est défini par $1/p+1/p'=1$.*

Preuve. Soit q (avec $q<p$) et soient $w_1, \ldots, w_l$ des blocs consécutifs disjoints sur (x_n). On pose:

$$\forall k\in\mathbf{N},\quad A_k=\left\{n\colon \left(\frac{\sum_i |w_i|^q}{N^{k+1}}\right)^{1/q} \leqq |w_n| \leqq \left(\frac{\sum_i |w_i|^q}{N^k}\right)^{1/q}\right\}.$$

Soit $|A_k|$ le cardinal de A_k. On a:

$$\Big(\sum_i |w_i|^q\Big)^{1/q} \geqq \Big(\sum_{i\in A_k} |w_i|^q\Big)^{1/q} \geqq |A_k|^{1/q} \frac{\big(\sum_i |w_i|^q\big)^{1/q}}{N^{(k+1)/q}}.$$

D'où: $|A_k|\leqq N^{k+1}$. Et $|A_k|\lambda_{|A_k|}\leqq N^{k+1}\lambda_{N^{k+1}}\leqq N^{k+1}(\lambda_N)^{k+1}$ (par la proposition 6. (4) et le lemme 4). Mais on a aussi:

$$\Big|\sum_i w_i\Big| \leqq \sum_{k=0}^{+\infty}\Big|\sum_{i\in A_k} w_i\Big| \leqq \sum_{k=0}^{+\infty} |A_k|\lambda_{|A_k|}\frac{\big(\sum_i |w_i|^q\big)^{1/q}}{N^{k/q}} \leqq \sum_{k=0}^{+\infty}\frac{N^{k+1}N^{-(k+1)/q'}}{N^{k/q}}\Big(\sum_i |w_i|^q\Big)^{1/q},$$

$$\Big|\sum_i w_i\Big| \leqq \frac{N^{1/q}}{N^{1/q-1/p}}\Big(\sum_i |w_i|^q\Big)^{1/q}.$$

Ce qui achève la démonstration puisque on a alors pour tout $a\in S$

$$\Big|\sum_i a_i w_i\Big| \leqq C \sup |w_i| \Big(\sum_i |a_i|^q\Big).$$

Terminons la démonstration du théorème 9: d'après la remarque qui suit le lemme 4, (x_n) est B-convexe sur des blocs si et seulement si $\lambda_n \to 0$, donc si, et seulement si, il existe un réel p' tel que $\lambda_N = 1/N^{1/p'}$ pour un entier $N>1$. On applique le lemme 5 et on obtient le résultat annoncé. On peut aussi montrer la

Proposition 7. *Les propriétés suivantes sont équivalentes:*

(1) *(x_n) est B-convexe sur des blocs.*

(2) *$\forall \lambda$ fini, F ne contient pas de l_n^1 λ-uniformément sur des blocs de (x_n).*

(3) *$\exists \lambda>1$, F ne contient pas de l_n^1 λ-uniformément sur des blocs de (x_n).*

Cette dernière assertion revient à dire que l^1 n'est pas finiment représenté en blocs dans (x_n).

Remarques. 1) Si F est B-convexe, alors (x_n) est B-convexe sur des blocs, mais c_0 n'est pas B-convexe et sa base canonique est B-convexe sur des blocs.

2) (x_n) est B-convexe sur des blocs dans F si et seulement si (e_n) est B-convexe sur des blocs dans J; mais F B-convexe n'implique pas J B-convexe (considérer $F=l^2$).

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Ortholattis linéarisables

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I. Préliminaires

On rencontre souvent dans différentes branches des mathématiques la méthode de « linéarisation ». Cela signifie que, étant donné une structure mathématique quelconque, on lui associe une structure linéaire en termes duquel on peut formuler d'intéressantes propriétés pour la structure initiale. Ceci nous permet d'utiliser les résultats les plus profonds de l'analyse fonctionnelle pour la solution de certains problèmes non linéaires.

Dans ce qui suit nous allons introduire un type d'ortholattis, dits linéarisables, auxquels nous pouvons appliquer la méthode de « linéarisation » au sens général esquissé ci-dessus.

Nous n'avons pas l'intention de donner ici un exposé complet de la théorie des ortholattis linéarisables. Nous définissons seulement les notions les plus élémentaires, ensuite nous examinons la question de leur caractérisation algébrique.

Tout d'abord nous faisons quelques remarques sur la terminologie.

Si L est un ortholattis quelconque, par L^* désignons l'ensemble de toutes les fonctions p de L dans l'intervalle $[0, 1]$ vérifiant les axiomes suivants:

($\mathrm{E_I}$) $p(1)=1$, où par **1** on note le plus grand élément de L.

($\mathrm{E_{II}}$) Pour tout système fini orthogonal $(e_i)_{i\in I}$ d'éléments de L soit $p(\bigvee_{i\in I} e_i)= = \sum_{i\in I} p(e_i)$.

Les éléments de L^* seront appelés *états* sur L; cette dénomination est motivée par les applications de la théorie des ortholattis en physique mathématique.

Nous dirons qu'un ortholattis L est *séparé* si l'ensemble L^* sépare les points de L, c'est-à-dire pour tous $e, f \in L$, $e \neq f$ il existe $p \in L^*$, tel que $p(e) \neq p(f)$. Il est facile de voir qu'un ortholattis séparé est nécessairement orthomodulaire, mais

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en général la réciproque n'est pas vraie. Par ailleurs, presque tous les ortholattis qu'on rencontre dans les applications sont séparés.

Définition. On dira que l'ortholattis séparé L est *linéarisable* s'il existe un espace normé réel E et une application $w\colon L\to E$, tels que

($\mathrm{OL_I}$) w est additive, c'est-à-dire pour $e, f\in L$, $e\perp f$ on a $w(e\vee f)=w(e)+w(f)$;

($\mathrm{OL_{II}}$) $\sup\{\tau(w(e))\,|\,\tau\in E', \|\tau\|\leqq 1, \tau\circ w\in L^*\}=\|w(e)\|=1$ pour tout $e\in L$, $e\neq \mathbf{0}$, où par $\mathbf{0}$ on désigne le plus petit élément de L et E' est le dual topologique de l'espace normé E;

($\mathrm{OL_{III}}$) pour tout $p\in L^*$ il existe une fonctionnelle linéaire τ sur E, telle que $p=\tau\circ w$ et $\|\tau\|\leqq 1$.

Si (E, w) est une couple satisfaisant aux axiomes ($\mathrm{OL_I}$), ($\mathrm{OL_{II}}$) et ($\mathrm{OL_{III}}$), on dira qu'elle est une *linéarisation* de L.

Par la suite nous donnerons une caractérisation purement algébrique des ortholattis linéarisables. Pour le moment occupons-nous de leurs propriétés plus simples.

Si L est un ortholattis linéarisable et (E, w) est une linéarisation de L, alors l'application w est injective. En effet, si $e, f\in L$, $w(e)=w(f)$, alors pour toute fonctionnelle linéaire τ sur E on a $\tau(w(e))=\tau(w(f))$, d'où compte tenu de ($\mathrm{OL_{III}}$) on déduit que $p(e)=p(f)$ pour tout $p\in L^*$; mais alors $e=f$, car l'ortholattis L est séparé.

D'autre part, en désignant par E_0 le complété du sous-espace normé de E engendré par l'image de w, on voit aisément que la couple (E_0, w) est également une linéarisation de L. Ceci montre que nous aurions pu énoncer la définition des ortholattis linéarisables admettant l'espace normé E complet et l'ensemble $w(L)$ total dans l'espace de Banach E.

Soit L un ortholattis linéarisable quelconque. Posons une linéarisation (E, w) de L admettant que E soit complet et l'ensemble $w(L)$ soit total dans E. Désignons par E' l'espace dual topologique de E et définissons l'ensemble

$$K := \{\tau\in E'\,|\,\tau\circ w\in L^*, \|\tau\|\leqq 1\}.$$

Il est clair que K est un sous-ensemble convexe dans l'espace vectoriel réel E', de plus il est faiblement compact. En effet, on vérifie sans peine que K est faiblement fermé et contenu dans la boule unité de E' qui est faiblement compact. Considérons maintenant l'application

(1) $$K\to L^*;\ \tau\mapsto\tau\circ w.$$

Cette application est bijective; en effet, elle est injective car l'ensemble $w(L)$ est total dans E, d'autre part de l'axiome ($\mathrm{OL_{III}}$) il découle qu'elle est surjective. Par ailleurs, il est évident que L^* peut être considéré comme un ensemble convexe dans l'espace vectoriel produit $\mathbf{R}^L$ et d'après le théorème de Tikhonov il est compact pour la topologie produite de $\mathbf{R}^L$. Celá étant, dans la suite nous considérerons l'ensemble des états L^* comme un ensemble convexe compact dont le structure est induite par celle d'espace localement convexe produit $\mathbf{R}^L$. Or, il est évident que l'application (1) est un homéomorphisme entre les espaces topologiques compacts K et L^*, conservant tous les combinaisons convexes finies. Ceci montre qu'on peut identifier entre eux les ensembles convexes compacts K et L^*. Plus loin nous fairons usage de ce résultat.

Remarquons que nous avons de nombreux exemples pour des ortholattis linéarisables. Dans la suite nous verrons que l'ortholattis de projecteurs d'une C^*-algèbre de Baer (cf. [1], 1.3) est linéarisable. Par conséquent, tous les ortholattis de von Neumann (en particulier: tous les ortholattis hilbertiens) sont linéarisables. Il en est de même pour les ortholattis boréliens des espaces topologiques séparés. Plus loin nous verrons que tout ortholattis distributif est linéarisable.

II. Deux lemmes

Dans ce numéro nous prouvons deux lemmes nécessaires pour la suite. D'abord introduisons quelques notations.

Si X est un ensemble, par 1_X désignons l'application associant à tout élément de X le nombre 1.

Étant donné un espace topologique compact X, on désigne par $C(X; \mathbf{R})$ l'espace vectoriel réel des applications numériques continues définies dans X muni de la structure définie par la sup-norme.

Soit K un ensemble convexe compact dans l'espace vectoriel topologique réel X. Désignons par $A(K; \mathbf{R})$ le sous-espace vectoriel de $C(K; \mathbf{R})$ dont les éléments conservent toutes les combinaisons convexes finies d'éléments de K. Si σ est une fonctionnelle linéaire continue sur X, alors $(\sigma + \lambda 1_X)|_K \in A(K; \mathbf{R})$ pour tout $\lambda \in \mathbf{R}$. A noter que $A(K; \mathbf{R})$ est un sous-espace vectoriel fermé de l'espace de Banach $C(K; \mathbf{R})$.

Lemme 1. *Soient K un ensemble convexe compact dans l'espace localement convexe réel séparé X et $\mathscr{E}$ un sous-espace vectoriel de $C(K; \mathbf{R})$, tel que $\mathscr{E} \subset A(K; \mathbf{R})$ et $1_K \in \mathscr{E}$. Munissons $\mathscr{E}$ de la structure d'espace normé induit par celle de $C(K; \mathbf{R})$. Alors, pour toute fonctionnelle linéaire μ sur $\mathscr{E}$ les propositions suivantes sont équivalentes:*

(a) *Il existe $p \in K$ tel que pour la mesure de Radon δ_p concentrée en le point p soit $\delta_p|_\varepsilon = \mu$.*

(b) $\|\mu\| = \mu(1_K) = 1$.

Démonstration. Il est évident que (a) entraîne (b). Pour établir l'implication inverse, prenons une fonctionnelle linéaire μ sur $\mathscr{E}$ vérifiant (b). D'après le théorème de Hahn—Banach il existe une fonctionnelle linéaire $\bar{\mu}$ sur $C(K; \mathbf{R})$ prolongeant μ et ayant la même norme. Or, $\bar{\mu}(1_K)=\mu(1_K)=1=\|\mu\|=\|\bar{\mu}\|$, donc un résultat bien connu dans la théorie de la mesure nous dit que la mesure de Radon $\bar{\mu}$ sur l'espace compact K est positive (cf. [4], Ch. V, § 5, n°5, prop. 9). Par suite, de la convexité de K il découle qu'on peut prendre le barycentre de la mesure $\bar{\mu}$; c'est-à-dire le point $p\in K$ bien déterminé par la condition suivante: pour toute fonctionnelle linéaire continue σ sur l'espace localement convexe X on a

$$\sigma(p) = \int_K \sigma(p')\, d\bar{\mu}(p')$$

(cf. [4], Ch. IV, § 7, n°1, cor. de la prop. 1).

Prouvons maintenant que $\delta_p|_{\mathscr{E}}=\mu$. Soit $\varphi\in\mathscr{E}$ arbitraire. En appliquant un résultat de Mokobodzki (cf. [3], Ch. XI, § 1, T6) on obtient l'existence d'une suite de fonctionnelles linéaires continues $(\sigma_n)_{n\in\mathbf{N}}$ sur X et d'une suite numérique $(\lambda_n)_{n\in\mathbf{N}}$, telles que la suite des fonctions $(\lambda_n 1_X+\sigma_n)_{n\in\mathbf{N}}$ converge vers φ uniformément sur K. Par la définition du point p on a

$$\int_K (\lambda_n+\sigma_n(p'))\, d\bar{\mu}(p') = \lambda_n+\sigma_n(p)$$

pour tout $n\in\mathbf{N}$, donc

$$\delta_p|_{\mathscr{E}}(\varphi) = \varphi(p) = \lim_{n\to\infty} (\lambda_n+\sigma_n(p)) = \lim_{n\to\infty} \int_K (\lambda_n 1_X+\sigma_n)\, d\bar{\mu} =$$

$$= \int_K \lim_{n\to\infty} (\lambda_n+\sigma_n(p'))\, d\bar{\mu}(p') = \int_K \varphi\, d\bar{\mu} = \bar{\mu}(\varphi) = \mu(\varphi).$$

Celà achève la démonstration du lemme.

Lemme 2. *Soient K un ensemble convexe compact dans un espace vectoriel topologique réel séparé et $(\varphi_i)_{i\in I}$ un système fini de $A(K; \mathbf{R})$, tel que*

$$\text{(i)}\quad \|\varphi_i\| = 1 \quad (i\in I) \quad \text{et} \quad \text{(ii)}\quad \sum_{i\in I} |\varphi_i| \leqq 1,$$

où $\|\cdot\|$ désigne la norme d'espace de Banach $C(K; \mathbf{R})$. Alors, pour tout système $(\lambda_i)_{i\in I}\in\mathbf{R}^I$ on a

$$\Big\|\sum_{i\in I} \lambda_i\varphi_i\Big\| = \max_{i\in I} |\lambda_i|.$$

Démonstration. Si $p\in K$, alors

$$\Big|\Big(\sum_{i\in I} \lambda_i\varphi_i\Big)(p)\Big| \leqq \big(\max_{i\in I} |\lambda_i|\big) \sum_{i\in I} |\varphi_i(p)| \leqq \max_{i\in I} |\lambda_i|, \quad \text{donc} \quad \Big\|\sum_{i\in I} \lambda_i\varphi_i\Big\| \leqq \max_{i\in I} |\lambda_i|.$$

Réciproquement, choisissons un indexe $k \in I$ arbitraire. Alors, de (i) il découle que $\sup_{p \in K} |\varphi_k(p)| =: \|\varphi_k\| = 1$, ainsi en vertu du théorème de Weierstrass il existe $p_k \in K$, tel que $|\varphi_k(p_k)| = 1$. Mais alors (ii) entraîne que pour tout $i \in I$: $|\varphi_i(p_k)| = \delta_{ik}$, par suite

$$|\lambda_k| = \Big|\sum_{i \in I} \lambda_i \varphi_i(p_k)\Big| \leqq \Big\|\sum_{i \in I} \lambda_i \varphi_i\Big\|, \quad \text{donc} \quad \max_{k \in I} |\lambda_k| \leqq \Big\|\sum_{i=I} \lambda_i \varphi_i\Big\|.$$

III. Caractérisation des ortholattis linéarisables

On sait qu'un *sous-ortholattis* B de l'ortholattis L est un sous-ensemble de L contenant $\mathbf{1}$, tel que pour tous $e, f \in B$ on a $e \wedge f^{\perp} \in B$; munit de la structure induite par celle de L, B se transforme en un ortholattis.

Rappelons qu'un *orthohomomorphisme* d'un ortholattis B dans un autre L est une application u de B dans L satisfaisant aux conditions suivantes:

$$u(\mathbf{1}) = \mathbf{1}, \quad u(e^{\perp}) = u(e)^{\perp}, \quad u(e \vee f) = u(e) \vee u(f) \quad (e, f \in L).$$

Dans ce numéro nous étudierons la question suivante. Soit L un ortholattis séparé. A quelles conditions supplémentaires doit satisfaire la structure de L, pour que l'ortholattis L soit linéarisable? En d'autres termes, comment caractériser les ortholattis linéarisables parmi tous les ortholattis séparés? Telle caractérisation est donnée par le théorème suivant.

Théorème. *Si L est un ortholattis séparé, les propositions suivantes sont équivalentes:*

(a) *Pour tout $e \in L$, $e \neq \mathbf{0}$ il existe $p \in L^*$, tel que $p(e) = 1$.*

(a') *Pour tout $e \in L$, $e \neq \mathbf{0}$ on a $\sup_{p \in L^*} p(e) = 1$.*

(b) *L est linéarisable.*

(c) *Pour tous les sous-ortholattis distributifs B de L et $p \in B^*$ il existe $\bar{p} \in L^*$ prolongeant p à L tout entier.*

(c') *Si B est un ortholattis distributif, u: $B \to L$ un orthohomomorphisme, $p \in B^*$ et*

$$\{b \in B \mid u(b) = \mathbf{0}\} \subset \{b \in B \mid p(b) = 0\},$$

alors il existe $\bar{p} \in L^*$ mettant le diagramme suivant commutatif:

$$\begin{tikzcd} {[0,1]} & \\ B \arrow[u, "p"] \arrow[r, "u"'] & L \arrow[ul, "\bar{p}"'] \end{tikzcd}$$

Démonstration. Nous avons vu que l'ensemble L^* est un convexe compact dans l'espace localement convexe produit $\mathbf{R}^L$. Si $e \in L$, l'application $L^* \to \mathbf{R}$; $p \mapsto p(e)$ est continue pour la topologie de L^*, ainsi de la compacité de L^* il s'ensuit que (a')⇒(a); c'est-à-dire les propositions (a) et (a') sont équivalentes.

D'autre part, (c) entraîne (c'). En effet, si les hypothèses de (c') sont vérifiées, l'ensemble $u(B)$ est un sous-ortholattis distributif de L et il existe une application $p_0\colon u(B) \to [0, 1]$ et une seule, telle que $p_0 \circ u = p$. Il est clair que $p_0 \in (u(B))^*$, ainsi en supposant (c) on déduit l'existence d'un $\bar{p} \in L^*$, tel que $\bar{p}|_{u(B)} = p_0$; c'est-à-dire $\bar{p} \circ u = p$. Ceci montre que les propositions (c) et (c') sont équivalentes.

On vérifie sans peine que (c)⇒(a). En effet, prenant un $e \in L$, $e \neq \mathbf{0}$ on définit B comme le sous-ortholattis de L engendré par l'ensemble $\{e\}$. Or, $B = \{\mathbf{0}, \mathbf{1}, e, e^\perp\}$, donc B est distributif et il résulte de (c) que l'état p sur B défini par les équations: $p(e) := 1$, $p(e^\perp) := 0$ a un prolongement $\bar{p} \in L^*$ sur L tout entier. Alors $\bar{p}(e) = 1$, donc (c) entraîne (a).

Pour la démonstration de (a')⇒(b), définissons l'application $w\colon L \to C(L^*; \mathbf{R})$; $e \mapsto (p \mapsto p(e))$. Soit E le sous-espace vectoriel de $C(L^*; \mathbf{R})$ engendré par l'ensemble $w(L)$. Alors la couple (E, w) est l'une des linéarisations de L. En effet, de l'axiome ($\mathrm{E_I}$) découle ($\mathrm{OL_I}$). Si $p \in L^*$ alors désignant par τ_p la restriction sur E de l'application $C(L^*; \mathbf{R}) \to \mathbf{R}$; $\varphi \mapsto \varphi(p)$, on définit une fonctionnelle linéaire sur E, telle que $\|\tau_p\| \leqq 1$ et $\tau_p \circ w = p \in L^*$, donc ($\mathrm{OL_{III}}$) est vrai. Enfin, pour tout $e \in L$, $e \neq \mathbf{0}$ on a

$$\sup_{p \in L^*} p(e) = \|w(e)\| \geqq \sup\{\tau(w(e)) \mid \tau \in E', \|\tau\| \leqq 1, \tau \circ w \in L^*\} \geqq$$

$$\geqq \sup\{\tau_p(w(e)) \mid p \in L^*\} = \sup_{p \in L^*} p(e),$$

donc (a') entraîne ($\mathrm{OL_{II}}$).

Ainsi, il nous reste à prouver l'implication (b)⇒(c). Soit L un ortholattis linéarisable et soit (E, w) une linéarisation de L. Admettons que E est complet et $w(L)$ est total dans E. Soient B un sous-ortholattis distributif de L et $p \in B^*$ arbitraire. Désignons par E' le dual topologique de l'espace de Banach E et définissons l'ensemble $K := \{\tau \in E' \mid \tau \circ w \in L^*, \|\tau\| \leqq 1\}$. Nous avons vu que K est un sous-ensemble convexe, faiblement compact de E' et que l'application $K \to L^*$; $\tau \mapsto \tau \circ w$ est un homéomorphisme affine entre les ensembles convexes compacts K et L^*. Pour tout $e \in L$ soit $\hat{e}\colon K \to \mathbf{R}$; $\tau \mapsto \tau(w(e))$. Alors $\hat{e} \in A(K; \mathbf{R})$ $(e \in L)$ et l'application $L \to A(K; \mathbf{R})$; $e \mapsto \hat{e}$ est bien injective, car l'ortholattis L est séparé. D'autre part, si $\tau \in K$, alors $\tau \circ w \in L^*$ entraîne que $\hat{\mathbf{1}}(\tau) := \tau(w(\mathbf{1})) = (\tau \circ w)(\mathbf{1}) = 1$, c'est-à-dire $\hat{\mathbf{1}} = 1_K$. Soient $\hat{L} := \{\hat{e} \mid e \in L\}$ et $\hat{B} := \{\hat{e} \mid e \in B\}$. Désignons par $\mathscr{E}$ le sous-espace vectoriel normé de l'espace de Banach $C(K; \mathbf{R})$ engendré par l'ensemble $\hat{L}$. Puisque l'application $e \mapsto \hat{e}$ est injective, il existe une application $\hat{p}\colon \hat{B} \to [0, 1]$ et une seule, telle que $\hat{p}(\hat{e}) = p(e)$ $(e \in B)$.

En conclusion, nous avons un espace normé réel $\mathscr{E}$, un sous-ensemble $\hat{B}$ de $\mathscr{E}$ et une application $\hat{p}$: $\hat{B}\to\mathbf{R}$. Nous allons montrer qu'il existe une fonctionnelle linéaire μ sur $\mathscr{E}$, telle que $\mu|_{\hat{B}}=\hat{p}$ et $\|\mu\|\leqq 1$. Conformément à la résolution générale du problème des moments, pour l'existence de telle fonctionnelle linéaire il faut et il suffit que pour tous systèmes finis $(e_i)_{i\in I}$ et $(\lambda_i)_{i\in I}$ de B et $\mathbf{R}$, respectivement, soit

$$\Big|\sum_{i\in I}\lambda_i\hat{p}(\hat{e}_i)\Big| \leqq \Big\|\sum_{i\in I}\lambda_i\hat{e}_i\Big\|$$

où par $\|\ \|$ on désigne la norme de $\mathscr{E}$ qui est égale à la sup-norme.

Donc, soit $(e_i)_{i\in I}$ un système fini d'éléments de B, tel que $e_i\neq\mathbf{0}$ $(i\in I)$. Il résulte de la distributivité de B et du théorème de représentation de Stone qu'on peut traiter B comme une algèbre d'ensembles. Il s'ensuit l'existence d'un système fini orthogonal $(f_j)_{j\in J}$ d'éléments de B, tel que pour tout $i\in I$ il existe un sous-ensemble non vide J_i de J vérifiant l'égalité: $e_i=\bigvee_{j\in J_i} f_j$. Évidemment, on peut supposer que $f_j\neq\mathbf{0}$ $(j\in J)$.

Pour tous $(\lambda_i)_{i\in I}\in\mathbf{R}^I$ et $\tau\in K$ on a

$$\Big(\sum_{i\in I}\lambda_i\hat{e}_i\Big)(\tau) := \sum_{i\in I}\lambda_i\tau\big(w(e_i)\big) = \sum_{i\in I}\lambda_i(\tau\circ w)(e_i) = \sum_{i\in I}\lambda_i(\tau\circ w)\Big(\bigvee_{j\in J_i} f_j\Big) =$$
$$= \sum_{i\in I}\Big(\sum_{j\in J_i}\lambda_i(\tau\circ w)(f_j)\Big) = \sum_{j\in J}\Big(\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big)\tau\big(w(f_j)\big) = \Big(\sum_{j\in J}\Big(\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big)\hat{f}_j\Big)(\tau),$$

car $\tau\circ w\in L^*$, donc

$$\sum_{i\in I}\lambda_i\hat{e}_i = \sum_{j\in J}\Big(\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big)\hat{f}_j.$$

D'autre part, pour tout $j\in J$ on a $\hat{f}_j\in A(K;\mathbf{R})$, $\|\hat{f}_j\|=1$ et $\sum_{j\in J}|\hat{f}_j|=\widehat{\bigvee_{i\in I} e_i}\leqq \mathbf{1}_K$. Ainsi, prenant en considération le Lemme 2, on obtient aisément les inégalités suivantes:

$$\Big\|\sum_{i\in I}\lambda_i\hat{e}_i\Big\| = \Big\|\sum_{j\in J}\Big(\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big)\hat{f}_j\Big\| = \max_{j\in J}\Big|\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big| \geqq \Big|\sum_{j\in J}\Big(\sum_{\substack{i\in I\\ j\in J_i}}\lambda_i\Big)p(f_j)\Big| =$$
$$= \Big|\sum_{i\in I}\Big(\sum_{j\in J_i}\lambda_i p(f_j)\Big)\Big| = \Big|\sum_{i\in I}\lambda_i p\Big(\bigvee_{j\in J_i} f_j\Big)\Big| = \Big|\sum_{i\in I}\lambda_i p(e_i)\Big| = \Big|\sum_{i\in I}\lambda_i\hat{p}(\hat{e}_i)\Big|.$$

Ceci montre qu'il existe une fonctionnelle linéaire μ sur l'espace normé $\mathscr{E}$, telle que $\mu|\hat{B}=\hat{p}$ et $\|\mu\|\leqq 1$. On a alors $\mu(1_K)=\mu(\hat{\mathbf{1}})=\hat{p}(\hat{\mathbf{1}})=p(\mathbf{1})=1$, du lemme 1, appliqué au sous-espace vectoriel $\mathscr{E}$ de $A(K;\mathbf{R})$ il découle qu'il existe $\tau\in K$, tel que $\mu(\varphi)=\varphi(\tau)$ $(\varphi\in\mathscr{E})$. Cela signifie que pour tout $e\in B$ on a $\tau\big(w(e)\big)=:\hat{e}(\tau)= =\mu(\hat{e})=\hat{p}(\hat{e})=p(e)$, c'est-à-dire l'état $\bar{p}:=\tau\circ w\in L^*$ est un prolongement de p sur L tout entier.

Ceci achève la démonstration du théorème.

Appliquons ce théorème dans la démonstration de la proposition suivante.

Proposition 1. *Soient L un ortholattis linéarisable, B un ortholattis distributif, u: $B \to L$ un orthohomomorphisme et $p \in B^*$. Soit Γ un ensemble d'orthohomomorphismes de L dans lui-même dont les éléments sont deux-à-deux commutables et vérifient les équations: $\alpha \circ u = u$ $(\alpha \in \Gamma)$. Alors les propositions suivantes sont équivalentes:*

(a) *Il existe un état $\bar{p}$ sur L invariant par rapport à Γ (c'est-à-dire $\bar{p} \circ \alpha = \bar{p}$ pour tout $\alpha \in \Gamma$), tel que $p = \bar{p} \circ u$.*

(b) *Pour tout $b \in B$, $u(b) = \mathbf{0}$ entraîne $p(b) = 0$.*

Démonstration. De toute évidence (a)⇒(b), donc on doit prouver (b)⇒(a). Le théorème précédent entraîne que l'ensemble $K := \{\bar{p} \in L^* \mid p = \bar{p} \circ u\}$ n'est pas vide. Désignons par X l'espace vectoriel topologique réel séparé, dont l'espace vectoriel sous-jacent est l'espace produit $\mathbf{R}^L$ et dont la topologie est égale à la topologie produite. Il est aisé de voir que K est un sous-ensemble convexe compact de X. Pour tout $\alpha \in \Gamma$ on désigne par $\hat{\alpha}$ l'application de X dans lui-même qui fait correspondre à tout $\varphi \in X$ l'application $\varphi \circ \alpha$. Il est clair que pour tout $\alpha \in \Gamma$ la fonction $\hat{\alpha}$ est une application linéaire continue sur l'espace localement convexe X. D'autre part, les éléments de l'ensemble d'opérateurs $\{\hat{\alpha} \mid \alpha \in \Gamma\}$ sont deux-à-deux commutables. D'après le théorème de Markov—Kakutani (cf. [2], Ch. II, § 4, Application) on obtient qu'il existe $\bar{p} \in K$, tel que $\bar{p} = \hat{\alpha}(\bar{p}) = \bar{p} \circ \alpha$ pour tout $\alpha \in \Gamma$, et l'état $\bar{p}$ ainsi obtenu est bien l'état cherché.

Pour conclure, nous indiquerons deux classes importantes des ortholattis linéarisables.

Proposition 2. (a) *Tout ortholattis distributif B est linéarisable.*

(b) *Si A est une C^*-algèbre de Baer, l'ortholattis $L(A)$ de projecteurs de A est linéarisable.*

Démonstration. (a) Soit B un ortholattis distributif arbitraire. En vertu du théorème de représentation de Stone il existe un ensemble Ω et une algèbre d'ensembles $\mathscr{B}$ dans Ω, tels que les ortholattis B et $\mathscr{B}$ sont orthoisomorphes. Soit u: $B \to \mathscr{B}$ un orthoisomorphisme entre B et $\mathscr{B}$. Pour tout $\omega \in \Omega$ définissons l'application D_ω de la manière suivante:

$$D_\omega\colon B \to \{0, 1\}; \quad b \mapsto \begin{cases} 1, & \omega \in u(b), \\ 0, & \omega \notin u(b). \end{cases}$$

On en déduit sans peine que pour tout $\omega \in \Omega$ l'application D_ω est un état sur B.

Si $e, f \in B$, $e \neq f$, alors $\big(u(e) \setminus u(f)\big) \cup \big(u(f) \setminus u(e)\big) \neq \emptyset$ et choisissant un élément ω arbitraire de cet ensemble, on voit que $D_\omega(e) \neq D_\omega(f)$; c'est-à-dire l'ortholattis B est séparé.

D'autre part, si $e \in B$, $e \neq \mathbf{0}$, alors $u(e) \neq \emptyset$ et pour tout $\omega \in u(e)$ on a $D_\omega(e) = 1$. D'après le Théorème ceci montre que B est linéarisable.

(b) Soit A une C^*-algébre de Baer. On désigne par $L(A)$ l'ortholattis des projecteurs de A. Remarquons d'abord que si τ est une fonctionnelle linéaire positive sur A, $\tau \neq 0$, alors l'application $p_\tau := (\tau/\|\tau\|)|_{L(A)}$ est un état sur $L(A)$.

Si $e, f \in L(A)$, $e \neq f$, alors d'après le théorème de Hahn—Banach il existe une fonctionnelle linéaire continue τ sur A, telle que $\tau(e) \neq \tau(f)$. On sait que toute fonctionnelle linéaire continue sur A peut être écrite sous la forme d'une combinaison **C**-linéaire de fonctionnelles linéaires positives (cf. [5], C^*-algèbres, § 2, 2.6.4), ainsi on peut supposer que τ est positive. Dans ce cas on a $\tau \neq 0$ et $p_\tau(e) \neq p_\tau(f)$; c'est-à-dire l'ortholattis $L(A)$ est séparé.

D'autre part, si $e \in L(A)$ et $e \neq \mathbf{0}$, alors il existe une fonctionnelle linéaire positive τ sur A, telle que $\|\tau\| = 1$ et $\tau(e)^{1/2} = \|e\| = 1$ (cf. [6], Theorem 12.39). Mais alors $p_\tau(e) = \tau(e) = 1$ et d'après le Théorème, l'ortholattis $L(A)$ est linéarisable.

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On tightness of random sequences

WINFRIED STUTE

Let $(\xi_n)_n$ be a sequence of random elements in a complete separable metric space (X, d), defined on some probability space $(\Omega, \mathscr{A}, \mathbf{P})$. In many situations, particularly in statistical large sample theory, it is required to show that the laws $\mathscr{L}(\xi_n)$, $n \geqq 1$, converge weakly to some specified (Borel) measure μ. For this a general device is to guarantee that $\mathscr{L}(\xi_n)$, $n \geqq 1$, has at least one cluster point and, in a second step, that there is at most one of such points. While uniqueness may be shown by applying general methods for identifying weak limits (cf. BILLINGSLEY [1]), the existence part usually takes account of Prohorov's theorem. Accordingly, it remains to prove that ξ_n, $n \geqq 1$, is uniformly tight:

(1) for given $\varrho > 0$ there exists some compact subset K_ϱ of X such that $\mathbf{P}(\xi_n \notin K_\varrho) \leqq \varrho$ for all $n \geqq 1$.

Apart from stochastic arguments, to find such a K_ϱ, one has to characterize the (relatively) compact subsets of X. This might cause some difficulties due to the fact that such a description needs a far reaching investigation of the topology induced by d. In many cases, however, there exists a (closed) subspace X_0 of X such that

(2) the ξ_n's, as $n \to \infty$, concentrate more and more on X_0, so that a possible limit distribution is supported by X_0.

(3) the relative topology induced on X_0 admits a simpler characterization of compactness.

An important example we have in mind is the space $X = D[0, 1]$ of right-continuous functions on $[0, 1]$ with left-hand limits, endowed with the Skorohod topology (cf. BILLINGSLEY [1]). The class of processes with paths in D contains appropriate versions of partial sum, empirical and quantile processes. In each case the limit process may be chosen so as to have continuous paths, i.e. we may take $X_0 = C[0, 1]$, the space of continuous functions on $[0, 1]$. As a matter of fact the Skorohod topology on C coincides with the topology of uniform convergence.

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Thus a characterization of compactness in X_0 is obtained from the classical Arzela—Ascoli Theorem. Identification of the limit of course relies on the convergence of the finite dimensional distributions.

In this paper a simple method for proving tightness is proposed which is based on appropriate X_0-valued transformations $T_\varepsilon(\xi_n)$, $\varepsilon>0$, of ξ_n, $n\geqq 1$.

Proposition 1. *Assume that, for each* $\varepsilon>0$, $T_\varepsilon\colon X\to X_0$ *is a measurable transformation such that*

(4) $T_\varepsilon(\xi_n)$, $n\geqq 1$, *is tight in* (X_0, d) *for each* $\varepsilon>0$,

(5) $\lim\limits_{\varepsilon\to 0}\limsup\limits_{n\to\infty}\mathbf{P}\big(d(T_\varepsilon(\xi_n),\xi_n)\geqq\eta\big)=0$ *for all* $\eta>0$.

Then ξ_n, $n\geqq 1$, *is tight in* (X, d), *and each cluster point* μ *of* $\mathcal{L}(\xi_n)$, $n\geqq 1$, *satisfies* $\mu(X_0)=1$.

Proof. Fix some $\eta>0$. By (4) we have, given $\varepsilon>0$,

$$\mathbf{P}\big(T_\varepsilon(\xi_n)\notin M^\eta\big)\leqq\eta \quad \text{for all} \quad n\geqq 1$$

for some finite $M=M(\eta,\varepsilon)\subset X_0$, where $M^\eta=\{x\in X\colon d(x, M)<\eta\}$ is the open η-neighborhood of M in X. For small enough $\varepsilon>0$ (5) implies

$$\mathbf{P}(\xi_n\notin M^{2\eta})\leqq 2\eta \quad \text{for all} \quad n\geqq n_0(\eta).$$

Since $\xi_1, \ldots, \xi_{n_0-1}$ are tight in (X, d), we may find some finite $M_0(\eta)\equiv M_0\supset M$ in X such that

$$\mathbf{P}(\xi_n\notin M_0^{2\eta})\leqq 2\eta \quad \text{for all} \quad n\geqq 1.$$

For K_ϱ we may then take the closure of the set $\bigcap\limits_{k\geqq 1} M_0^{\varrho 2^{-k}}$. To show that each cluster point μ is supported by X_0, assume w.l.o.g. that $\mathcal{L}(\xi_n)\to\mu$ weakly. Since X_0 is closed, $X_0^\eta\downarrow X_0$ as $\eta\downarrow 0$. Hence it remains to prove $\mu(X_0^\eta)=1$. As is well known, the set of η's for which X_0^η has a μ-null boundary forms a dense set in $(0,\infty)$. Hence it suffices to consider only such η's. In this case

$$\mu(X_0^\eta)=\lim_{n\to\infty}\mathbf{P}(\xi_n\in X_0^\eta).$$

That the right-hand side equals one now easily follows from (5) and the fact that $T_\varepsilon(\xi_n)\in X_0$ for all $\varepsilon>0$.

Let us show the usefulness of our approach by giving a straightforward proof of the following important result (cf. Billingsley [1], Theorem 15.5).

Proposition 2. *Let* ξ_n, $n\geqq 1$, *be a random sequence in* $D[0, 1]$ *such that*

(6) *for each* $\varrho>0$ *there exists some finite* $a>0$ *such that*

$$\mathbf{P}\big(|\xi_n(0)|\geqq a\big)\leqq\varrho \quad \textit{for all} \quad n\geqq 1.$$

(7) *for all* $\eta, \varrho>0$ *there exists some* $0<\delta<1$ *such that for all* $n \geqq n_0(\eta, \varrho)$

$$\mathbf{P}\big(\sup_{|t-s|\leqq\delta} |\xi_n(t)-\xi_n(s)| \geqq \eta\big) \leqq \varrho.$$

Then ξ_n, $n \geqq 1$, *is tight in* $(D[0,1], d)$, *and each cluster point* μ *satisfies* $\mu(C[0,1])=1$.

Proof. For $f \in D[0,1]$, put $f(t)=f(1)$ for $t>1$ and $f(t)=f(0)$ for $t<0$. Let K be a smooth nonnegative kernel function on the real line, integrating to one and vanishing outside some bounded interval. Put

$$Tf(t) \equiv \tilde{f}(t) = \int f(x)K(t-x)\,dx = \int f(t-y)K(y)\,dy, \quad 0 \leqq t \leqq 1.$$

Obviously, $\tilde{f} \in C[0,1]$. If $\sup_{|t-s|\leqq\delta} |f(t)-f(s)| < \eta$, we have for $|t-s| \leqq \delta$:

$$|\tilde{f}(t)-\tilde{f}(s)| \leqq \int |f(t-y)-f(s-y)|K(y)\,dy < \eta \int K(y)\,dy = \eta,$$

i.e. $\sup_{|t-s|\leqq\delta} |\tilde{f}(t)-\tilde{f}(s)|<\eta$ whenever $\sup_{|t-s|\leqq\delta} |f(t)-f(s)|<\eta$. Furthermore, if $|f(0)|<a$ and $\sup_{|t-s|\leqq\delta} |f(t)-f(s)|<\eta$, we obtain

$$\|f\| \equiv \sup_{0\leqq s\leqq 1} |f(s)| < a+\eta/\delta \equiv b < \infty$$

and thus $|\tilde{f}(0)| \leqq \|f\| < b$. It follows from (6) and (7) and the Arzela—Ascoli Theorem that $T(\xi_n)$, $n \geqq 1$, is tight in $C[0,1]$.

Now, we may let K depend on ε in such a way that the degree of smoothing decreases as $\varepsilon \to 0$. To be specific, let

$$K(x) = K_\varepsilon(x) = \varepsilon^{-1}K_0(x/\varepsilon),$$

where K_0 is a preassigned probability kernel vanishing outside some finite interval, say $[-1,1]$. Define

$$T_\varepsilon(f)(t) = \varepsilon^{-1}\int f(x)K_0\big((t-x)/\varepsilon\big)\,dx.$$

We already know that $T_\varepsilon(\xi_n)$, $n \geqq 1$, is tight in $C[0,1]$ for each $\varepsilon>0$. Furthermore,

$$\tilde{f}(t)-f(t) = \int_{\mathrm{supp}(K)} [f(t-y)-f(t)]K(y)\,dy,$$

whence

$$\sup_{0\leqq t\leqq 1} |\tilde{f}(t)-f(t)| \leqq \sup_{\substack{0\leqq t\leqq 1\\ y\in \mathrm{supp}(K)}} |f(t-y)-f(t)|.$$

For $K=K_\varepsilon$, we have $\operatorname{supp}(K)\subset[-\varepsilon,\varepsilon]$ and thus $\sup_{0\leqq t\leqq 1}|\tilde{f}(t)-f(t)|<\eta$ whenever $\sup_{|t-s|\leqq\varepsilon}|f(t)-f(s)|<\eta$. Observe that $d(\tilde{f},f)\leqq\sup_{0\leqq t\leqq 1}|\tilde{f}(t)-f(t)|$ and conclude that for $\varepsilon\leqq\delta$

$$\mathbf{P}\big(d(T_\varepsilon(\xi_n),\xi_n)\geqq\eta\big)\leqq\varrho,\quad n\geqq n_0(\eta,\varrho).$$

This shows (5) and completes the proof of the proposition.

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Bibliographie

B. Aulbach, Continuous and Discrete Dynamics near Manifolds of Equilibria (Lecture Notes in Mathematics, 1058), IX+142 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

These lecture notes are concerned with the following problem of the qualitative theory of differential and difference equations: which conditions assure that a solution of a differential (or a difference) equation converges to same point of the manifold of stationary solutions? In stability theory and in the theory of the asymptotic behaviour of solutions most attention has thus far been payed to the behaviour of solutions near isolated equilibria. The new direction of the study of continua of stationary solutions is motivated both from a theoretical and a practical point of view. Theoretically, the manifolds of stationary points are very important special cases of invariant manifolds that have been studied actively in the modern theory of differential equations. On the other hand, several model equations in mechanics, physics, economy, biology and medicine possess such manifolds. The basic selection model from population genetics for separated generations (Fisher—Wright—Haldane model) is discussed in the book as an application.

The continuous time case and the discrete time case are treated in two separated parts in a parallel manner. Actually, the book is well-organized and is written very clearly and precisely. Only basic knowledge of ordinary differential equations and some familiarity with some concepts of the qualitative theory of dynamical systems are prerequisite for understanding.

Summing up, these lecture notes can be recommended for mathematicians, users of mathematics and students in mathematics interested in the qualitative theory of differential and difference equations and its applications.

L. Hatvani (Szeged)

Automata, Languages and Programming, 11th Colloquium, Antwerp, Belgium, July 1984. Edited by J. Paredaens (Lecture Notes in Computer Science 172), VIII+528 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

In addition to the texts of the invited lectures, this volume contains 46 contributions selected from a total of 141 submitted papers.

The invited lecturers were R. Fagin (Topics in Database Dependency Theory) and A. L. Rosenberg (The VLSI Revolution in Theoretical Circles).

Other topics cover a wide range of theoretical computer science: automata, formal languages, analysis of algorithms, computability and complexity, program specification, semantics of programming languages, etc. Some papers give deep theoretical results in classical fields, others just outline new trends or make an attempt to meet new demands.

The volume is recommended to experts interested in theoretical aspects of computer science.

Z. Ésik (Szeged)

Banach Space Theory and its Applications, Proceedings, Bucharest, 1981. Edited by A. Pietsch, N. Popa and I. Singer (Lectures Notes in Mathematics, 991), X+302 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

This book contains the printed versions of lectures held at the GDR—Romanian seminar in Bucharest, 1981. The seminar was organized by the National Institute for Scientific and Technical Creation in collaboration with the Department of Mathematics of the University of Jena. The book consists of 26 papers concerning Banach space geometry and Banach space operator theory. Reader's familiarity with functional analysis and general topology is supposed.

L. Gehér (Szeged)

T. Banchoff—J. Wermer, Linear Algebra Through Geometry (Undergraduate Texts in Mathematics), X+257 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1983.

The main purpose of this book is to introduce students into elementary linear algebra by emphasizing the geometric significance of the subject. The first four chapters define vectors in 1, 2, 3 and 4 dimensions, give the geometry of vectors, introduce the concept of linear transformations and their matrix representations, give a connection between linear transformations and systems of linear equations and define determinants. Chapter 5 introduces finite dimensional vector spaces and investigates general systems of finitely many linear equations in homogeneous and inhomogeneous cases.

L. Gehér (Szeged)

C. Berg—J. P. R. Christensen—P. Ressel, Harmonic Analysis on Semigroups (Theory of Positive Definite and Related Functions), (Graduate Text in Mathematics, 100), X+289 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

One of the most fundamental problems of Fourier analysis is to determine the conditions under which a suitable function is just the Fourier resp. Laplace transform of a positive measure. Several famous results of Bochner, Bernstein—Widder and Hamburger can be considered as special cases of a general theorem on positive definite functions on Abelian semigroups with an involution. The purpose of this book is to give a systematic treatment of these functions from this general point of view.

After a few introductory chapters on topological vector spaces, Radon measures and integral representations, the detailed exposition is given in chapters 4—8. Beside the mentioned general theorems the central topics of the book are Schoenberg-type theorems, moment problems and the Hoeffding inequality. The fundamental result of Schoenberg (asserting that to each continuons function $\varphi\colon \mathbf{R}\to\mathbf{C}$ with the property that $\varphi\circ\|\cdot\|_n$ is a positive definite function on $(\mathbf{R}^n, \|\cdot\|_n)$ for all n, there exists a finite nonnegative measure on $\mathbf{R}_+$ with the Laplace transform $\varphi(\sqrt{t})$) is proved in a quite abstract form here, replacing both the real line and the half-line $\mathbf{R}_+$ by arbitrary Abelian semigroups with neutral element. The Hoeffding-type inequalities serve for a probabilistic characterization of the positive (resp. negative) definite functions.

The book is written in a very clear and enjoyable style and can serve as an excellent textbook for an advanced graduate course. It contains many exercises and detailed historical comments as well.

Z. I. Szabó (Szeged)

M. Berger—P. Pansu—J. P. Berry—X. Saint-Raymond, Problems in Geometry (Problem Books in Mathematics), VIII+266 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

Only very few problem books are available for teaching geometry, so all such collections give great pleasure for teachers, especially those as excellent as the present one. Most of the problems contained in the volume are chosen from the book *Geometry* by Marcel Berger.

The problems cover a very large scale of geometry such as tillings, affine spaces, projective spaces, euclidean vector spaces, triangles, spheres, circles, convex sets, polytopes, quadrics, conics, elliptic and hyperbolic geometry. The book is divided into three parts. The first part is devoted to a summary of the notions, the second one contains the suggestions and hints and the third one provides the solutions.

The volume is of interest to students as well as to secondary school teachers.

Z. I. Szabó (Szeged)

Bifurcation Theory and Applications, Montecatini, Italy, 1983. Edited by L. Salvadori (Lecture Notes in Mathematics, 1057), VI+233 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

Undoubtedly, bifurcation theory is one of the central topics of the modern theory of differential equations. This is shown also by the great number of monographs and textbooks published nowadays on this topic — some of them were reviewed in these Acta, too.

These lecture notes contain the subject-matter of an international summer course held at Montecatini, Italy, June 24—July 2, 1983, organized by the Centro Internationale Mathematico Estivo (C. I. M. E.) Foundation. The main lecturers were outstanding experts in the field: S. Busenberg, Bifurcation phenomena in Biomatematics; I. J. Duistermaat, Bifurcation of periodic solutions near equilibrium points of Hamiltonian systems; J. K. Hale, Introduction to dynamic bifurcation; G. Iooss, Bifurcation and transition to turbulence in Hydrodynamics.

The two common characteristic features of the lectures were: stressing the importance of the connections between stability and bifurcation problems, and showing the role of bifurcation theory in approaching the analysis of natural phynomena. Take just a simple problem as a foretaste. Many models in the theory of epidemics lead to systems of parametrized nonlinear ordinary differential equations. A threshold value α_0 of the parameter α is sought for such that if $\alpha \leq \alpha_0$, then the identically zero function is the only stable non-negative solution, while if $\alpha > \alpha_0$, a nontrivial stable positive solution exists. This solution can be identified with the persistence of an endemic level disease.

L. Hatvani (Szeged)

R. Bott—L. W. To, Differential Forms in Algebraic Topology (Graduate Texts in Mathematics, 82), XII+331 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

Although there are many textbooks in algebraic topology, only a few of them can present the intuitive foundations of this very far developed theory. This excellent book concentrates the attention of the reader on smooth topology and thus can give a treatment of some very intuitive and geometric fundamental questions of this theory. Only a good knowledge of linear algebra, advanced calculus and general topology is assumed but some acquaintance with simplicial complexes, differential geometry and homotopy groups can be helpful. Chapter I starts with a rapid introduction to the Grassmann calculus of exterior differential forms on manifolds. There is given the "computable" definition of de Rham cohomology theory, the treatment of Poincaré duality and its various extensions, such as the Thom isomorphism. Chapter II is devoted to the study of the Čech-de Rham complex using the techniques of spectral sequences as an extension of the Mayer—Vietoris principle. There is a detailed discussion of the topology of sphere bundles. In Chapter III the spectral sequences are treated in a more formal manner. There is given a review of homotopy theory before the discussion of the application of spectral sequences to this theory.

In Chapter IV an introduction to characteristic classes is presented. The self-contained treatment of Chern and Pontrjagin classes is illustrated with interesting computations of concrete examples and applications. Lastly, the geometry of the universal bundle is discussed.

This very nice book is intended "to open some of the doors to the formidable edifice of modern algebraic topology". It can be highly recommended to everybody who is interested in algebraic or differential topology.

Péter T. Nagy (Szeged)

D. Bump, Automorphic Forms on GL (3, R) (Lecture Notes in Mathematics, 1083), XI+ 184 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo.

The book is a useful professional reading for researchers and students with special interests in the theory of automorphic forms on reductive Lie groups or in analytic number theory. The special importance of general linear groups on this field is based upon their natural connection with L-series associated with general automorphic forms. Although the theory concerning automorphic forms on $GL(2)$ can be viewed as classical, our knowledge about the case of $GL(n)$ is still quite lacunary because, as the author remarks, most principal questions e.g. the theory of Ramanujan sums, the coefficients arising from the Hecke algebra, the Whittaker functions and Eisenstein series become much richer when passing to $GL(3)$, moreover they only begin to show their full ramification on $GL(3)$. Since the last decade there has been a considerable development in the theory of automorphic forms on $GL(3)$ due to Jacquet, Piatetski—Shapiro, Shalika, Kostant, Goldfeld, Friedberg, Shintani, Terras, Vinogradov, Thakhtadzyan, Imai, Goodman, Wallach and the author. The book provides a well-arranged and relatively self-contained unified approach to their works in the mentioned directions with an introductory survey of the classical 2-dimensional case for motivation. It should also be mentioned that mainly complete proofs are given almost free of representation theory and emphasizing the computational aspects, furthermore the book includes new results, too.

L. L. Stachó (Szeged)

B. Chandler—W. Magnus, Combinatorial Group Theory: A Case Study of the History of Ideas (Studies in the History of Mathematics and Physical Sciences, 9), VIII+ 234 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1982.

To give a complete definition of combinatorial group theory is not an easy task. As a first approach it may be characterized as the theory of groups which are given by generators and defining relations or, as we would say today, by a presentation. The 1882 paper of W. von Dyck is the first one in which generators and defining relations are not only introduced as new concepts, but are also used effectively for mathematical research, but it is highly probable, that the germs of these ideas can be traced back to earlier authors. Nevertheless, we can state, that Dyck's paper was the initial point of this theory.

Starting with a report on Dyck's paper mentioned above, the book describes the unfolding of combinatorial group theory. On the basis of the nature of research in this field the authors distinguish two periods. Part I of the book covers the period from 1882 to 1918, and Part II from 1918 to 1945, but this second part contains some outlook to the contemporary research, too.

Part I, The Begining of Combinatorial Group Theory contains the following main topics: Dyck's group-theoretical studies, the theory of discontinuous groups, the fundamental groups of topological spaces, precursors of later developments (arithmetically defined linear groups of higher dimensions, geometrical constructions, braid groups, finite groups etc.).

Part II, The Emergence of Combinatorial Group Theory as an Independent Field, deals with the developments during the period from 1918 to 1945. This period starts with a paper of J. Nielson in which problems of combinatorial group theory were investigated and solved which do not show an obvious dependence on problems in topology or in other fields. From this time, although it had not ceased to have its original stimulating effects, the field began to develop its own problems and its own methods.

For the period from 1918 to 1937 the authors selected seven topics which they considered to be the most important contributions to the field. These are the following: free groups and their authomorphisms, the Reidemeister—Schreier method, free products and free products with amalgamations, one-relator groups, metabelian groups and related topics, commutator calculus and lower central series, varieties of groups (this latest topic started essentially with a 1937 paper of B. H. Neumann). The arrangement of these seven topics is chronological with respect to the first papers in each section, but the later literature is considered as well. The subsequent chapters mostly deal with later developments and the impact of mathematical logic.

In both parts some chapters are dealing not with mathematics as such, but with phenomena relevant to mathematical research.

The prerequisite for Part I except Chapter I.6 is a not too rigorous but sound knowledge of the fundamentals and basic terms of algebra. Especially group theory is necessary. The technical prerequisites for Part II are somewhat higher than those for Part I, although the definitions and technical terms are given.

Lajos Klukovits (Szeged)

Combinatorial Theory, Proceedings, Schloss Ranischholzhausen, Germany, 1982. Ed. by D. Jungnickel and K. Vedder (Lecture Notes in Mathematics, 969), VI+326 pages, Springer-Verlag, Berlin—Heidelberg—New York.

This volume contains the proceedings of a conference on combinatorial theory that took place at Schloss Ranischholzhausen in May to mark the 375th anniversary of the Universität Giessen. The book is a selection of the invited lectures and the contributed talks. The 21 papers cover the whole range of Combinatorics. The reader can find new results on regular sets, partitioning problems, Dedekind numbers, optimal coverings, etc.

The wide range yields a good overwiew of this fast developing and diverging field of mathematics.

G. Galambos (Szeged)

R. Cooke, The Mathematics of Sonya Kovalevskaya, IX+235 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

Is S. Kovalevskaya a serious mathematician of the 19^{th} century or more of a heroine of the women's movement? At the present time there exist different opinions. In general one knows something about her life which was packed full of exciting events, and perhaps about the theorem of Cauchy—Kovalevskaya. This is insufficient to answer the above question in any direction. In her life famous mathematicians as Hermite, Picard, Chebyshev, Mittag-Leffler, and first of all Weierstrass considered Kovalevskaya as one of the best mathematicians of the world. Now her role in the mathematical life is a little obscure. The negative comments came partly from Felix Klein. One of the problems is an error in Kovalevskaya's paper on the Lamé equations discovered by Volterra. One can consider this mistake as a "disaster" or a starting point of further research. Choose the suitable for you. Another problem is that her papers are written in Weierstrass' style, therefore one can hardly decide how much of the work is her own. Be it as it may, again, Klein expressed great admiration that Kovalevskaya achieved so much in her short life.

In this book the author focuses attention on some important parts of the late 19th century mathematics. He makes us deeply acquainted with the history of problems studied by Kovalevskaya. This work is the first complete exposition of her mathematical work and seems to be suitable for developing the reader's own view about Kovalevskaya as a mathematician.

L. Pintér (Szeged)

C. W. Curtis, Linear Algebra (An Introductory Approach), (Undergraduate Texts in Mathematics), IX+337 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

This book is the fourth edition of a textbook designed for upper division courses in linear algebra. It does not suppose an earlier course, so it may be useful for students. To understand the theoretical results, there are several examples of concrete problems in almost every section. The author attributes great importance to these examples, and presents not only numerical exercises, but theoretical ones as well. To encourage students to develop their own procedures for checking their work, not all the solutions are given.

The main feature of the book is that it provides an introduction to the axiomatic methods of modern algebra. The first five chapters discuss basic properties of vector spaces. This is followed by the theory of a single linear transformation. Chapter 8 deals with dual vector spaces and multilinear algebra. The last chapter of the theoretical part of the book discusses orthogonal and unitary transformations.

The interested reader can find some applications of linear algebra in Chapter 10. Anyway, the survey of applications is the most useful part of the book which is a good introduction to linear algebra for those not having any preliminary knowledge in this field of mathematics.

G. Galambos (Szeged)

Differential Geometry of Submanifolds, Proceedings, Kyoto, 1984. Edited by K. Kenmotsu (Lecture Notes in Mathematics, 1090), VI+132 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The present volume contains the text of most of the lectures presented by young Japanese mathematicians at the Conference on Differential Geometry of Submanifolds held at the Research Institute of Mathematical Sciences of Kyoto University of January 23—25, 1984. The proceedings include 12 papers written by the following authors: Kause A. (Estimates for Solutions of Poisson Equations and their Application to Submanifolds); Miyaoka R. (Taut Embeddings and Dupin Hypersurfaces); Adachi T., Sunada T. (Geometric Bounds for the Number of Certain Harmonic Mappings); Ohnita Y. (The First Standard Minimal Immersions of Compact Irreducible Symmetric Spaces); Takakuwa S. (A Hypersurfaces with Prescribed Mean Curvature); T. Ohsawa (Holomorphic Embedding of Compact S. P. C. Manifolds into Complex Manifolds as Real Hypersurfaces); Koiso M. (The Stability and the Gauss Map of Minimal Surfaces in $\mathbf{R}^3$); Kitagawa Y.

(Compact Homogeneous Submanifolds with Parallel Mean Curvature); Watanabe K. (Sure les ensembles nodaux); Mashimo K. (On Some Stable Minimal Cones in $\mathbf{R}^7$); Naitoh H. (Symmetric Submanifolds of Compact Symmetric Spaces); Kenmotsu K. (Gauss Maps of Surfaces with Constant Mean Curvature, Appendix).

Z. I. Szabó (Szeged)

J. Dixmier, General Topology (Undergraduate Texts in Mathematics), VII+140 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

The experienced reader curiously opens the book because to write a text on general topology for undergraduate students is not an easy task. Topology is the axiomatic study of the notions of limit and continuity. Thus, it demands a very high level of abstract thinking and gives a generous experience not only in the theory of real-valued functions of a real variable but in the advanced branches of mathematical analysis, too. The author, who is an outstanding analyst, has performed this not easy task excellently. The treatment is well-chosen. It is written in the general spirit of Bourbaki, but it is elementary and easily accessible. The basic concepts are motivated appealingly to the reader's experience in elementary analysis. Already on the first pages of the book the author introduces the notions of the open and closed sets in metric spaces and shows their basic properties. For students this gives a good base to the definition of the abstract topological space.

The book contains valuable discussions of some topics which are not always found in topology books (numerical functions, Stone—Weierstrass Theorem, normed spaces, infinite sums).

J. L. Kelley's General Topology is generally referred to as What Every Analyst Should Know. The present book can be recommended as What Every Mathematician Should Know.

L. Hatvani (Szeged)

R. D. Driver, Why Math? (Undergraduate Texts in Mathematics), XIV+233 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

There exist only few mathematical books which are useful to almost everyone, even to students who don't like mathematics. In my opinion this is such a book. Requiring only a little algebra and geometry, the author presents various interesting problems and examples mainly taken from the real world. These problems seem to be suitable to make the readers understand the advantage of mathematics in the everyday life, and so to make them wonder to solve similar problems. They hardly realize and deal with real mathematical questions. Besides the examples, teachers find interesting methods too.

Chapter headings are: Arithmetic review, Prime numbers and fractions, The Pythagorean Theorem and square roots, Elementary equations, Quadratic polynomials and equations, Powers and geometric sequences, Areas and volumes, Galilean relativity, Special relativity, Binary arithmetic, Sets and counting, Probability, Cardinality. Finally, here are some problems chosen a little accidentally of the book (surely the reader will find more interesting ones):

1. How long might it take you to factor the integer 38009 without the aid of a calculator or a computer?

2. A baseball is thrown straight upwards from a height of 7 feet with an initial velocity of 50 feet per second. Find the maximum height of the baseball and the time when this height is reached.

3. If the value of an investment increases by 26% in 2 years, what is the equivalent effective annual rate of interest?

4. A police radar signal reflected by an approaching car returns at a frequency 2×10^{-7}% higher than the transmitted frequency. Find the speed of the car.

5. How can a sailboat sail at right angles to the wind direction?

6. If a moon is traveling at 99,9% of the speed of light, how long would it live and how far could it travel from the viewpoint of a stationary observer?

L. Pintér (Szeged)

B. A. Dubrovin—A. T. Fomenko—S. P. Novikov, Modern Geometry. Methods and Applications (Part I. The Geometry of Surfaces, Transformation Groups, and Fields), (Graduate Texts in Mathematics, 93), XV+464 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

Modernization of the teaching of geometry in the Soviet Union started in 1971. This modernization was directed mainly towards the applications of the subject. The present book is the most significant representation of this trend. It is the translation of the first volume of a three-volume series published originally under the same title by Moscow Nauka in 1979.

The volume covers the following topics of differential geometry: The theory of surfaces, the algebraic and differential calculus of tensors and fields, and the calculus of variation. The last two chapters are devoted to physical applications, especially to the study of Hamiltonian structures, Yang-Mills fields and gauge transformations.

The book is one of the best introductions to modern differential geometry. Its material is explained very clearly and enjoyably. It is useful for mathematical students, teachers and physicists as well.

Z. I. Szabó (Szeged)

H. D. Ebbinghaus—J. Flum—W. Thomas, Mathematical Logic (Undergraduate Texts in Mathematics), IX+216 pages, Springer-Verlag, New York—Heidelberg—Berlin, 1984.

This introductory text is translated by A. S. Ferebee from the German original „Einführung in die mathematische Logik", published by Wissenschaftliche Buchgesellschaft, Darmstadt, 1978.

The volume consists of two parts. PART A provides a good, easily comprehensible, self-contained introduction to the fundamental notions and results of first order logics. After defining syntax and semantics, a sequential inference system and a Henkin-type proof of its completeness are presented with extreme care. Compactness and Löwenheim—Skolem properties are then derived. The last chapter of PART A titled "The Scope of First Order Logic" discusses with clarity how these formal tools are applied in the foundation of other branches of mathematics.

PART B starts by introducing some extensions of first order logic, namely second order logics and certain infinitary ones. Then Church's Theorem on undecidability of first order logics, Trachtenbrot's Theorem, incompleteness of second order logics, and Gödel's Incompleteness Theorems are stated and proved in detail.

In the last two chapters, the Ehrenfeucht—Fraïssè characterization of (first order) elementary equivalence and Lindström's Theorem stating that no proper extension of first order logic admits both the compactness and the Löwenheim-Skolem properties are included.

The whole text is well structured for the use of students and is written in a clear, appealing style.

P. Ecsedi-Tóth (Szeged)

D. B. Fuks—V. A. Rokhlin, Beginner's Course in Topology (Universitext), XI+516 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The material of this book is based on a rather lengthy course of lectures held at the Leningrad and Moscow Universities, and presents an introduction into the most fundamental topics of topology. The book is divided into 5 chapters. Chapter 1 is devoted to general topological spaces. The most important concepts and results are given, including homotopies. The following chapters deal with spaces of special characters. Chapter 2 defines and investigates cellular spaces, especially simplicial spaces, and their topological properties. Chapter 3 gives the fundamental concepts of topological manifolds and special manifolds. At the the end of this chapter, the simplest structure theorems can be found. Chapter 4 deals with bundles with and without group structure, and with vector bundles. Chapter 5 studies homotopy groups. Starting with the general theory, the homotopy groups of spheres, classical manifolds and cellular spaces are investigated.

The book is highly recommended to anyone interested in topology and having some familiarity with higher mathematics.

L. Gehér (Szeged)

C. George, Exercises in Integration (Problem Books in Mathematics), X+ 550 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

This book contains over 200 exercises in Lebesgue integration and its applications in analysis (convolution, Fourier transforms, trigonometric series). Students can deal with the problems usefully if, after having worked seriously upon a problem, seek some pointers from the solution, or compare it with their own. Teachers will find this book an important supplement completing their collection of problems on Lebesgue integration and will discover some new original solutions. Finally, as the author says: "In this book researches will find some results that are not always treated in courses on integration; they are either properties whose use is not as universal as those which usually appear and which are therefore found scattered about in appendices in various works, or are results that correspond some technical lemmas which *I* have picked up in recent articles on a variety of subjects: group theory, differential games, control theory, probability, etc; ..."

J. Németh (Szeged)

H. Gericke, Mathematik in Antike und Orient, XII+292 Seiten, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

This book of four chapters is the material of the first one of the author's two semester course on the history of mathematics (the second one is devoted to the western mathematics).

The first chapter, the Pre-Greek Mathematics, starts with a brief account of prehistoric mathematics, the mathematical content of the megalitic monuments. After it we can read about the Babylonian algebra, geometry and stronomy, the Rhind and Moscow papyri, and the Egyptian calendar and astronomy. The latest part deals with some problems from the Indian Śulvasūtras.

The second chapter sets up some important questions of the Greek mathematics: the development of the deductive method (the Pythagorean school and the influence of the Elean philosophy), incommensurable line segments, the Eudoxos' theory of magnitudes, the quadrature of the parabola by Archimedes, conic sections, theory of numbers (primes, Pythagorean triples, figurate numbers), algebraic problems from Diophantus' Arithmetica, the development of sciences (astronomy, the theory of motions), etc.

The third chapter deals with Oriental mathematics. Here we can read about the famous Chinese "Nine Chapters of the Mathematical Art", and the works of Aryabhata, Brahmagaupta and Bhaskara II. The chapter ends with a short outline of the mathematics of the Moslem countries (e.g. the work of Al-Khwarizmi, the cubic equations, the parallel postulate).

Chapter 4, Biographical and Bibliographical Notes, is very useful for teaching purposes. It contains an extensive bibliography for each chapter and short biographies of the mathematicians mentioned in the text.

This book is recommended not only to students and teachers of mathematics but everybody who wants to read a short but substantial historical outline of the ancient and Oriental mathematics.

Lajos Klukovits (Szeged)

K. Itô, Introduction to Probability Theory, X+213 pages, Cambridge University Press, Cambridge—London—New York—New Rochelle—Melbourne—Sydney, 1984.

This book is an English translation of the first four chapters of Itô's book, Probability Theory (in Japanese, Iwanami-Shoten, Tokyo, 1978), based on courses of probability at different universities. By reading these chapters we can well conclude that it would be desirable to have an English translation of the complete book, which must be just as enjoyable as these translated chapters are.

The first chapter deals with finite trials in order to acquaint the reader with some of the ideas of probability. It is written with mathematical accuracy, but it does not require any knowledge of higher mathematics. The rest is based on measure theory and analysis, assuming that the reader is familiar with them. The second chapter treats properties of probability spaces and measures. It contains, for example, the extension theorem, direct products of probability measures, the Luzin theorem and the Lebesgue decomposition.

The main aim of Chapter 3 is to rigorously define the concept of probability for general trials in terms of measure theory. Itô first defines conditional probability with respect to decompositions of the sample space (Kolmogorov's definition), and then explains Doob's definition followed by a discussion of the relationship between these two definitions. Properties of sums of independent random variables are discussed in the last chapter. Convergent and divergent series of random variables, strong laws of large numbers, Lindeberg's central limit theorem are included.

At the end of each section several problems are presented with hints for solution to help the reader to understand the material involved. Unfortunately, these four chapters of the book do not contain any references, leaving the reader without orientation concerning further parts of probability theory.

Students and researchers who like modern, abstract and rigorous mathematics will enjoy this book, and it can be also used as a text.

Lajos Horváth (Szeged and Ottawa)

I. M. James, General Topology and Homotopy Theory, VIII+248 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

This book gives a good glimpse of the basic material of the subject in a convenient form. The text is as self-contained as possible, and can be understood easily. The subject is developed in eight chapters. The first two chapters contain the fundamental knowledge concerning categories and topological spaces. The third chapter is connected with the various categories associated with the basic category of topology, which is of central importance. Chapter 4 contains an outline of the basic theory of topological groups and, in particular, topological transformation groups. The last four chapters are devoted to various aspects of homotopy theory. In Chapter 5 the notion of homotopy of maps is introduced, the homotopic classification of maps and the classification of the points of a space into path-components are studied. In Chapter 6 the notions of a fibration and of its dual, a cofibration are defined, based on homotopy lifting property and homotopy extension property, respectively. These are of fundamental importance in the theory of classification of maps by homotopy. Chapter 7 deals with separation axioms and discusses various concepts which are associated with them. In the final chapter the extension problem for mappings has been considered using the notion of cofibration.

Only familiarity with the theory of point-set topology is supposed.

L. Gehér (Szeged)

A. J. E. M. Janssen—P. van der Steen, Integration Theory (Lecture Notes in Mathematics, 1078), V+224 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The main purpose of this book is to present and compare various ways to introduce Lebesgue-integration. This presentation is mainly based on lectures of N. G. de Brujin. The following topics are treated among others: the Riemann and the Riemann—Stieltjes integral; measurable functions and measurable sets; L^p spaces; measure spaces and integration; approximation properties of measurable sets and measurable functions; the Riesz approximation theorem; Baire sets and Baire functions; the Radon—Nikodym theorem; continuous linear functionals on L^pspaces; the Fubini—Stone and Tonelli—Stone theorems; the Fourier transform in $L^2(R)$.

The book contains a large set of exercises. In some cases the exercises extend the theory while several examples may help the reader to understand the theory more deeply. There are hardly any prerequisites for studying the book: a course on introductory calculus is sufficient.

J. Németh (Szeged)

S. Kantorovitz, Spectral Theory of Banach Space Operators (Lecture Notes in Mathematics, 1012), VIII+179 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The material of this book is based on lectures given at various universities in 1981. The main purpose is to get a spectral analysis and operational calculus for a rather large class of Banach space operators. The book contains results obtained in the last few years. The text is divided into the following 14 chapters: Operational calculus, Examples, First reduction, Second reduction, Volterra elements, The family $S+\varrho V$, Convolution operators in L^p, Some regular semigroups, Similarity, Spectral analysis, The family $S+\varrho V$, S unbounded, Similarity (continued), Singular C^n operators, Local analysis.

Familiarity with Banach space theory and Banach algebra theory is supposed.

L. Gehér (Szeged)

M. H. Karwan—V. Lotfi—J. Telgen—S. Zionts, Redundancy in Mathematical Programming (A State-of-the-Art Survey), (Lecture Notes in Economics and Mathematical Systems, 206), VIII+285 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

Redundancy in mathematical programming is common, being generally brought about by the lack of complete knowledge about the system of constraints and the desire on the part of the problem formulator not to omit essential elements of the formulation. This volume presentsts an up-to-date survey of methods for identifying and removing redundancy and presents the results of extensive empirical tests on these methods. Based on the results of these tests, recommendations for improvements of the methods are given and some of the improvements are tested.

The first chapter considers the phenomena of redundancy as it arises in formulating mathematical problems, and a comprehensive survey of the literature on it is presented. The second chapter contains the mathematical foundations and notation. The succeeding chapters discuss several methods of identifying redundant constraints in linear programming (Chapter 3—Chapter 15). Chapter 16 deals with the improvements and extensions of the methods discussed earlier.

This book has a theoretical value because it holds together the main methods on redundancy, and it is valuable from the practical point of view as well, because the results of extensive empirical tests on moderate size linear programming problems provide a means of comparing the various methods.

G. Galambos (Szeged)

B. S. Kashin—A. A. Saakjan, Orthogonal series (Russian), 496 pages, Nauka, Moscow, 1984.

This excellent book collects together, in a unified treatment, a lot of important results in the field achieved during the last 25 years mainly by Russian authors. Among others, the following basic facts have been revealed:

(i) A number of statements of the properties of the classical trigonometric system have a common nature so as they remain valid for a wide class of orthonormal systems (e.g., for uniformly bounded or complete ONS).

(ii) There are certain questions where nonclassical ONS (e.g., the Franklin system) behave better than the classical ones.

(iii) The study of general function systems is often reduced to the study of ONS (e.g., in the case of the so-called factorization theorems).

The book is not intended to comprise all new essential results in the field. Some themes are not included at all or appear only in the form of remarks. The choice of the material presented was definitely influenced by the topics of the seminar by D. E. Menshov and P. L. Ul'janov on the theory of functions of a real variable, which has been working at the State University of Moscow for many years.

A certain portion of the theorems in the book has not been presented in any monograph yet and even experts will find new pieces of information for themselves. Nevertheless, the authors were guided by the rule that a graduate student could understand everything without special efforts. In other words, the authors prove all assertions whose proofs lie outside the standard university material at the Soviet universities. In order to emphasize the essence of the methods, the results are usually not presented in their most general forms.

The reader is only assumed to be familiar with the elementary facts in the theory of functions of a real or a complex variable, and in functional analysis; approximately to such an extent what is contained, e.g. in the books "Elements of the theory of functions and functional analysis" by A. N. Kolmogorov and S. V. Fomin and "A short course in the theory of analytic functions" by A. I. Markushevich. In addition, certain supplementary information is added in two Appendices to the end of the book.

The book consists of ten chapters, Remarks on Notations, two Appendices, Notes, List of References involving 201 items, and a short Subject Index. These Notes provide detailed accounts of the priority as well as comments and additional remarks concerning the results included in the text.

Ch. 1 contains the fundamental notions of completeness, totality, minimality, biorthogonality, bases, unconditional bases, etc. and interrelations among them. Ch. 2 is a concise summarization of stochastic independent functions. Ch. 3 deals with the Haar system whose basic properties play a key role in Chs. 8—10. The main emphasis is laid on the study of unconditional convergence of Fourier—Haar series. Ch. 4 gives a brief account of the trigonometric system including the proof of the Littlewood conjecture.

Ch. 5 is a good summary of the chief points in the theory of the Hilbert transform. The famous theorem of C. Fefferman that the dual space of Re $\mathscr{H}^1$ is the BMO is also presented with a proof using the notion of atoms. (Re $\mathscr{H}^1$ consists of those functions $f \in L^1(-\pi, \pi) = L^1$ for which the conjugate function $\tilde{f}$ also belongs to L^1.)

In Ch. 6 the Faber—Schauder and the Franklin systems are studied in detail. The result of P. Woytaszczyk that the Franklin system is an unconditional basis in the nonperiodic space $\mathscr{H}(0, 1)$ is presented, as well. A consequence is that Re $\mathscr{H}^1$ also possesses an unconditional basis.

Ch. 7 deals with the questions of orthogonalization of systems by means of extension of their domain to a larger set, and with the famous factorization theorems of E. M. Nikishin and B. Maure.

Ch. 8 collects the most traditional theorems on the a.e. convergence of general orthogonal series, in particular, the Menshov—Rademacher theorem, the influence of the order of magnitude of the Lebesgue functions, etc.

Ch. 9 is devoted to divergence problems. For example, they deal with the theorem of P. L. Ul'janov and A. M. Olevskiĭ according to which there is no complete ONS of unconditional convergence for l^2, and the theorems of S. V. Bochkarev according to which any uniformly bounded ONS exhibits almost the same divergence behavior as the classical trigonometric system.

Ch. 10 offers a delicate selection of the vast subject of the representation of (not necessarily a.e. finite) measurable functions by orthogonal series using a certain convergence notion (convergence in measure, a.e. convergence, etc.).

The above short account can hardly give a right impression of the wealthiness of this well-written book. The authors have been quite successful in including the most useful results in the field. The book is highly recommended to both beginners and experts in Classical Analysis who want to make acquaintance with the up-to-date methods and results in the theory of orthogonal series. It will certainly stimulate new researches in this area as well as various applications in Functional Analysis, Probability, etc.

F. Móricz (Szeged)

R. Lidl and G. Pilz, Applied Abstract Algebra (Undergraduate Texts in Mathematics), XVIII + 545 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

The purpose of this volume is to give a systematic introduction to applications of modern algebra at an advanced undergraduate level. A first course on abstract algebra is prerequisite, although most of the concepts needed to understand the material are explained in detail.

The book is organized around 3 topics: Boolean algebras, finite fields, and semigroups. Applications of Boolean algebras include switching circuits and simplification methods in propositional logic. Additional material demonstrates the use of Boolean algebras in topology and in probability theory. Coding theory, combinatorial applications, algebraic cryptopraphy, and linear recurring sequences comprise the core of the material on applications of finite fields. The combinatorial applications are Hadamard matrices, balanced incomplete block design, Steiner systems, and Latin squares. Fast adding and Pólya's theory of enumeration give further topics on finite fields. Semigroups and their relation to automata and formal languages form the subject ot the last main area. The discussion on automata culminates in the Krohn—Rhodes decomposition theory.

Many illustrative examples and exercises help the reader in attaining new concepts and methods.

The last chapter is devoted to detailed solutions of the exercises. Except for the last one, all chapters end with historical notes. Several computer programs are contained in an appendix.

The book is warmly recommended to everyone who is familiar with the material of a standard first algebra course and wants to get acquainted with the highlights of applied modern algebra.

Z. Ésik (Szeged)

Padé Approximation and its Applications, Proceedings, Bad Honnef, Germany, 1983. Edited by H. Werner, H. J. Bünger (Lecture Notes in Mathematics, 1071), VI+264 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

Several topics are considered nowadays in the field of Padé approximation. Their common property is that functions (on one or more variable) are approximated by means of rational fractions (continued fractions and other expressions) or that related techniques (acceleration of convergence of sequences) are used in applications. Recently, it has become a visible trend towards the treatment of multivariate problems.

This volume (which is the ninth one in a series started in 1972 in Canterbury) is not an introduction to the Padé approximation. It contains some results concerning special problems. From the content; M. de Bruin: Some Convergence Results in Simultaneous Rational Approximation to the Set of Hypergeometric Functions $\{{}_1\mathcal{F}_1(1;c_i;z)\}_{i=1}^n$. A. Cuyt: The Mechanism of the Multivariate Padé Process. A. Iserles: Order Stars and the Structure of Padé-Tableaux. F. Lambert, M. Musette: Solitary Wawes, Padeons and Solitons. A. Magnus: Riccati Acceleration of Jacobi Continued Fractions and Laguerre—Hahn Orthogonal Polynomials.

Some results are useful in problems of the numerical application of approximation theory including their implementations on a personal computer, although several papers present new methods more or less in an experimental stage of the development.

G. Németh (Budapest)

M. H. Protter—H. F. Weinberger, Maximum Principles in Differential Equations, VII+261 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

It was a long time ago when the maximum principle became one of the best known and most useful tools of the theory of differential equations. Perhaps the root of the maximum principle is the elementary fact that a function f: $[a, b]\to R$ with $f''>0$ assumes its maximum value at one of the endpoints of $[a, b]$. The main advantages of the principle are its relatively short simple and descriptive forms and its easy and natural applicability in various fields. By the principle we have information about the solutions without knowing them explicitly. Therefore the principle is an important tool in the approximation of the solutions. Although the maximum principle is used especially for partial differential equations, it has various attractive and simple applications to ordinary differential equations too, furnishing a natural introduction.

The second, third and finally the fourth chapter are devoted to elliptic, parabolic and hyperbolic partial differential equations, respectively. The book contains classical as well as modern results. (This is a corrected reprint of the second printing originally published in 1967.) The material is chosen and organized in such a way that this work may be recommended not only to mathematicians but also to physicists, chemists, engineers and students, who are interested in differential equations. Surely the readers will enjoy the interesting exercises ending the sections.

L. Pintér (Szeged)

H. Reinhard, Equations différentielles, XIV+446 pages, Gauthier-Villars, Paris, 1982.

There are many books discussing the theory of ordinary differential equations. Some of them made a significant effect on the further development of the theory by raising new ideas. But after a while the effect decreases. The new problems and the new results require a new presentation of the fundamentals. Some results and methods which were new a few years ago are nowadays frequently used means and fundamental theorems. This gives a possibility for a partly new discussion of the fundamentals.

In addition to providing a frame for deep understanding, a work that puts the fundamentals in new light may help the reader in getting acquainted with modern results and becoming able to read up-to-date articles.

In my opinion this is such a book. The notions and theorems are well thought out and illustrated by several examples. Surely this book will be very interesting and useful for both mathematicians (also specialists find interesting ideas in the work) and non-mathematicians, e.g. engineers, who are interested in the applications.

L. Pintér (Szeged)

G. de Rham, Differentiable Manifolds (Grundlehren der mathematischen Wissenschaften, 266), X+167 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

The book is the English translation of Georges de Rham's classic work on differential manifolds published originally by Hermann (Paris) in 1955. It gives a coherent exposition of the theory of differential forms on a manifold and harmonic forms on a Riemannian space.

The first two chapters give an introduction to the elements of differential manifolds and the theory of differential forms, while Chapter III is devoted to the study of currents means of which the relationship between differential forms and chains can be described. This part contains the classical de Rham's cohomology theory. The last chapter deals with the Hodge theory of harmonic differential forms on a Riemannian space. There is given a new proof for Hodge's fundamental theorem. In addition, Kodaira's decomposition theorem and an interesting theorem of A. Andreotti and E. Vesentini is considered too.

A new introduction of S. S. Chern is added to this English translation, where the last sentence reads: "I believe, however, that in his enthusiasm for new results a mathematician will be well-advised to stop at this landmark, where he will have a lot to learn both on the mathematics and on the mathematical style."

Z. I. Szabó (Szeged)

A. Rényi, A Diary on Information Theory, 192 pages, Akadémiai Kiadó, Budapest, 1984.

Many people nourish the notion of remoteness of mathematics and mathematicians from the rest of society, and frequently enough they are proven right in their belief. There are however a number of exemptions from this assumed scenario, and Rényi was certainly one of them. Devoted to his research as he was, he also managed to find time for writing on mathematics for the joy of sharing it with others. It gives one great pleasure to know that he had succeeded eminently in his strive to do so in books like *Dialogues on Mathematics* and *Letters on Probability*.

The first part of the present book, entitled: "On the Mathematical Notion of Information (Diary of a university student)", was intended to be a continuation of these two books. Unfortunately, it was left unfinished at his untimely death. The final chapter of the text given here was completed from Rényi's own notes by one of his students, Gyula Katona. Just like that of the other two, the intent of the present work is to explain what mathematics is, or can be, in a literary, enjoyable style. An imaginary university student, Bonifác Donát, sends him his diary which, after reading it, Rényi proposes to publish without any modification. The result is a deeply probing masterpiece, just like its predecessors.

This book also contains a number of other popular articles by Rényi, most of which appeared in various journals. They are on games of chance and probability, on the teaching of probability, on variations on a theme by Fibonacci, and on the mathematical theory of graphs and trees. Their overall message is again the theme of how one can come to like and appreciate mathematics. While some mathematical knowledge is certainly of help to read some of them, their deep insight and style of writing are often convincing enough even without understanding all the technical details which themselves are also explained in a series of footnotes by Gyula Katona.

The translator Zsuzsanna Makkai-Bencsáth of Montreal, and translation editors Marietta and Tom Morry of Ottawa, Canada have done an excellent job in rendering the original Hungarian into English. Students of any trade of life should find it to their liking, a gift from someone who cared for them.

Miklós Csörgő (Ottawa)

W. Ruppert, Compact Semitopological Semigroups: An Intrinsic Theory (Lecture Notes in Mathematics, 1079), 260 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1984.

A semigroup S with a topology such that only the left and right translations are supposed to be continuous are called semitopological semigroups (for in topological semigroups the multiplication map $S\times S\to S$ is continuous). They occur naturally in the context of selfmaps of a topological space and operator semigroups on a locally convex vector space, endowed with the weak or strong topology. The theory of compact semitopological semigroups has been intensively studied for the last 25 years. The present Lecture Notes give an introduction and a systematic treatment of this theory directed towards the intrinsic structural topological and algebraic approach, avoiding the use of methods borrowed from functional analysis. Chapter I is devoted to the basic facts of the semitopological theory. Especially the set $E(S)$ of all idempotents and the existence of minimal idempotents are investigated. Various methods for the construction of semigroups are introduced and used for the study of examples and counterexamples. Chapter II deals with the study of the subset of $S\times S$ for the semitopological semigroup S where the multiplication map $S\times S\to S$ is jointly continuous. Using these results various structural assertions are formulated. In Chapter III the semitopological compactifications of locally compact topological groups are investigated. These questions are closely connected with the theory of weakly almost periodic functions and the corresponding compactifications of topological groups. There are given interesting applications in this direction. Chapter IV is devoted to the study of the structure of semitopological semigroups with identity 1 defined on a compact connected subset of a Euclidean manifold such that 1 is an inner point. In an Appendix the author gives a survey on the most interesting actual open questions and problems of semitopological semigroup theory.

At the end of each chapter there is a summary of main results as well as additional comments and references.

The reader is supposed to be familiar with some basic knowledge in semigroup theory and general topology. We recommend these notes to everybody working in related fields of mathematics.

Péter T. Nagy (Szeged)

Seminar on Nonlinear Partial Differential Equations, Edited by S. S. Chern (Mathematical Sciences Research Institute Publications, Volume 2), 373 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

This book consists of 18 lectures held at a seminar organized by the Mathematical Sciences Research Institute for graduate students and mathematicians working in other branches of mathematics. But the lectures are instructive for users of partial differential equations too, e.g. for physicists, chemists and engineers.

Perhaps just the same is valid for the nonlinear partial differential equations that S. A. Antman says in the introduction of this lecture about nonlinear elasticity: "There are many reasons why nonlinear elasticity is not widely known in the scientific community: (i) It is basically a new science whose mathematical structure is only now becoming clear. (ii) Reliable expositions of the theory often take a couple of hundred pages to ge to the heart of the matter. (iii) Many expositions are written in a complicated indicial notation that boggles the eye and turns the stomach."

The lectures of this seminar — many of them are real gems — will surely win students and others over to partial differential equations. In general the lectures present the question, the essential results, problems and accurate bibliography. As is seen, the lecturers' aim is the clear exposition of ideas that lead the readers to the understanding of the results.

To give a short foretaste we enumerate the titles of some lectures: An introduction to Euler's Equations for an incompressible fluid (A. J. Chorin), A walk through partial differential equations (F. John), Remarks on zero viscosity limit for nonstationary Navier—Stokes flows with boundary (T. Kato), Free boundary problems in mechanics (J. B. Keller), Shock waves, increase of entropy and loss of information (P. D. Lax), Analytical theories of vortex motion (J. Neu), Applications of the maximum principle (M. H. Protter), Minimax methods and their application to partial differential equations (P. H. Rabinowitz), Equations of plasma physics (A. Weinstein).

L. Pintér (Szeged)

S. Shelah, Proper Forcing (Lecture Notes in Mathematics, 940), XXIX+496 pages, Springer-Verlag, Berlin—Heidelberg—New York, 1982.

The author's main concern here is to apply forcing techniques. Quoting his words: "We adopted an approach I heard from Baumgartner and may have been used by others: not proving that forcing works, rather take axiomatically that it does and go ahead to applying it. As a result we assume only knowledge of naive set theory (except some isolated points later on in the book). The idea of this approach is that otherwise when the student learns what is axiomatic set theory and how you can show by forcing that *CH* may fail (and that *CH* holds by learning something on *L*) the course is finished. But he has only a vague idea of the rich possibilities in forcing, and no idea how to use them".

Central to the discussions are independence results mostly on small uncountable cardinals; more generally, the volume concentrates on developing new methods for independence proofs, thus providing a good basis for further research in this important area of set theory.

The book is excellently written; in fact, it can be used in a number of different ways: as a text book on applications of forcing or as a source of recent results; it seems to be useful for experts and for (graduate) students of set theory. Moreover, I am quite sure, it is an enjoyable reading to everyone interested in modern independence research.

P. Ecsedi-Tóth (Szeged)

I. M. Sigal, Scattering Theory for Many-Body Quantum Mechanical Systems (Lecture Notes in Mathematics, Vol. 1011), IV+130 pages, Springer-Verlag, Berlin—Heidelberg—New York—Tokyo, 1983.

Quantum mechanical many body scattering theory is an important field from the practical point of view of physics. On the other hand it is an interesting application of the well elaborated spectral theory of Hilbert space self-adjoint operators. Though all the problems of scattering theory are far from beeing fully solved at present, this book — based mainly on the authors own work — summarizes some very important results: the existence and asymptotic completeness of the N particle scattering wave operators. The mathematical compactness makes this book comprehensive mainly for mathematicians working in the field of Hilbert space theory, one can miss only some remarks connecting the rigorous results with physical facts.

M. G. Benedict (Szeged)

I. Vincze, Mathematische Statistik mit industriellen Anwendungen, 502 Seiten (in zwei Bänden), Akadémiai Kiadó, Budapest, 1984.

Das ist die zweite, erweiterte deutschsprachige Auflage des Buches. Da der Umfang des Buches geringfügig vergrößert wurde, war es möglich, ein neues Kapitel über die Theorie der Entscheidungsfunktionen einzufügen, den Begriff der Robustheit und einige Probleme der Weibull-Verteilung zu besprechen und den Shapiro—Wilk-Test anzugeben.

In der Einleitung wird der Gegenstand der Wahrscheinlichkeitsrechnung angegeben und die wichtigsten Aufgaben der Wahrscheinlichkeitsrechnung und der mathematischen Statistik definiert. Im zweiten Kapitel werden die — später benötigten — wahrscheinlichkeitstheoretischen Hilfsmittel behandelt, wobei man neben den elementaren Begriffen auch die Grenzwertsätze und die Gesetze der großen Zahlen diskutiert. Einige wichtige Wahrscheinlichkeitsverteilungen sind auch angegeben. Im nächsten Teil werden die Grundlagen der Stichprobenentnahme behandelt. So werden z. B. einfache, zwei — und mehrstufige und sequentielle Verfahren verglichen, und die geordnete Stichprobe definiert. Anschließend werden die wichtigsten Stichprobenfunktionen und Ergebnisse (Satz von Gliwenko, die Kolmogorowschen und Smirnowschen Grenzverteilungsätze) angegeben.

Das vierte Kapitel beschäftigt sich mit der Theorie der statistischen Schätzungen. Besonders die Punktschätzungen werden sehr eingehend behandelt (die Begriffe der erwartungstreuen Schätzung, der Wirksamkeit der Schätzung, der konsistenten und stark konsistenten Schätzungen, der suffizienten Schätzungen werden definiert, die Cramer—Raosche Ungleichung ist angegeben und auch die wichtigsten Methoden zur Erstellung von statistischen Schätzfunktionen sind enthalten). Daneben wird auch die Intervallschätzung und die Robustheit noch kurz besprochen.

Das fünfte Kapitel ist den Fragen der Prüfung von statistischen Hypothesen gewidmet. Neben einer sehr überschaubaren Einleitung von Grundlagen (parametrische und nichtparametrische Methoden, Fehler erster und zweiter Art, Gütefunktion, Testvergleiche) findet man eine große Auswahl von Tests, die für verschiedenen Fragen ausgearbeitet wurden. Zu den meisten Methoden sind einfache Beispiele hinzugefügt; dadurch ist das Verstehen der Methoden erleichtert. Die nächsten zwei kurzen Kapitel geben die wichtigsten Aufgaben des sequentiellen Stichprobenverfahrens und die Grundbegriffe der Theorie der Entscheidungfunktionen an. Beide Gebiete gehören zu den wichtigsten Teilen der modernen Statistik, und so bilden diese Kapitel eine gute Ergänzung zu den klassischen Methoden.

Im zweiten Band des Buches findet man drei weitere Kapitel: eins über Varianzanalyse, eins über Korrelations- und Regressionsanalyse und eins über die statistischen Methoden der Qualitätskontrolle. Sowohl bei der Varianzanalyse als auch bei der Korrelations- und Regressionsanalyse werden mehrere Aufgabe gestellt, theoretisch gelöst und an praktischen Beispielen demonstiert. Das letzte Kapitel zeigt, wie man aus mehreren in Frage kommenden Methoden eine geeignete Wahl treffen kann oder zu einem sich in der Praxis ergebenden Problem eine geeignete Methode entwerfen kann. Die Forderung nach Wirtschaftlichkeit führt oft nicht zu den wirksamsten Methoden, besonders dann, wenn die Methode sehr häufig anzuwenden ist, z. B. in der statistischen Qualitätskontrolle.

Das Buch strebt eine mathematisch präzise Behandlung des Stoffes an. Eine große Anzahl der Methoden ist angegeben und mit praktischen Beispielen sehr verständlich veranschaulicht. Das Buch stellt eine gute Einleitung in die Statistik dar für all jene die die Grundmethoden dieses Fäches in der Praxis anwenden wollen.

J. Csirik (Szeged)

K. Yosida, Operational Calculus (A Theory of Hyperfunctions) (Applied Mathematical Sciences, 55), X+170 pages, Springer-Verlag, New York—Berlin—Heidelberg—Tokyo, 1984.

The operational calculus is known as a frequently applied and successful technique in solving linear ordinary differential equations with constant coefficients as well as the telegraph equation that includes both the wave and the heat equations with constant coefficients.

The rapid development of this technique started with O. Heaviside's work and was motivated by his research in electromagnetic theory. In the further development and exact mathematical foundation J. Mikusiński's contribution was fundamental mainly by inventing the theory of convolution quotients. His book: "Operational Calculus" is widely read and available in several languages.

As the author claims in the Preface: "The aim of the present book is to give a simplified exposition as well as an extension of Mikusiński's operational calculus". The simplification means mainly the presentation of a plain proof of the Titchmarsh convolution theorem and the fact that the author need not rely upon the Titchmarsh convolution theorem for solving linear ordinary differential equations with constant coefficients. The extension relates to the calculus itself as well as a new application to the Laplace differential equation. The part, devoted to the applications to partial differential equations, discusses a number of problems concerning physics and engineering.

The book is written in an elegant, concise manner, in a lucid style. A number of examples and exercises enlighten the abstract concepts. This book is warmly recommended to everybody (including both students and researchers) who is interested in this mathematical discipline as well as to physicists and engineers who want to apply this technique to their problems.

E. Durszt (Szeged)

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